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Small Values and Chung's Laws of the Iterated Logarithm for Spatial Surface of Operator Fractional Brownian Motion

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Abstract: The multivariate Gaussian random fields with matrix-based scaling laws are widely used for inference in statistics and many applied areas. In such contexts interests are often in symmetry and in the rates of change of spatial surfaces in any given direction. This article analyzes the almost sure sample function behavior for operator fractional Brownian motion, including multivariate fractional Brownian motion. We obtain the estimations of small ball probability and the strongly locally nondeterministic for operator fractional Brownian motion in any given direction. Applying these estimates we obtain Chung's laws of the iterated logarithm for spatial surfaces of operator fractional Brownian motion. Our results show that the precise rates of change of spatial surfaces are completely determined by the self-similarity exponent.

Keywords: operator fractional Brownian motion; small ball probability; operator self-similarity; Chung's law of the iterated logarithm

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1. Introduction

Let $X = \{X(t) = (X_1(t), ..., X_p(t)), t \in \mathbb{R}\}$ be an operator fractional Brownian motion with exponent D, that is, X is a mean zero Gaussian process in $\mathbb{R}^p$, has stationary increments and is operator self-similar with exponent D, $X(0) = \mathbf{0}$ a.s. We will use the following definition for operator self-similarity, which corresponds to that of operator self-similar random fields of Sato [31]. An $\mathbb{R}^p$-valued random field $X = \{X(t), t \in \mathbb{R}\}$ is said to be operator self-similar if there exists an $D \in L(\mathbb{R}^p)$, where $L(\mathbb{R}^p)$ is the set of linear operators on $\mathbb{R}^p$, such that for all $c > 0$,

$$X(c\,\cdot) \overset{\mathrm{d}}{=} c^D X(\cdot), \tag{1}$$

where $X \overset{\mathrm{d}}{=} Y$ means the processes X and Y have the same finite dimensional distributions and $c^D = \sum_{k=0}^{\infty} \frac{1}{k!}(\ln c)^k D^k$.

An operator fractional Brownian motion has been introduced in the seminal papers of Laha and Rohatgi [14], Hudson and Mason [11], Maejima and Mason [19] and Didier and Pipiras [8] as extensions to the class of fractional Brownian motion. If $d = 1$, X is the standard fractional Brownian motion. If the operator self-similar exponent $D = \mathrm{diag}(H_1, ..., H_p)$ is a diagonal operator, where $H_i \in (0,1)$ for all $1 \le i \le p$, then X is referred to multivariate fractional Brownian motion. See Stoev and Taqqu [33], Lavancier et al. [15,16] and Coeurjolly et al. [3] for more information about multivariate fractional Brownian motion.

The cross-covariance structure of multivariate fractional Brownian motion induced by the operator self-similarity and the stationarity of the increments has been first studied in [16], Theorem 2.1, without having recourse to the Gaussian assumption. Amblard et al. [1] have parameterized this covariance structure in a more simple way as follows.

Firstly, the i-th component of the multivariate fractional Brownian motion is a fractional Brownian motion with exponent $H_i \in (0,1)$, $1 \le i \le p$. The cross covariances are given in the following proposition.

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Proposition 1. *([16]). The cross covariances of the multivariate fractional Brownian motion satisfy the following representation, for all $(i,j) \in \{1,...,p\}^2$, $i \neq j$,*

(1) If $H_i + H_j \neq 1$, there exist $\sigma_i > 0$, $\sigma_j > 0$, $(\rho_{i,j}, \eta_{i,j}) \in [-1,1] \times \mathbb{R}$ with $\rho_{i,j} = \rho_{j,i} = \mathrm{corr}(X_i(1), X_j(1))$ and $\eta_{i,j} = -\eta_{j,i}$ such that

$$\begin{aligned}\mathbb{E}[X_i(s)X_j(t)] = \frac{\sigma_i\sigma_j}{2}\{&(\rho_{i,j} + \eta_{i,j}\mathrm{sign}(s))|s|^{H_i+H_j} + (\rho_{i,j} - \eta_{i,j}\mathrm{sign}(t))|t|^{H_i+H_j} \\ &- (\rho_{i,j} - \eta_{i,j}\mathrm{sign}(t-s))|t-s|^{H_i+H_j}\}.\end{aligned} \tag{2}$$

