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On Semiclassical Gravity and Horizon Area

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Abstract: The classical Einstein's Field Equation has been derived previously by demanding that the Clausius relation $dS = \delta Q/T$ holds. In this letter, we show that the semiclassical Einstein's Field Equation can be recovered using the generalized entropy S_{gen} by using the non-equilibrium thermodynamics where the Clausius relation is replaced by the entropy balance relation $dS = \delta Q/T + d_iS$, where d_iS is some internal entropy. Moreover, since the quantum fields violate the classical focusing, we, therefore, have to use quantum focusing in this case. Using then Jacobson's idea of thermodynamics of spacetime we recover the semiclassical equation of motion. We, therefore, in a sense also show the validity of the semiclassical approximation, a crucial approach for establishing a number of important ideas such as the Hawking effect.

Keywords: semiclassical gravity; generalized entropy; quantum expansion; thermodynamics of causal horizon; non-equilibrium thermodynamics of spacetime

1. Introduction

The connection between gravity and thermodynamics has long been explored. The first hints came from black hole thermodynamics when Hawking [1] showed that the area of a horizon(A) is a non-decreasing function of time.

$$\frac{dA}{dt} \geq 0 \tag{1}$$

Later, Bekenstein [2] proposed the equality of horizon area and entropy as

$$S = \gamma A \tag{2}$$

where γ is a constant of proportionality. Hawking [3] fixed this constant of proportionality by deriving the black hole temperature as[1]

$$\gamma = \frac{1}{4G\hbar} \tag{3}$$

where G is the universal gravitational constant and $\hbar$ is the reduced Planck's constant. In 1995, Jacobson [4] derived Einstein's field equation from this proportionality of horizon area and entropy and the fundamental thermodynamic result $dQ = TdS$ under equilibrium conditions. Some works under the non-equilibrium conditions were also presented later [5–7]. Jacobson took Q as the heat flux across a causal horizon while temperature T given by Unruh temperature [8] as observed by a local Rindler observer inside the horizon as

$$T = \frac{\hbar k}{2\pi} \tag{4}$$

where k is the local acceleration. He also assumed the proportionality of horizon area and entropy. Using these ingredients, he was able to derive Einstein's field equation in the form

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar\eta}T_{ab} \tag{5}$$

[1] In this paper we set $k_B = c = 1$

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where the constant $\eta = (4G\hbar)^{-1}$. The connection between gravity and thermodynamics was further explored by Padmanabhan [9–19] with some closely related follow up articles from his collaborators [20–23]. Recently, Verlinde [24] combined thermodynamic arguments and the holographic principle [25,26] and argued that gravity is an entropic force arising from the underlying microscopic theory to maximize its entropy. Very recently in 2016, Bousso et al. [27] conjectured the Quantum Focusing(QFC) which conjectures that the quantum expansion(Θ) given by

$$\Theta = \lim_{\mathcal{A}\to 0} \frac{4G\hbar}{\mathcal{A}} \frac{dS_{gen}}{d\lambda} \tag{6}$$

where

$$S_{gen} = \frac{A}{4G\hbar} + S_{out} \tag{7}$$

A is the area of the horizon and S_{out} is the entropy of matter outside the horizon, which cannot decrease

$$\frac{d\Theta}{d\lambda} \leq 0 \tag{8}$$

This definition of quantum expansion Θ allows us to generalize the classical focusing theorem to the semiclassical sector. The QFC implies a Quantum Bousso Bound which has already gathered a considerable amount of evidence [27–34]. It also implies the quantum singularity theorems [35,36], the generalized second law of causal horizons and holographic screens [37] and a new property of nongravitational theories, the Quantum Null Energy Condition(QNEC) [27,38–40]. Arvin [41] presented a restricted quantum focusing which he argued is sufficient to derive all known essential implications of the quantum focusing and also proved it on brane-world semiclassical gravity theories. In this letter, using Jacobson's approach and the QFC we show that the generalized entropy S_{gen} can be used to recover the semiclassical equation of motion. In the next section, we present our idea elaborately.

