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Polynomial recurrence for SDEs with a gradient-type drift, revisited

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Abstract: Polynomial recurrence bounds for a class of stochastic differential equations with a gradient type drift and an additive Wiener process are studied without Lyapunov functions.

Keywords: stochastic differential equation; gradient type drift; polynomial recurrence.

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1. Introduction

Let us consider a stochastic differential equation in R^d

$$dX_t = dB_t + \nabla U(X_t) dt \quad (1)$$

with initial data

$$X_0 = x. \quad (2)$$

Here B_t , $t \geq 0$ is a d -dimensional Brownian motion, X_t takes values in R^d , U is a symmetric non-positive function, $U(0) = 0$ and $\lim_{|x| \rightarrow \infty} U(x) = -\infty$. This function U is assumed to be locally bounded and locally in C^1 . The aim of this paper is to establish recurrence properties of the Markov process X_t , which are the usual preliminary steps to ergodicity, existence and uniqueness of its invariant probability measure, and to the bounds in the Law of Large Numbers type theorems, as well as to the bounds for the beta-mixing rate (cf. [21]). In this paper the goal is to establish some polynomial bound for the hitting time to some compact in \mathbb{R}^d by the process X and the moment bounds for the marginal distribution of the process X_t itself; the issues related to the invariant measure are left till further studies. This hitting time bound would not depend on the first derivatives of the function U , even though the drift in the SDE is of the gradient type. This may look a bit unusual because the drift in the SDE (1) is of the form $\nabla U(x)$. Such a problem – about bounds not depending explicitly on ∇U – was posed and in some particular case solved in [22]. Earlier, some other results in this area of SDEs with a gradient type drift were established by S.Ya. Makhno in his articles [13,14] and in the monograph [15]; more precisely, in [13–15] the drift is of the usual form, but the assumptions are stated for the integrated drift, which does correspond to our setting after an easy reformulation. Here we extend, and relax, and also correct some of the assumptions from [22], with the main aim to replace the assumptions of the limit type (see (7) in what follows) to the asymptotic inequalities (see (9) in what follows).

It is known that the rate of convergence to the invariant distribution as well as the rates for certain mixing coefficients may be derived from the estimates of the type

$$\mathbb{E}_x \tau^k \leq C(1 + |x|^m), \quad (3)$$

along with

$$\sup_{t \geq 0} \mathbb{E}_x |X_t|^m \leq C(1 + |x|^{m'}), \quad (4)$$

for some $k > 1$, $m, m', C > 0$, where $\tau = \inf(t \geq 0 : |X_t| \leq K)$ for some $K > 0$, see, e.g., [8,20], et al. Here the value of m in (3) and in the left hand side of (4) should be the same. In particular, for SDEs (1) it may be derived from (3) and (4) that

$$\text{var}(\mu_t^x - \mu^{inv}) \leq P(|x|)(1+t)^{-k'}, \quad (5)$$

with some k' and with some polynomial function P , at least, with a bounded function ∇U . Moreover, similar bounds may be established for the beta-mixing coefficient on the basis of the recurrence properties; this is known to be quite useful in various limit theorems (cf. [7]) as well as in the extreme value theory, cf. [11]. However, we do not pursue this goal here; certain applications will be studied in a separate paper.

The bounds like (3) under various assumptions were obtained for various classes of processes by many authors, see, in particular, [1,8,10], [17], [21], et al., and the references therein; yet, for SDEs all assumptions were usually – except the paper [22] – stated in terms of ∇U . See also [4,16] where stronger sub-exponential bounds were established under another standing assumption. In [20] and [21] a recurrence condition

$$-p = \limsup_{|x| \rightarrow \infty} (\nabla U(x), x) < 0$$

was used to get bounds like (5) and naturally these bounds also depend on some norm of ∇U . Here the problem was to find some analogue of the latter condition in terms of the limiting behaviour of the function U itself, as in [22] but under further relaxed assumptions.

In general, it seems to be a rare case where recurrence bounds may be established without using Lyapunov functions. Moreover, it is even not clear how to construct them for SDEs of such a type. Of course, **after** the first moment bound is established (the inequality (12) in what follows), it provides itself a Lyapunov function, but the most standard way of using them is the opposite, and here it is apparently not applicable.