(2) If $H_i + H_j = 1$, there exist $\sigma_i > 0$, $\sigma_j > 0$, $(\tilde{\rho}_{i,j}, \tilde{\eta}_{i,j}) \in [-1,1] \times \mathbb{R}$ with $\tilde{\rho}_{i,j} = \tilde{\rho}_{j,i} = \mathrm{corr}(X_i(1), X_j(1))$ and $\tilde{\eta}_{i,j} = -\tilde{\eta}_{j,i}$ such that

$$\mathbb{E}[X_i(s)X_j(t)] = \frac{\sigma_i\sigma_j}{2}\{\tilde{\rho}_{i,j}(|s| + |t| - |s-t|) + \tilde{\eta}_{i,j}(t\ln|t| - s\ln|s| - (t-s)\ln|t-s|)\}. \tag{3}$$

Remark 1. Note that coefficients $\rho_{i,j}, \tilde{\rho}_{i,j}, \eta_{i,j}, \tilde{\eta}_{i,j}$ depend on the parameters (H_i, H_j). Assuming the continuity of the cross covariances function with respect to the parameters (H_i, H_j), the expression (3) can be deduced from (2) by taking the limit as $H_i + H_j$ tends to 1, noting that $((s+1)^H - s^H - 1)/(1-H) \to s\ln|s| - (s+1)\ln|s+1|$ as $H \to 1$. We obtain the following relations between the coefficients: as $H_i + H_j \to 1$

$$\rho_{i,j} \sim \tilde{\rho}_{i,j} \quad \text{and} \quad (1 - H_i - H_j)\eta_{i,j} \sim \tilde{\eta}_{i,j}.$$

This convergence result can suggest a reparameterization of coefficients $\eta_{i,j}$ in $(1 - H_i - H_j)\eta_{i,j}$.

The multivariate models evoke several applications where matrix-based scaling laws are expected to appear, such as in long range dependent time series (see, e.g., [2,5,6,9,12,22]) and queueing systems (see, e.g., [7,13,20,21]). Like fractional Brownian motion in the univariate setting, operator fractional Brownian motion is a natural starting point in the construction of estimators for operator self-similar processes due to its tight connection to stationary fractional processes and its being Gaussian (on the general theory of operator self-similar processes, see [4,11,14,19]). The fractal nature for operator fractional Brownian motion such as the Hausdorff dimension of the image and graph, and spatial surface properties such as hitting probabilities, transience, and the characterization of polar sets were studied by Mason and Xiao [24].

The purpose of this paper is to investigate the rates of change of spatial surfaces for operator fractional Brownian motion in any given direction. We obtain the estimations of small ball probability and the prediction error for operator fractional Brownian motion in any given direction. Applying these estimates we investigate its small values and prove its Chung's law of the iterated logarithm. A Chung's law of the iterated logarithm for multivariate fractional Brownian motion is derived from it as a consequence. Our results show that the rates of change of spatial surfaces for operator fractional Brownian motion in any given direction are completely determined by the self-similarity exponent.

Our method of proof relies heavily on the multivariate regular variation theory developed by Meerschaert [25,26], Meerschaert and Scheffler [28,29], Seneta [32] and Wang [36], which is the key ingredient of the proof of our main results (see Sections 2 and 3).

We use the notations $f_t \sim g_t$ if $\lim f_t/g_t = 1$, $f_t \asymp g_t$ if there exits a constant $K > 0$ such that $K^{-1} \leq \liminf f_t/g_t \leq \limsup f_t/g_t \leq K$. All constants K appearing in this paper (with or without subscript) are positive and may not necessarily be the same in each occurrence. More specific constants in Section i will be denoted by $K_{i,1}, K_{i,2}, ...$ For $x \in \mathbb{R}$, let $\log x = \ln(x \vee e)$, $\log\log x = \ln\ln(x \vee e^2)$.