2. Recovering the Semiclassical Einstein's Field Equation

Hawking [3] showed that black holes could evaporate, thus their horizon area can decrease via Hawking radiation. One of the key assumptions he made is the validity of semiclassical Einstein's field equation in the semiclassical regime given by

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} \langle T_{\mu\nu} \rangle \tag{9}$$

where $G_{\mu\nu}$ is the Einstein tensor and $\langle T_{\mu\nu} \rangle$ is the expectation value of the energy-momentum tensor. We show how to recover this critical equation, thus establishing the validity of the semiclassical approximation under the assumption that the fluctuations in T_{ab} are negligible. Jacobson [4] derived Einstein's Field Equation from the proportionality of entropy and horizon area together with the fundamental thermodynamic relation $dQ = TdS$. The strongest evidence of the equality between the horizon area and entropy comes from the black hole physics [2,3] and from the holographic principle [25,26] which states that the information about the volume of space is stored on its boundary. Now, consider the following substitution to get a semiclassical result

$$A \to 4G\hbar S_{gen} \tag{10}$$

The argument for doing the above substitution is as follows: In the classical case, the entropy change is given by $dS = \eta\delta\mathcal{A}$ where $\delta\mathcal{A}$ is the area variation of a pencil of generators of the causal horizon $\mathcal{H}$ but semiclassically $\delta\mathcal{A}$ can be negative, so we should use $dS = \delta S_{gen}$ which is always positive. The intuitive argument is that classically we demand $d\theta/d\lambda \leq 0$ to impose a condition on spacetime curvature thereby leading to the classical equation of motion but semiclassically we should use the

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quantum focusing $d\Theta/d\lambda \leq 0$ to impose a condition on the spacetime curvature (since quantum fields violate classical focusing but even in this case quantum focusing holds) which should give the semiclassical equation of motion. In the semiclassical case, we work with the Quantum Expansion(Θ) which allows for the natural generalization of the classical focusing theorem to the semiclassical regime. Next, we precisely define our local causal horizon: the local neighborhood of any arbitrary spacetime point p can be viewed as a flat spacetime using the Equivalence principle. Through p we consider a spacelike 2-surface element Σ. The past horizon of Σ is called the local causal horizon which can be thought of as a local Rindler horizon passing through point p. Considering a local Rindler horizon through p, we can take a local (boost) killing vector χ^a, generating the horizon. To the past of Σ the heat flux can be defined as the boost energy across the horizon as

$$\delta Q = \int_{\mathcal{H}} \langle T_{ab} \rangle \chi^a d\Sigma^b \tag{11}$$

where we need to consider the angle brackets of the energy-momentum tensor for the semiclassical case. Here χ^a is a local boost Killing field generating the horizon $\mathcal{H}$. The relation between affine parameter λ and killing parameter v is generally given as $\lambda = -e^{-v}$. But as we show here, in the semiclassical case, we should use instead $\lambda = -e^{-2v}$ so as to recover the semiclassical equation. This can be understood as the vanishing of quantum expansion Θ to zeroth order in λ must occur at twice the rate to correctly impose the condition on spacetime curvature. Therefore, we get $\chi^a = -2k\lambda k^a$ and δQ takes the form

$$\delta Q = -2k \int_{\mathcal{H}} \lambda \langle T_{ab} \rangle k^a k^b d\lambda d\mathcal{A} \tag{12}$$

where k^a is the tangent vector to the horizon generator for an affine parameter λ. We can also write

$$\delta S_{gen} = \eta \int_{\mathcal{H}} \Theta d\lambda d\mathcal{A} \tag{13}$$

where $\eta = (4G\hbar)^{-1}$. Since in the semiclassical case, the surface (cross-section of the local causal horizon) is contracting (and also possibly shearing) at p, we cannot apply the equilibrium thermodynamic relation $dQ = TdS$, instead, we use the Clausius definition of entropy in the non-equilibrium case as [5,7,42]

$$dS = \frac{dQ}{T} + \delta N \tag{14}$$

where the rate of change of entropy is written as

$$dS = d_e S + d_i S \tag{15}$$

$d_e S$ is the rate of change of entropy exchange with the surroundings while $d_i S$ is the entropy developed internally in the system as a result of being out of equilibrium. It is important to note that $d_i S$ is a non-negative quantity in accordance with the second law. δN is the uncompensated heat which shows the amount of entropy associated with the heat which is intrinsic to the system when it undergoes an irreversible process. Let us expand Θ around p

$$\Theta = \Theta_p + \lambda \left.\frac{d\Theta}{d\lambda}\right|_p + \mathcal{O}(\lambda^2) \tag{16}$$

