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Article

On the Stabilization of the Solution to the Initial-Boundary Value Problem for One-Dimensional Isothermal Equations of Viscous Compressible Multicomponent Media Dynamics

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Abstract: An initial-boundary value problem is considered for one-dimensional isothermal equations of the dynamics of viscous compressible multicomponent media, which are a generalization of the Navier–Stokes equations. We prove the stabilization of the solution to the initial-boundary value problem while the time tends to infinity, without simplifying assumptions for the structure of the viscosity matrix, except for the standard physical requirements of symmetry and positive definiteness.

Keywords: viscous compressible medium; multicomponent flows; stabilization of solution

1. Statement of the initial-boundary value problem and formulation of the main result

Modeling the motion of multicomponent media, and hence the solution of the resulting mathematical problems, is an interesting and relatively little-studied problem of both physics and mathematics. Until now, a unified approach to this problem has not been developed, and there is no advanced mathematical theory about the existence, uniqueness, and properties of solutions to initial-boundary value problems that arise in this modeling. The purpose of this article does not include a detailed review of this problem as a whole; to a certain extent, an idea of it can be obtained from monographs [1,2], as well as from the reviews given in the articles [3,4]. The aim of the article is a mathematical study of the asymptotic behavior of the solution at $t \rightarrow +\infty$ of the following initial-boundary value problem for one-dimensional isothermal equations of viscous compressible multicomponent media:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \quad (1)$$

$$\rho \frac{\partial u_i}{\partial t} + \rho v \frac{\partial u_i}{\partial x} + K \frac{\partial \rho}{\partial x} = \sum_{j=1}^N v_{ij} \frac{\partial^2 u_j}{\partial x^2}, \quad i = 1, \dots, N, \quad (2)$$

$$\rho|_{t=0} = \rho_0(x), \quad u_i|_{t=0} = u_{0i}(x), \quad i = 1, \dots, N, \quad (3)$$

$$u_i|_{x=0} = u_i|_{x=1} = 0, \quad i = 1, \dots, N. \quad (4)$$

Here ρ , u_i , $i = 1, \dots, N$ are respectively the density and velocity of the medium components, which are sought functions of time $t \in [0, T]$, $0 < T < +\infty$, and of the point x of the flow domain $\Omega = \{x \in \mathbb{R} \mid 0 < x < 1\}$, $v = \frac{1}{N} \sum_{i=1}^N u_i$ is and average velocity of the medium, $N \geq 2$ is the number of components, $K = \text{const} > 0$, and constant viscosity coefficients v_{ij} , $i, j = 1, \dots, N$ from a symmetric matrix $\mathbf{N} > 0$.

As can be seen, equations (1), (2) are a generalization of the one-dimensional isothermal Navier—Stokes equations for the multicomponent case. Equations (2) contain higher order derivatives of the velocities of all components, since, unlike the Navier—Stokes equations, in which viscosity is a scalar, in the multicomponent case, due to the composite structure of viscous stress tensors, the viscosities form a matrix whose entries are responsible for viscous friction. Diagonal elements are responsible for viscous friction within each component, while off-diagonal elements are responsible for friction between components. This does not allow us to automatically extend the known results for the Navier—Stokes equations to the multicomponent case. In the case of a diagonal viscosity matrix, the equations are connected possibly via the lower order terms only. In this paper, we consider a more complex case of an off-diagonal viscosity matrix. Stabilization of solutions to one-dimensional Navier—Stokes equations is studied in [5–9]. Unique solvability and asymptotic behavior (as $t \rightarrow +\infty$) of the solution to the considered one-dimensional equations of viscous compressible multicomponent media in the polytropic case is studied in [10–12]. Similar questions for related one-dimensional models of multicomponent media are discussed in [13–20].

It will be useful to use mass Lagrangian coordinates in the study of the equations (1), (2). Let us consider t and $y(x, t) = \int_0^x \rho(s, t) ds$ as new independent variables. Then the system (1), (2) takes the form

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial y} = 0, \quad v = \frac{1}{N} \sum_{j=1}^N u_j, \quad (5)$$

$$\frac{\partial u_i}{\partial t} + K \frac{\partial \rho}{\partial y} = \sum_{j=1}^N v_{ij} \frac{\partial}{\partial y} \left(\rho \frac{\partial u_j}{\partial y} \right), \quad i = 1, \dots, N. \quad (6)$$

After that, the flow domain Ω transforms to the domain $\tilde{\Omega} = \{y \in \mathbb{R} \mid 0 < y < d\}$, where $d = \int_0^1 \rho_0(x) dx > 0$, and the initial and boundary conditions arrive at the form

$$\rho|_{t=0} = \tilde{\rho}_0(y), \quad u_i|_{t=0} = \tilde{u}_{0i}(y), \quad i = 1, \dots, N, \quad (7)$$

$$u_i|_{y=0} = u_i|_{y=d} = 0, \quad i = 1, \dots, N. \quad (8)$$

The main result of the article is the following assertion.