The paper consists of three sections: this introduction; the main results split in two parts, the earlier results and the new ones; and the proofs.

2. Main results

2.1. Earlier results

Let us recall briefly some earlier results from [22] where a little more general SDE was considered, with the drift of the form $b(x) - \nabla U(x)$. In this paper we assume $b \equiv 0$, and use $+\nabla U$ instead of $-\nabla U(x)$; naturally, the assumptions will be rewritten accordingly, for example, in our case $U(x) \rightarrow -\infty$ in place of $U(x) \rightarrow +\infty$ in [22]. This reminder will be useful for comparison – that is, to see what is, indeed, new here – and because close but relaxed assumptions will be assumed in the sequel. Assume

$$\sup_{x, x': |x-x'| \leq 1} (U(x) - U(x')) < \infty. \quad (6)$$

Since the equation (1) uses only the gradient of U , the particular value of U at the origin is not important; for example, without loss of generality we may assume $U(0) = 0$.

The function U is also assumed to possess a central symmetry, i.e., it only depends on the value of $|x|$ at each point, and the function $V(u) := U(x)$ for $|x| = u$ here is assumed to be in the class $C^1[0, \infty)$. (NB: Apparently, at the origin it follows that $V'(0) = 0$, otherwise the gradient $\nabla U(0)$ may not exist.) In [22] the bounds were established under the recurrence condition

$$\lim_{\xi \rightarrow \infty} \frac{V(\xi)}{\ln \xi} + d = -p < 0, \quad (7)$$

and under certain relationships between the constants. In particular, the bound (3) was established under the condition $p > 1/2$ for any $0 < k < p + 1/2$ and with any $m = 2k + \varepsilon$ ($\varepsilon > 0$); the inequality (4) was shown to be valid with any $m < 2p - 1$ and $m' = m + 2\varepsilon$.

Remark 1. Corrections in the paper [22] are as follows. (1) It was erroneously assumed in [22] that the function U may have the form $U(x) = U^1(x) + U^2(x)$ with a special requirement on U^2 . This is wrong: U^2 should be identically zero, and all further assumptions on U^1 should be understood as the assumptions on U itself. (2) Also, instead of the reference on the elliptic Harnack inequality in [5], or elsewhere there should be a reference on the parabolic one, see [12, Theorem 6.27]. Also, the dominating process y_t was chosen in [22] in a non-optimal way, see (14) and (15) in what follows for a better choice. The assumption (9) is weaker than (7) in the case $p_1 = p_2 = p$. However, the main difference is the assumption on U in an interval form (9) in the next subsection instead of the limit (7) assumed in [22].

2.2. New result

Below $[a]$ denotes the integer value of $a \in \mathbb{R}^1$.

Theorem 1. Assume the condition $U(x) = U(x')$ for any pair x, x' such that $|x| = |x'|$, and let the assumption (6) hold true, and let ∇U be locally bounded. Let $V(\xi) := U(x)$ for any x such that $|x| = \xi$, and let there exist two constants p_1 and p_2 such that

$$1/2 < p_2 \leq p_1 \quad (8)$$

and

$$p_2 \leq \frac{-V(\xi)}{\ln \xi} - \frac{d-1}{2} \leq p_1, \quad (9)$$

for all $\xi > 0$ which are large enough, $\xi \geq K$ where $K > 0$ is arbitrary. Then the bound

$$\sup_{t \geq 0} \mathbb{E}_x |X_t|^m \leq C(1 + |x|^{m'}) \quad (10)$$

holds true with any

$$m < 2p_1 + 1, \quad (11)$$

and with

$$m' = m + 2(p_1 - p_2).$$

Moreover, for any positive integer value of $k < 1 + \frac{2p_2 - 1}{2(1 + p_1 - p_2)} = \frac{2p_1 + 1}{2(1 + p_1 - p_2)}$ and¹ $m = 2k(1 + p_1 - p_2)$, the bound holds,

$$\mathbb{E}_x \tau^k \leq C(1 + |x|^m). \quad (12)$$

Moreover the Markov process (X_t) possesses an invariant probability measure μ ; this measure admits the bound

$$\int |x|^\ell \mu(dx) < \infty, \quad \forall \ell < 2p_2 - 1. \quad (13)$$

Remark 2. The inequality (12) is satisfied with $k = 1$ and some appropriate m iff $p_2 > 1/2$, which is a stading assumption (8).