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2. Methodology

2.1. Spectral index function and exponential operators

From the Jordan decomposition's theorem (see [10] p. 129 for instance), as done in [29] for the study of operator-self-similar Gaussian random fields, there exists a real invertible $p \times p$ matrix P such that $E = P^{-1}DP$ is of the real canonical form, which means that E is composed of diagonal blocks which are either Jordan cell matrix of the form

$$\begin{pmatrix} v & 0 & \dots & 0 \\ 1 & v & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & v \end{pmatrix}$$

with v a real eigenvalue of D or blocks of the form

$$\begin{pmatrix} \Lambda & 0 & \dots & \dots & 0 \\ I_2 & \Lambda & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & I_2 & \Lambda \end{pmatrix} \text{ with } \Lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

where the complex numbers $a \pm ib$ $(b \neq 0)$ are complex conjugated eigenvalues of D.

Let us recall that the eigenvalues of D are denoted by $v_j, j = 1, ..., d$ and that $0 < a_j = \Re(v_j) < 1$ for $j = 1, ..., d$. There exist $J_1, ..., J_d$, where each J_j is either a Jordan cell matrix or a block of the form (4), and P a real $p \times p$ invertible matrix such that

$$D = P \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_d \end{pmatrix} P^{-1}.$$

We can assume that each J_j is associated with the eigenvalue v_j of D and that

$$0 < a_0 \leq a_2 \leq \cdots \leq a_d < 1.$$

If $v_j \in \mathbb{R}$, J_j is a Jordan cell matrix of size $\tilde{l}_j = l_j \in \mathbb{N} \setminus \{0\}$. If $\lambda_j \in \mathbb{C} \setminus \mathbb{R}$, J_j is a block of the form (4) of size $\tilde{l}_j = 2l_j \in 2\mathbb{N} \setminus \{0\}$. Then for any $t > 0$,

$$t^D = P \begin{pmatrix} t^{J_1} & 0 & \dots & 0 \\ 0 & t^{J_2} & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & t^{J_d} \end{pmatrix} P^{-1}.$$

We denote by $(e_1, ..., e_p)$ the canonical basis of $\mathbb{R}^p$ and set $f_j = Pe_j$ for every $j = 1, ..., p$. Hence, $(f_1, ..., f_p)$ is a basis of $\mathbb{R}^p$. For all $j = 1, ..., d$, let

$$V_j = \text{span}\Big(f_k; \sum_{i=1}^{j-1} \tilde{l}_i + 1 \leq k \leq \sum_{i=1}^{j} \tilde{l}_i\Big).$$

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Then, each V_j is a D-invariant set and $\mathbb{R}^p = V_1 \oplus \cdots \oplus V_d$ is a direct sum decomposition of $\mathbb{R}^p$ into D-invariant subspaces. We may write $D = D_1 \oplus \cdots \oplus D_p$, where $D_i : V_i \to V_i$ and every eigenvalue of D_i has real part equal to a_i. The matrix for D in an appropriate basis is then block-diagonal with p blocks, the ith block corresponding to the matrix for D_i.

Let $\lambda_i = a_i^{-1}$ so that $\lambda_1 > \cdots > \lambda_d$. Let $\lambda(\theta) : \mathbb{R}^p \setminus \{0\} \to \{\lambda_1, \cdots, \lambda_d\}$ be the spectral index function, that is,

$$\lambda(\theta) = \lambda_i = 1/a_i \quad \text{for all } \theta \in L_i \setminus L_{i-1}, \tag{5}$$

where $L_i = V_1 \oplus \cdots \oplus V_i$ and $V_1, ..., V_d$ is the spectral decomposition $\mathbb{R}^p = V_1 \oplus \cdots \oplus V_d$ with respect to D. Choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^p$ such that $V_i \perp V_j$ for $i \neq j$, and let $\|x\| = \sqrt{\langle x, x \rangle}$ be the associated Euclidean norm. The operator norm of the linear operator A on $L(\mathbb{R}^p)$ is defined by

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1\}.$$

We first state several useful facts about the operator norm and exponential operators whose proofs are easy (see, e.g., [29] or [36] for their proofs) and will be used to our proofs.