Theorem 1. Assume that $\rho_0 \in W_2^1(0, 1)$, $\rho_0 > 0$, $u_{0i} \in W_2^1(0, 1)$, $i = 1, \dots, N$. Then $\rho \rightarrow d$, $u_i \rightarrow 0$, $i = 1, \dots, N$, as $t \rightarrow +\infty$, in the norm of the space $W_2^1(0, 1)$.

Proof. Let us obtain a priori estimates for the solution to the problem (1)–(4), which would be uniform in t , and from which the stabilization of the solution follows.

2. A priori estimates

Let us multiply the equations (2) by u_i , summarize in i and integrate in y , thus, using (1), we obtain

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^N \int_0^1 \rho u_i^2 dx + \sum_{i,j=1}^N v_{ij} \int_0^1 \left(\frac{\partial u_i}{\partial x} \right) \left(\frac{\partial u_j}{\partial x} \right) dx = -K \sum_{i=1}^N \int_0^1 u_i \left(\frac{\partial \rho}{\partial x} \right) dx. \quad (9)$$

Since $N > 0$, the second summand in the left-hand side of (9) satisfies the inequality¹

$$\sum_{i,j=1}^N v_{ij} \int_0^1 \left(\frac{\partial u_i}{\partial x} \right) \left(\frac{\partial u_j}{\partial x} \right) dx \geq C_1(N) \sum_{i=1}^N \int_0^1 \left(\frac{\partial u_i}{\partial x} \right)^2 dx. \quad (10)$$

Multiplying (1) by $\ln \rho - \ln d$ and integrating in x , we obtain for the right-hand side of (9)

$$-K \sum_{i=1}^N \int_0^1 u_i \left(\frac{\partial \rho}{\partial x} \right) dx = KN \sum_{i=1}^N \int_0^1 \rho \left(\frac{\partial v}{\partial x} \right) dx = -KN \frac{d}{dt} \int_0^1 (\rho \ln \rho - (\ln d + 1)\rho + d) dx. \quad (11)$$

Thus, using (10) and (11), from (9) we derive

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^N \int_0^1 \rho u_i^2 dx + KN \frac{d}{dt} \int_0^1 (\rho \ln \rho - (\ln d + 1)\rho + d) dx + C_1 \sum_{i=1}^N \int_0^1 \left(\frac{\partial u_i}{\partial x} \right)^2 dx \leq 0. \quad (12)$$

In the Lagrangian coordinates (t, y) the formula (12) looks like

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^N \int_0^d u_i^2 dy + KN \frac{d}{dt} \int_0^d \frac{1}{\rho} (\rho \ln \rho - (\ln d + 1)\rho + d) dy + C_1 \sum_{i=1}^N \int_0^d \rho \left(\frac{\partial u_i}{\partial y} \right)^2 dy \leq 0. \quad (13)$$

From (13), after integration in t , we obtain the estimate

$$\sum_{i=1}^N \int_0^d u_i^2 dy + \int_0^d \frac{\rho \ln \rho - (\ln d + 1)\rho + d}{\rho} dy + \sum_{i=1}^N \int_0^t \int_0^d \rho \left(\frac{\partial u_i}{\partial y} \right)^2 dy d\tau \leq C_2, \quad (14)$$

where $C_2 = C_2(C_1, K, N, d, \inf_{(0,d)} \tilde{\rho}_0, \sup_{(0,d)} \tilde{\rho}_0, \{ \|\tilde{u}_{0i}\|_{L_2(0,d)} \})$. Note that all terms in the left-hand side of (14) are nonnegative.