Remark 3. Under the assumptions of the theorem this invariant measure is apparently unique. It will be rigorously derived in further publications along with the rate of convergence to the invariant regime, for which rate the polynomial recurrence plays a crucial role.

Remark 4. Since ∇U is assumed locally bounded, the solution of the equation (1) is strong and pathwise unique, at least, locally in time (see, for example, [18]). As it follows from the finiteness almost surely of the hitting time τ , the solution does not explode and, hence, it remains strong and pathwise unique for all $t \geq 0$.

¹ Hence, any $m < 2p_1 + 1$ is available, as stated in (11).

3. Proof of theorem 1

1. Comparison to a solution for a 1D equation with reflection. Similarly to [22], after an application of Itô's formula to $d|X_t|$ and due to the comparison theorems for SDEs with reflection as, for example, in [9, Proofs] one gets,

$$|X_t| \leq y_t, \quad (14)$$

$$dy_t = d\bar{w}_t + \left(\frac{d-1}{2y_t} - V'(y_t) \right) dt + d\varphi_t \equiv d\bar{w}_t - \bar{V}'(y_t) dt + d\varphi_t, \quad (15)$$

where

$$\bar{w}_t = \int_0^t 1(|X_s| > 0) \frac{X_s dW_s}{|X_s|}$$

is a 1-dimensional Wiener process, and y_t is a (strong and pathwise unique) solution of the SDEs above with a non-sticky boundary condition at (any) point $K > 0$ so that $y_t \geq K$ for all t , φ is its local time at K , $\bar{V}'(y) = V'(y) - d/y$ for $y > 0$; in other words, we let

$$\bar{V}(y) = V(y) - \frac{d-1}{2} \ln y, \quad y > 0.$$

Notice that the condition (9) may be rewritten in the form

$$\zeta^{2p_2} \leq \exp(-2\bar{V}(\zeta)) \leq \zeta^{2p_1}, \quad \zeta \geq K. \quad (16)$$

Indeed, let us show (14): by Itô's formula for $|X_t| \neq 0$, denoting for convenience $Y_t := \sum_k (X_t^k)^2$ we have,

$$\begin{aligned} dY_t &= d \sum_k (X_t^k)^2 = \sum_k d(X_t^k)^2 = \sum_k 2X_t^k dX_t^k + \frac{1}{2} \sum_k 2(dX_t^k)^2 \\ &= 2|X_t| \underbrace{\sum_k \frac{X_t^k dW_t^k}{|X_t|}}_{=d\bar{w}_t} + \left(2|X_t| \sum_k \frac{X_t^k}{|X_t|} U'_{X^k}(X_t) + d \right) dt \\ &= |X_t| d\bar{w}_t + \left(2|X_t| \left\langle \frac{X_t}{|X_t|}, \nabla U(X_t) \right\rangle + d \right) dt = |X_t| d\bar{w}_t + (2|X_t| V'(|X_t|) + d) dt. \end{aligned}$$

Therefore, still for $|X_t| \neq 0$,

$$(dY_t)^2 = 4Y_t dt = 4|X_t|^2 dt,$$

and so,

$$\begin{aligned} d|X_t| &= d(Y_t)^{1/2} = \frac{1}{2}Y_t^{-1/2}dY_t + \frac{1}{2} \times \frac{1}{2} \times \left(-\frac{1}{2}\right)Y_t^{-3/2}(dY_t)^2 \\ &= \frac{1}{2} \left(\frac{2|X_t|d\bar{w}_t + (2|X_t|V'(|X_t| + d)dt)}{|X_t|} \right) - \frac{4|X_t|^2 dt}{8|X_t|^3} \\ &= d\bar{w}_t + \left(V'(|X_t|) + \frac{d}{2|X_t|} - \frac{1}{2|X_t|} \right) dt \\ &= d\bar{w}_t + \left(V'(|X_t|) + \frac{d-1}{2|X_t|} \right) dt = d\bar{w}_t + \bar{V}'(|X_t|)dt. \end{aligned}$$

Thus, where $|X_t| \geq K$, its stochastic differential has the same form as the stochastic differential of y_t , but unlike y_t , the process $|X_t|$ with positive probability takes the values less than K . By well-known comparison theorems,

$$P_x(|X_t| \leq y_t, t \geq 0) = 1.$$

NB: Here it is important that both solutions are strong.