i) $\|x\|/\|A^{-1}\| \leq \|Ax\| \leq \|A\| \cdot \|x\|$ for all $A \in L(\mathbb{R}^p)$ and all $x \in \mathbb{R}^p$;
ii) $\|AB\| \leq \|A\| \cdot \|B\|$ for all $A, B \in L(\mathbb{R}^p)$;
iii) If $A \in L(\mathbb{R}^p)$ and $s, t > 0$, then $s^A t^A = (st)^A$;
iv) If $A \in L(\mathbb{R}^p)$ and $t > 0$, then $t^{-A} = (1/t)^A = (t^A)^{-1}$;
v) If $AB = BA$ and $t > 0$, then $t^A t^B = t^{A+B}$.
vi) If $t \geq 1$, then $K_{2,1} t^{a_1 - \epsilon} \leq \|t^D\| \leq K_{2,2} t^{a_d + \epsilon}$ for any $0 < \epsilon < a_1$;
vii) If $0 < t < 1$, then $K_{2,3} t^{a_d + \epsilon} \leq \|t^D\| \leq K_{2,4} t^{a_1 - \epsilon}$ for any $0 < \epsilon < a_1$;

Let $U = (u_{ij})$ be a real invertible $d \times d$ matrix U such that $\mathrm{Cov}(X(1)) = UU^*$. In fact, it is symmetric and holds whenever $\mathrm{Cov}(X(1))$ is invertible. For any vector $\theta \in \Gamma := \mathbb{R}^p \setminus \{\mathbf{0}\}$, define $\varphi : (0, \infty) \to (0, \infty)$ by $\varphi(x) = \varphi(x, \theta) = \|(x^D U)^* \theta\|$, where A^* denotes the transpose of the matrix or vector A.

Lemma 1. *Let $\theta \in \mathbb{R}^p$ be an unit vector. Then, for any $\epsilon > 0$ and $s > 0$,*

$$\frac{\varphi(hs)}{\varphi(s)} \geq \begin{cases} K_{2,5} h^{\frac{1}{\lambda(\theta)} - \epsilon} & \text{if } h \geq 1 \\ K_{2,6} h^{a_1 + \epsilon} & \text{if } h \leq 1 \end{cases} \quad \text{and} \quad \frac{\varphi(hs)}{\varphi(s)} \leq \begin{cases} K_{2,7} h^{\frac{1}{\lambda(\theta)} + \epsilon} & \text{if } h \geq 1 \\ K_{2,8} h^{a_1 - \epsilon} & \text{if } h \leq 1 \end{cases} \tag{6}$$

Proof. The proof of both cases is similar, so we only proof the case $h \geq 1$. For any $\theta \in \Gamma$, there exists a unique $1 \leq i \leq d$ such that $\theta \in L_i \setminus L_{i-1}$, where $L_i = V_1 \oplus \cdots \oplus V_i$ and $V_1, ..., V_d$ is the spectral decomposition $\mathbb{R}^p = V_1 \oplus \cdots \oplus V_d$ with respect to D. Moreover, for any $\theta \in L_i \setminus L_{i-1}$, there exist $\theta_j \in V_j$, $\theta_j \neq 0$, $1 \leq j \leq i$, such that $\theta = \theta_1 + \cdots + \theta_i$ and $(t^D U)^* \theta = (t^{D_1} U_1)^* \theta_1 + \cdots + (t^{D_i} U_i)^* \theta_i$, where $D_1, .., D_d$ is the spectral decomposition of D and $U_1, .., U_d$ is the spectral decomposition of U. Then, for any $h \geq 1$,

$$\frac{\varphi^2(h, \theta)}{\varphi^2(h, \theta_i)} = \frac{\|(h^D U)^* \theta\|^2}{\|(h^{D_i} U_i)^* \theta_i\|^2} = 1 + \sum_{j=1}^{i-1} \frac{\|(h^{D_j} U_j)^* \theta_j\|^2}{\|(h^{D_i} U_i)^* \theta_i\|^2}.$$

Noting that every eigenvalue of D_j has real part equal to a_j, by Facts i), ii) and vi), we have that for any $\epsilon > 0$ and $1 \leq j \leq i$,

$$\|(h^{D_j} U_j)^* \theta_j\| \leq \|(h^{D_j})^* \theta_j\| \cdot \|U_j\| \leq K_{2,9} \|h^{D_j}\| \leq K_{2,10} h^{a_j + \epsilon} \tag{7}$$

and

$$\|(h^{D_j} U_j)^* \theta_j\| \geq \|(h^{D_j})^* \theta_j\| / \|U_j^{-1}\| \geq K_{2,11} \|\theta_j\| / \|(1/h)^{D_j}\| \geq K_{2,12} h^{a_j - \epsilon}. \tag{8}$$