Let us rewrite the equations (6) as

$$\frac{1}{N} \sum_{j=1}^N \tilde{v}_{ij} \frac{\partial u_j}{\partial t} + \frac{K}{N} \left(\sum_{j=1}^N \tilde{v}_{ij} \right) \frac{\partial \rho}{\partial y} = \frac{1}{N} \frac{\partial}{\partial y} \left(\rho \frac{\partial u_i}{\partial y} \right), \quad i = 1, \dots, N, \quad (15)$$

where $\tilde{v}_{ij}, i, j = 1, \dots, N$ are the entries of the matrix $\tilde{\mathbf{N}} = \mathbf{N}^{-1}$. Summarizing (15) in i , we obtain

$$\frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \frac{\partial u_j}{\partial t} + \tilde{K} \frac{\partial \rho}{\partial y} = \frac{\partial}{\partial y} \left(\rho \frac{\partial v}{\partial y} \right), \quad (16)$$

where $\tilde{K} = \frac{K}{N} \left(\sum_{i,j=1}^N \tilde{v}_{ij} \right) > 0$. The equations (5) imply that

$$\rho \frac{\partial v}{\partial y} = - \frac{\partial \ln \rho}{\partial t}. \quad (17)$$

¹ Everywhere, $C_k, k \in \mathbb{N}$ stand for various positive constants depending on the values given in brackets, but independent of t .

Substituting $\rho \frac{\partial v}{\partial y}$ from (17) into the equality (16), we arrive at the relation

$$\frac{\partial^2 \ln \rho}{\partial t \partial y} + \tilde{K} \frac{\partial \rho}{\partial y} = -\frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \frac{\partial u_j}{\partial t}. \quad (18)$$

Let us multiply (18) by $\frac{\partial \ln \rho}{\partial y}$ and integrate in y , then we obtain the equality

$$\frac{1}{2} \frac{d}{dt} \int_0^d \left(\frac{\partial \ln \rho}{\partial y} \right)^2 dy + \tilde{K} \int_0^d \rho \left(\frac{\partial \ln \rho}{\partial y} \right)^2 dy = -\frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \int_0^d \left(\frac{\partial u_j}{\partial t} \right) \left(\frac{\partial \ln \rho}{\partial y} \right) dy. \quad (19)$$

The right-hand side of (19) can be converted via integration by parts and using (8), (17):

$$\begin{aligned} -\frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \int_0^d \left(\frac{\partial u_j}{\partial t} \right) \left(\frac{\partial \ln \rho}{\partial y} \right) dy &= -\frac{d}{dt} \left(\frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \int_0^d u_j \left(\frac{\partial \ln \rho}{\partial y} \right) dy \right) + \\ &+ \frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \int_0^d \rho \left(\frac{\partial u_j}{\partial y} \right) \left(\frac{\partial v}{\partial y} \right) dy. \end{aligned} \quad (20)$$

Hence from (19) it follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^d \left(\frac{\partial \ln \rho}{\partial y} \right)^2 dy + \tilde{K} \int_0^d \rho \left(\frac{\partial \ln \rho}{\partial y} \right)^2 dy &= -\frac{d}{dt} \left(\frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \int_0^d u_j \left(\frac{\partial \ln \rho}{\partial y} \right) dy \right) + \\ &+ \frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \int_0^d \rho \left(\frac{\partial u_j}{\partial y} \right) \left(\frac{\partial v}{\partial y} \right) dy. \end{aligned} \quad (21)$$

Let us integrate (21) in t :

$$\begin{aligned} \frac{1}{2} \int_0^d \left(\frac{\partial \ln \rho}{\partial y} \right)^2 dy + \tilde{K} \int_0^t \int_0^d \rho \left(\frac{\partial \ln \rho}{\partial y} \right)^2 dy d\tau &= \\ &= \frac{1}{2} \int_0^d \frac{1}{\tilde{\rho}_0^2} \left(\frac{\partial \tilde{\rho}_0}{\partial y} \right)^2 dy - \frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \int_0^d u_j \left(\frac{\partial \ln \rho}{\partial y} \right) dy + \\ &+ \frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \int_0^d \frac{\tilde{u}_{0j}}{\tilde{\rho}_0} \left(\frac{\partial \tilde{\rho}_0}{\partial y} \right) dy + \frac{1}{N} \sum_{i,j=1}^N \tilde{v}_{ij} \int_0^t \int_0^d \rho \left(\frac{\partial u_j}{\partial y} \right) \left(\frac{\partial v}{\partial y} \right) dy d\tau. \end{aligned}$$

The last equality and (14) lead to the estimate

$$\int_0^d \left(\frac{\partial \ln \rho}{\partial y} \right)^2 dy + \int_0^t \int_0^d \rho \left(\frac{\partial \ln \rho}{\partial y} \right)^2 dy d\tau \leq C_3, \quad (22)$$

where $C_3 = C_3(C_2, \tilde{\mathbf{N}}, \tilde{K}, N, \inf_{(0,d)} \tilde{\rho}_0, \|\tilde{\rho}_0\|_{W_2^1(0,d)}, \{\|\tilde{u}_{0i}\|_{L_2(0,d)}\})$.