2. Invariant density and its moments. The invariant density $f(x)$ of the process X_t is well known: it has a form $f(x) = c \exp(2U(x))$. Indeed, we may check the invariance equation $L^*f = 0$:

$$\begin{aligned} \frac{1}{c} \left(\frac{1}{2} \Delta f(x) - \operatorname{div}(f \nabla U) \right) &= \frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} \exp(2U(x)) - \sum_i \frac{\partial}{\partial x_i} (U_{x_i}(x) \exp(2U(x))) \\ &= \sum_i \left\{ \frac{1}{2} \frac{\partial}{\partial x_i} (2U_{x_i}(x) \exp(2U(x))) - \frac{\partial}{\partial x_i} (U_{x_i}(x) \exp(2U(x))) \right\} = 0. \end{aligned}$$

Further, for any finite dimension $d \geq 1$ we have,

$$\begin{aligned} \int_{|x|=1}^{\infty} |x|^m \exp(2U(x)) dx &= \int_{|x|=1}^{\infty} |x|^m \exp(2V(|x|)) dx \leq \int_{|x|=1}^{\infty} |x|^m |x|^{-2(p_2+(d-1)/2)} dx \\ &= \int_1^{\infty} r^{m-2(p_2+(d-1)/2)} r^{d-1} dr = \int_1^{\infty} r^{m-2p_2} dr. \end{aligned}$$

The last integral converges iff $m - 2p_2 < -1$, that is,

$$\boxed{m < 2p_2 - 1.}$$

Recall that the uniqueness of the invariant measure is not emphasized here, but see remark 3. The value of m may be integer, or non-integer here.

3. An upper bound for the normalizing constant. The invariant density $f(y)$ of the process y_t on the half-line $[|x|, \infty)$ with the reflection barrier $K = |x|$ has a form

$$C(|x|) \exp(2\bar{V}(y)), \quad y > |x|, \quad (17)$$

which can be easily verified by a direct computation of the stationarity equation $\bar{L}^*f = 0$ that follows in a standard way from Itô's formula with expectations for any $g(y_t)$ for $g \in C_0^2$ (with a compact support and with $g(|x|) = 0$), where \bar{L} is the generator of (y_t) and \bar{L}^* is its

adjoint with respect to Lebesgue's measure. Notice that at this stage it is irrelevant whether or not this invariant distribution of the process y_t is unique (in fact, it is, although, we do not use it and, hence, do not pursue this goal).

The normalizing identity implies the estimation from above under the condition $2p_1 > 1$ (it coincides with $2p_1 > d$ just for $d = 1$),

$$C(|x|) = \left(\int_{|x|}^{\infty} \exp(2\bar{V}(y)) dy \right)^{-1} \leq \left(\int_{|x|}^{\infty} \xi^{-2p_1} dy \right)^{-1} = (2p_1 - 1)|x|^{2p_1-1}, \quad (18)$$

for the values of $|x| \geq K$ where the assumptions (16) are valid. For smaller values of $|x|$ below K , convergence of the integral $\int_{|x|}^{\infty} \exp(2\bar{V}(y)) dy$ cannot be destroyed because in any finite neighbourhood of zero the function $\exp(2\bar{V}(y))$ is assumed bounded by one, see the condition (6). Naturally, the integral $\int_{|x|}^{\infty} \xi^{-2p_1} dy$ increases when $|x|$ decreases, so that for smaller values of $|x|$ we have also smaller values of $C(|x|)$. Hence, in all cases

$$C(|x|) \leq (2p_1 - 1)|x|^{2p_1-1} \vee C_0 \quad (19)$$

with some finite C_0 . Also note for the sequel that due to the assumption (9), the density $f(y)$ admits the bound $f(y) = c \exp(2\bar{V}(y)) \leq cy^{-2p_2}$ for $y \geq K$ and, hence, integrates some power function: namely, under the condition $p_2 > 1/2$ we have,

$$\int |y|^\ell f(y) dy \leq C(|x|) \int_{|x|}^{\infty} |y|^{\ell-2p_2} dy < \infty, \quad \forall \ell < 2p_2 - 1. \quad (20)$$

Note that the range for possible values of ℓ here coincides with that for ℓ in (13). This prompts that if we had no explicit formulae for the invariant distribution of the process (X_t) but only for the dominating one (y_t) , then still the right order for its finite moments could have been obtained using the technique based on Harris – Khasminskii's method.