In the semiclassical regime, we have [27]

$$\frac{d\Theta}{d\lambda} = -\frac{1}{2}\theta^2 - \zeta^2 - R_{ab}k^a k^b + \frac{d}{d\lambda}\left(\frac{4G\hbar}{\mathcal{A}} S'_{out}\right) \tag{17}$$

Therefore Eq.(13) becomes

$$\delta S_{gen} = \eta \int_{\mathcal{H}} \left[\Theta_p - \lambda \left(\frac{\theta^2}{2} + \zeta^2 + R_{ab}k^a k^b - \frac{d}{d\lambda} \frac{4G\hbar}{\mathcal{A}} S'_{out} \right) \Big|_p \right] d\lambda d\mathcal{A} \tag{18}$$

Since at zero order in λ, $\Theta_p = 0$, we have

$$\theta_p = -\frac{4G\hbar}{\mathcal{A}} S'_{out} \Big|_p \tag{19}$$

We therefore get

$$\delta S_{gen} = \eta \int_{\mathcal{H}} \left[-2\lambda \left(\frac{\theta^2}{2} + \zeta^2 + R_{ab}k^a k^b \right) \right] d\lambda d\mathcal{A} \tag{20}$$

From this, we find

$$d_i S = -2\eta \int_{\mathcal{H}} \lambda \left(\frac{\theta^2}{2} + \zeta^2 \right) d\lambda d\mathcal{A} \tag{21}$$

and

$$d_e S = -2\eta \int_{\mathcal{H}} \lambda R_{ab} k^a k^b d\lambda d\mathcal{A} \tag{22}$$

Thus, the internal entropy production rate $d_i S$, in this case, is twice that of the classical one and has contributions from both scalar and tensorial degrees of freedom. To see why $d_e S$ and $d_i S$ take these expressions can be understood by expressing Eq.(21) in terms of the Killing parameter v as

$$d_i S = 4\eta \int_{\mathcal{H}} dv d\mathcal{A} \left(\frac{\theta^2}{2} + \zeta^2 \right) \geq 0 \tag{23}$$

as required by the second law (since the affine parameter λ and Killing parameter v on the horizon are related by $\lambda = -e^{-2v}$). Eq.(22) then follows from Eq.(18). Thus, given the entropy change $dS = \delta S_{gen}$, where $dS = d_i S + d_e S$, then Eq.(22) and Eq.(21) are unique choices for $d_e S$ and $d_i S$ respectively. Therefore, at first order in λ, equating $d_e S = dQ/T$ we obtain

$$\langle T_{ab} \rangle k^a k^b = \frac{\hbar \eta}{2\pi} R_{ab} k^a k^b \tag{24}$$

On the other hand, we have $d_i S = \delta N$ as the internal entropy associated with some irreversible dissipation occurring at the Rindler wedge. Eq.(24) implies

$$\frac{2\pi}{\hbar \eta} \langle T_{ab} \rangle = R_{ab} + f g_{ab} \tag{25}$$

for some function f. Local conservation implies $\langle T_{ab} \rangle$ is divergenceless and we therefore recover semiclassical equation of motion

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar \eta} \langle T_{ab} \rangle \tag{26}$$

where Λ is a constant identified as the cosmological constant.

3. Conclusion

In this letter, we showed that the semiclassical Einstein's field equation can be recovered in the semiclassical case using the generalized entropy S_{gen}. Since S_{gen} is a cutoff independent quantity, it reveals information about the full quantum gravity theory and shows that the semiclassical approximation can be trusted as long as the fluctuations in T_{ab} are negligible.

Conflicts of Interest: The author declares no conflict of interest.

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