Let note that the equation (5) implies, for every $t \in [0, T]$, the existence of a point $\tilde{y}(t) \in [0, d]$ such that

$$\rho(t, \tilde{y}(t)) = d. \quad (23)$$

Since

$$\ln \rho(t, y) = \ln \rho(t, \tilde{y}(t)) + \int_{\tilde{y}(t)}^y \frac{\partial(\ln \rho(t, s))}{\partial s} ds,$$

then by the Hölder inequality, taking into account (22) and (23), we deduce that

$$|\ln \rho(y, t)| \leq |\ln d| + \sqrt{d} \left\| \frac{\partial \ln \rho}{\partial y} \right\|_{L_2(0, d)} \leq C_4(C_3, d),$$

and we immediately conclude that

$$0 < \frac{1}{C_5} \leq \rho(t, y) \leq C_5 < +\infty, \quad C_5 = C_5(C_4). \quad (24)$$

Hence, from (14), (22) and (24) we obtain the estimate

$$\sum_{i=1}^N \int_0^t \int_0^d \left(\frac{\partial u_i}{\partial y} \right)^2 dy d\tau + \int_0^d \left(\frac{\partial \rho}{\partial y} \right)^2 dy + \int_0^t \int_0^d \left(\frac{\partial \rho}{\partial y} \right)^2 dy d\tau \leq C_6(C_2, C_3, C_5). \quad (25)$$

In order to derive the next group of estimates, we multiply (6) by $\frac{\partial^2 u_i}{\partial y^2}$ and integrate in y , thus we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^d \left(\frac{\partial u_i}{\partial y} \right)^2 dy + \sum_{j=1}^N v_{ij} \int_0^d \rho \left(\frac{\partial^2 u_i}{\partial y^2} \right) \left(\frac{\partial^2 u_j}{\partial y^2} \right) dy = - \sum_{j=1}^N v_{ij} \int_0^d \left(\frac{\partial \rho}{\partial y} \right) \left(\frac{\partial^2 u_i}{\partial y^2} \right) \left(\frac{\partial u_j}{\partial y} \right) dy + \\ + K \int_0^d \left(\frac{\partial \rho}{\partial y} \right) \left(\frac{\partial^2 u_i}{\partial y^2} \right) dy. \end{aligned} \quad (26)$$

Let us summarize the equation (26) over i and integrate over t , then we obtain

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^N \int_0^d \left(\frac{\partial u_i}{\partial y} \right)^2 dy + \sum_{i,j=1}^N v_{ij} \int_0^t \int_0^d \rho \left(\frac{\partial^2 u_i}{\partial y^2} \right) \left(\frac{\partial^2 u_j}{\partial y^2} \right) dy d\tau = \frac{1}{2} \sum_{i=1}^N \int_0^d \left(\frac{\partial \tilde{u}_{0i}}{\partial y} \right)^2 dy - \\ - \sum_{i,j=1}^N v_{ij} \int_0^t \int_0^d \left(\frac{\partial \rho}{\partial y} \right) \left(\frac{\partial^2 u_i}{\partial y^2} \right) \left(\frac{\partial u_j}{\partial y} \right) dy d\tau + K \sum_{i=1}^N \int_0^t \int_0^d \left(\frac{\partial \rho}{\partial y} \right) \left(\frac{\partial^2 u_i}{\partial y^2} \right) dy d\tau. \end{aligned} \quad (27)$$

The left-hand side of (27) satisfies the estimate

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^N \int_0^d \left(\frac{\partial u_i}{\partial y} \right)^2 dy + \sum_{i,j=1}^N v_{ij} \int_0^t \int_0^d \rho \left(\frac{\partial^2 u_i}{\partial y^2} \right) \left(\frac{\partial^2 u_j}{\partial y^2} \right) dy d\tau \geq \\ \geq C_7(C_1, C_5) \left(\sum_{i=1}^N \int_0^d \left(\frac{\partial u_i}{\partial y} \right)^2 dy + \sum_{i=1}^N \int_0^t \int_0^d \left(\frac{\partial^2 u_i}{\partial y^2} \right)^2 dy d\tau \right). \end{aligned} \quad (28)$$