4. The inequality (10) with any real value $m < 2p_2 - 1$ and with $m' = m + 2(p_1 - p_2)$ (where m' is not necessarily integer either) follows from a direct calculation: since by the comparison theorem the process $y_t^{|x|}$ with $|x|$ large enough does not exceed the stationary version of the Markov process satisfying the same non-sticky reflection SDE with the reflection barrier at $|x|$, then

$$\begin{aligned} \mathbb{E}_x |X_t|^m &\leq \mathbb{E}_{|x|} |y_t|^m \leq C(|x|) \int_{|x|}^{\infty} \xi^m \exp(2\bar{V}(\xi)) d\xi \\ &\leq (C|x|^{2p_1-1} \vee C_0) \int_{|x|}^{\infty} \xi^m \xi^{-2p_2} d\xi \\ &\leq (C|x|^{m+2(p_1-p_2)+1-1}) \vee C_0 = (C|x|^{m+2(p_1-p_2)}) \vee C_0 \end{aligned}$$

(here the constants C and C_0 may be different on different lines and even on the same line), which is true for any x large enough, due to comparison theorems for the processes y_t with different initial data y_0 , see, e.g., [19] for bounded coefficients; this result naturally generalizes to the locally bounded ones in the situation where there is no explosion for (y_t) . For any $|x|$ – not necessarily small – this implies the bound (4), as required. Note that the drift in [19] was assumed bounded, or, at most, satisfying a linear growth condition; however, given that all solutions are strong and that they are defined on the infinite interval of time without explosion, this assumption can be dropped and replaced by a local

boundedness of \bar{V}' outside zero. Important is that the values or norms of the drift ∇U do not contribute to the constants in the final bound where only features of the function U itself will be used.

5. The inequality (12). This is the crucial part of the statement of the theorem. Denote $v^q(\xi) = \mathbb{E}_\xi \gamma^q$ for any integer $q \geq 0$, $\gamma = \inf(t : y_t \leq K)$; $v^0(\xi) \equiv 1$. Recall that \bar{L} denotes the generator of y_t , that is,

$$\bar{L}g(y) = \frac{1}{2} g''(y) + \bar{V}'(y)g'(y),$$

where $\bar{V}(y) = V(y) - \frac{(d-1) \ln y}{2}$, $y > 0$. By virtue of the identity

$$\left(\int_0^\gamma 1 dt \right)^q = q \int_0^\gamma \left(\int_0^t 1 ds \right)^{q-1} dt,$$

it follows,

$$v^q(y_0) = q \mathbb{E}_{y_0} \int_0^\gamma v^{q-1}(y_t) dt,$$

for any q such that the integral in the right hand side converges. In turn, by Itô's or Dynkin's formula this implies an equation

$$Lv^q = -qv^{q-1}, \quad (q \geq 1) \quad (21)$$

(cf. with [2, theorem 13.17] where the equation is explained differently and under a stronger assumption which guarantees some exponential moment of γ). Evidently, one boundary value for the latter equation is $v^q(K) = 0$. Concerning the "second boundary value" usual for a PDE of the second order, it is seemingly missing here. The justification of the formula for solution below can be done by the limiting procedure as follows. Let $N > K$ be the second boundary (later on N would go to infinity). Let $v_N^q(\xi) = \mathbb{E}_\xi \gamma_N^q$ for any integer $q \geq 0$, $\gamma_N = \inf(t : y_t^N \leq K)$, where the process y_t^N is a solution of the equation similar to (15) but with another non-sticky reflection at N . Recall that all solutions are strong and, hence, may be constructed on the same probability space; see, e.g., [19] for SDEs with one boundary, and results from this paper are easily extended for the case with two finite boundaries. Apparently, $y_t^N \leq y_t$ for any t and N , and $\gamma_N \uparrow \gamma$ as $N \uparrow \infty$. So, by the monotone convergence, $v_N^q \uparrow v^q$ for all values of q , no matter whether or not the limit v^q is finite. Then the sequence of the functions $v_N^q(\xi)$ satisfies the equations (21) with boundary conditions