Due to (25) and the inequalities ($i = 1, \dots, N$)

$$\left\| \frac{\partial u_i}{\partial y} \right\|_{C[0,d]}^2 \leq 2 \left\| \frac{\partial u_i}{\partial y} \right\|_{L_2(0,d)} \left\| \frac{\partial^2 u_i}{\partial y^2} \right\|_{L_2(0,d)}, \quad (29)$$

the second summand in the right-hand side of (27) satisfies the estimate

$$- \sum_{i,j=1}^N v_{ij} \int_0^t \int_0^d \left(\frac{\partial \rho}{\partial y} \right) \left(\frac{\partial^2 u_i}{\partial y^2} \right) \left(\frac{\partial u_j}{\partial y} \right) dy d\tau \leq \frac{C_7}{4} \sum_{i=1}^N \int_0^t \int_0^d \left(\frac{\partial^2 u_i}{\partial y^2} \right)^2 dy d\tau + C_8, \quad (30)$$

where $C_8 = C_8(C_6, C_7, \mathbf{N}, N)$. The third summand in the right-hand side of (27) can be estimated as follows:

$$K \sum_{i=1}^N \int_0^t \int_0^d \left(\frac{\partial \rho}{\partial y} \right) \left(\frac{\partial^2 u_i}{\partial y^2} \right) dy d\tau \leq \frac{C_7}{4} \sum_{i=1}^N \int_0^t \int_0^d \left(\frac{\partial^2 u_i}{\partial y^2} \right)^2 dy d\tau + C_9(C_6, C_7, K, N). \quad (31)$$

Thus, from (27), due to (28), (30) and (31), the estimate

$$\sum_{i=1}^N \int_0^d \left(\frac{\partial u_i}{\partial y} \right)^2 dy + \sum_{i=1}^N \int_0^t \int_0^d \left(\frac{\partial^2 u_i}{\partial y^2} \right)^2 dy d\tau \leq C_{10} \left(C_7, C_8, C_9, \left\{ \|\tilde{u}_{0i}\|_{W_2^1(0,d)} \right\} \right) \quad (32)$$

follows. Finally, integrating (26) in t , we arrive at the inequalities

$$\int_0^t \left| \frac{d}{d\tau} \int_0^d \left(\frac{\partial u_i}{\partial y} \right)^2 dy \right| d\tau \leq C_{11} (C_5, C_6, C_{10}, \mathbf{N}, K, N), \quad i = 1, \dots, N. \quad (33)$$

3. Stabilization of the solution with an unlimited increase in time

From (25) and (33), the following convergence follow

$$\left\| \frac{\partial u_i}{\partial y} \right\|_{L_2(0,d)} \rightarrow 0, \quad i = 1, \dots, N \quad (34)$$

as $t \rightarrow +\infty$. Differentiating (5) with respect to y and multiplying by $\frac{\partial \rho}{\partial y}$, using (29), we arrive at the estimate

$$\int_0^t \left| \frac{d}{d\tau} \int_0^d \left(\frac{\partial \rho}{\partial y} \right)^2 dy \right| d\tau \leq C_{12} (C_5, C_6, C_{10}, N). \quad (35)$$

Hence, as $t \rightarrow +\infty$, we have

$$\left\| \frac{\partial \rho}{\partial y} \right\|_{L_2(0,d)} \rightarrow 0. \quad (36)$$

Thus, it is proved that (see (23)) in the norm of $W_2^1(0, d)$

$$u_i \rightarrow 0, \quad i = 1, \dots, N, \quad \rho \rightarrow d \quad (37)$$

as $t \rightarrow +\infty$. It is easy to verify now that the same convergence takes place in the Euler variables in the norm of the space $W_2^1(0, 1)$. The theorem is proved. \square

4. Conclusions

For the system of differential equations of isothermal viscous compressible multicomponent media with an non-diagonal, symmetric, and positive-definite viscosity matrix, an analysis is made of the asymptotic behavior (as $t \rightarrow +\infty$) of the solution to the initial-boundary value problem in the case of one spatial variable. The main difficulty was to obtain a priori estimates. To overcome it, the mass Lagrangian coordinates have been used. As a result, new a priori estimates are obtained and stabilization of the solution of the initial-boundary value problem is proved.

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