$$v_N^q(K) = 0, \quad (v_N^q)'(N) = 0.$$

The formula for solution of such an equation reads,

$$v_N^q(\xi) = 2q \int_K^\xi \exp(-2\bar{V}(y_1)) dy_1 \int_{y_1}^N v_N^{q-1}(y_2) \exp(2\bar{V}(y_2)) dy_2, \quad K \leq \xi \leq N,$$

which may be verified by a direct calculation. Indeed, substituting $\xi = K$, we get $v_N^q(K) = 0$, and by taking the derivative, we can see that

$$(v_N^q)'(\xi)|_{\xi=N} = 2q \exp(-2\bar{V}(\xi)) \int_\xi^N v_N^{q-1}(y_2) \exp(2\bar{V}(y_2)) dy_2|_{\xi=N} = 0.$$

The equation itself follows from a little calculus as follows:

$$\begin{aligned}
 (v_N^q)'(\xi) &= 2q \exp(-2\bar{V}(\xi)) \int_{\xi}^N v_N^{q-1}(y_2) \exp(2\bar{V}(y_2)) dy_2, \\
 (v_N^q)''(\xi) &= -2q \exp(-2\bar{V}(\xi)) v_N^{q-1}(\xi) \exp(2\bar{V}(\xi)) \\
 &\quad - 4q \bar{V}'(\xi) \exp(-2\bar{V}(\xi)) \int_{\xi}^N v_N^{q-1}(y_2) \exp(2\bar{V}(y_2)) dy_2 \\
 &= -2q v_N^{q-1}(\xi) - 4q \bar{V}'(\xi) \exp(-2\bar{V}(\xi)) \int_{\xi}^N v_N^{q-1}(y_2) \exp(2\bar{V}(y_2)) dy_2, \\
 Lv_N^q(\xi) &= \frac{1}{2} (v_N^q)''(\xi) + \bar{V}'(\xi) (v_N^q)'(\xi) \\
 &= -q v_N^{q-1}(\xi) - 2q \bar{V}'(\xi) \exp(-2\bar{V}(\xi)) \int_{\xi}^N v_N^{q-1}(y_2) \exp(2\bar{V}(y_2)) dy_2 \\
 &\quad + \bar{V}'(\xi) \times 2q \exp(-2\bar{V}(\xi)) \int_{\xi}^N v_N^{q-1}(y_2) \exp(2\bar{V}(y_2)) dy_2 = -q v_N^{q-1}(\xi),
 \end{aligned}$$

as required. Uniqueness of solution for a linear ODE system is well-known.

Hence, by induction, the function $v^q(\xi)$ satisfies a representation using the function v^{q-1} ,

$$v^q(\xi) = 2q \int_K^{\xi} \exp(-2\bar{V}(y_1)) dy_1 \int_{y_1}^{\infty} v^{q-1}(y_2) \exp(2\bar{V}(y_2)) dy_2. \quad (22)$$

By another induction this implies the inequalities (recall that $v^0 \equiv 1$):

$$\begin{aligned}
 v^1(\xi) &= 2 \int_K^{\xi} \exp(-2\bar{V}(y_1)) dy_1 \int_{y_1}^{\infty} v^0(y_2) \exp(2\bar{V}(y_2)) dy_2 \\
 &= 2 \int_K^{\xi} \exp(-2\bar{V}(y_1)) dy_1 \int_{y_1}^{\infty} \exp(2\bar{V}(y_2)) dy_2 \leq 2 \int_K^{\xi} y_1^{2p_1} dy_1 \int_{y_1}^{\infty} y_2^{-2p_2} dy_2 \\
 &= C \int_K^{\xi} y_1^{2p_1-2p_2+1} dy_1 = C(\xi^{2(p_1-p_2)+2} - K^{2(p_1-p_2)+2}) \leq C\xi^{2(p_1-p_2)+2}, \quad (23)
 \end{aligned}$$

which is finite under the condition that $p_2 > 1/2$ (otherwise the inner integral diverges). Further,

$$\begin{aligned} v^2(\xi) &= 4 \int_K^{\xi} \exp(-2\bar{V}(y_1)) dy_1 \int_{y_1}^{\infty} v^1(y_2) \exp(2\bar{V}(y_2)) dy_2 \\ &\leq C \int_K^{\xi} \exp(-2\bar{V}(y_1)) dy_1 \int_{y_1}^{\infty} y_2^{2(p_1-p_2)+2} \exp(2\bar{V}(y_2)) dy_2 \\ &\leq C \int_K^{\xi} y_1^{2p_1} dy_1 \int_{y_1}^{\infty} y_2^{2(p_1-p_2)+2-2p_2} dy_2 \\ &= C \int_K^{\xi} y_1^{2p_1} dy_1 y_1^{2p_1-4p_2+3} = C(\xi^{4(p_1-p_2)+4} - K^{4(p_1-p_2)+4}) \\ &\leq C\xi^{4(p_1-p_2+1)}, \end{aligned}$$

where in the calculus it was assumed that $2p_1 - 4p_2 + 2 < -1$, that is, that $p_1 < 2p_2 - 3/2$, otherwise the inner integral in the calculus diverges. Since from the beginning $p_1 \geq p_2$, for the value of p_2 this means that compulsory $p_2 > 3/2$.

It looks plausible that the general bound for a (finite) v^q is provided by the formula

$$v^q(\xi) \leq C_q \xi^{2q(1+p_1-p_2)}. \quad (24)$$

The base being already established, let us show the induction step. Assume that for $q = n - 1$ the formula is valid with some constant C_{n-1} , that is,

$$v^{n-1}(\xi) \leq C_{n-1} \xi^{2(n-1)(1+p_1-p_2)}.$$

Then for $q = n$, as long as the integrals in the calculus below converge, we have,

$$\begin{aligned} v^n(\xi) &= 2n \int_K^{\xi} \exp(-2\bar{V}(y_1)) dy_1 \int_{y_1}^{\infty} v^{n-1}(y_2) \exp(2\bar{V}(y_2)) dy_2 \\ &\leq 2n \int_K^{\xi} y_1^{2p_1} dy_1 \int_{y_1}^{\infty} C_{n-1} y_2^{2(n-1)(p_1-p_2+1)} y_2^{-2p_2} dy_2 \\ &= C_n \int_K^{\xi} y_1^{2p_1} y_1^{2n-1+2(n-1)p_1-2np_2} dy_1 = C_n \int_K^{\xi} y_1^{2n-1+2np_1-2np_2} dy_1 \leq C_n \xi^{2n(p_1-p_2+1)}. \end{aligned}$$

Hence, indeed, by induction the formula (24) is established.

The values of q for which all the integrals in the calculus above converge for each $1 \leq n \leq q$ must satisfy the bound

$$2(q-1)(1+p_1-p_2) - 2p_2 < -1,$$

that is,

$$q < q_0 := 1 + \frac{2p_2 - 1}{2(1+p_1-p_2)} = \frac{1+2p_1}{2(1+p_1-p_2)}.$$

Recall that in this paper only integer values of q are used; however, q_0 introduced above is not necessarily integer, but in any case the inequality $q_0 > 1$ is equivalent to

$p_2 > 1/2$, which is necessary and sufficient for the finiteness of the first v^1 . This proves the last statement of the theorem. QED

4. Discussion

In most of earlier studies Lyapunov functions were used to check recurrent properties of a Markov process, either assumed “ad hoc”, or derived from some properties of the coefficients of an SDE. Here the method does not use Lyapunov functions at all because it is totally unclear how to construct them in the case under consideration. Instead, the method is based on comparison and on the system of equations like in [2, Theorem 13.17]; the recurrence bounds do not use any norm of the drift, but only some its integral characteristics. The only previous result of this sort to the best of the author’s knowledge was established by himself in 2000 (in Russian) and 2001 (English translation) [22], and some *other properties* related to convergence of SDE solutions to some limit if the coefficients converge in a weak integral sense were due to the works by S.Ya. Makhno [13–15]. The assumptions of the papers [22] were considerably extended in this paper, and certain corrections were made. Further studies may try to extend or waive the assumptions of the central symmetry on the drift, and include a multiplicative Wiener noise, and possibly work with non-integer values of q .

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Abbreviations

The following abbreviations are used in this manuscript:

SDE stochastic differential equation

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