Since $\epsilon > 0$ is arbitrary and $a_j < a_i$ for all $1 \le j \le i-1$, we have $\varphi(h,\theta) \asymp \varphi(h,\theta_i)$. Thus, by Facts ii) and vi), for any $\epsilon > 0$,

$$\frac{\varphi(hs)}{\varphi(s)} \asymp \frac{\varphi(hs,\theta_i)}{\varphi(s,\theta_i)} = \frac{\|(h^{D_i})^*(s^{D_i}U_i)^*\theta_i\|}{\|(s^{D_i}U_i)^*\theta_i\|} \le \|(h^{D_i})^*\| \le K_{2,13}h^{a_i+\epsilon}.$$

Similarly to the above inequality, we have

$$\frac{\varphi(s)}{\varphi(hs)} \le \|((1/h)^{D_i})^*\| \le K_{2,14}h^{-a_i+\epsilon}.$$

The proof is completed. □

Now we summarize some basic facts about Gaussian processes. Let $\{Z(t); t \in S\}$ be a Gaussian process. We provide S with the following metric

$$d(s,t) = \|Z(s) - Z(t)\|_2$$

where $\|Z\|_2 = (\mathbb{E}(Z^2))^{1/2}$. We denote by $N_d(S,\epsilon)$ the smallest number of open d-balls of radius ϵ needed to cover S and write $R = \sup\{d(s,t); s,t \in S\}$.

The following lemma is well known. It is a consequence of the Gaussian isoperimetric inequality and Dudley's entropy bound(see [35]).

Lemma 2. *There exists an absolute constant $C > 0$ such that for any $u > 0$, we have*

$$\mathbb{P}\Big(\sup_{s,t\in S} |Z(s) - Z(t)| \ge C(u + \int_0^R \sqrt{\log N_d(S,\epsilon)d\epsilon}\Big) \le \exp(-Ku^2). \tag{9}$$

Lemma 3. *Consider a function ψ such that $N_d(S,\epsilon) \le \psi(\epsilon)$ for all $\epsilon > 0$. Assume that for some constant $C > 0$ and all $\epsilon > 0$ we have*

$$\psi(\epsilon)/C \le \psi(\frac{\epsilon}{2}) \le C\psi(\epsilon).$$

Then

$$\mathbb{P}\Big(\sup_{s,t\in S} |Z(s) - Z(t)| \le u\Big) \ge \exp(-K\psi(u)). \tag{10}$$

This is proved in [34]; see also [30] and [18]. It gives an estimate for the lower bound of the small ball probability of Gaussian processes.

2.2. Strong local nondeterminism

Now we start to construct a moving average representation of operator fractional Brownian motion.

Lemma 4. *Let $D \in L(\mathbb{R}^p)$ be a linear operator with $0 < a_1, a_d < 1$. For $t > 0$, define*

$$X(t) = \int_{-\infty}^0 (t-x)^{D-\frac{1}{2}I} - (-x)^{D-\frac{1}{2}I}\mathbf{B}(dx) + \int_0^t (t-x)^{D-\frac{1}{2}I}\mathbf{B}(dx), \tag{11}$$

where $I \in L(\mathbb{R}^p)$ is the identity operator and $\{\mathbf{B}(s), -\infty < s < \infty\}$ is p-dimensional standard Brownian motion and i.i.d. components. Then the random field $X = \{X(t), t \in \mathbb{R}\}$ is an operator fractional Brownian motion with exponent D. Furthermore, X is isotropic in the sense that for every $t \in \mathbb{R}$,

$$X(t) \stackrel{\mathrm{d}}{=} |t|^D X(1), \tag{12}$$

and X has a version with continuous sample paths almost surely.

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Proof. The proof is similar to that for the stochastic integral representation of operator fractional Brownian motion given in Theorem 3.1 in [24], we omit the details. The proof is completed. □

The following result establishes the strongly locally nondeterministic for operator fractional Brownian motion in any given direction $\theta \in \Gamma$.

Lemma 5. *Let $X = \{X(t), t \in \mathbb{R}\}$ be an operator fractional Brownian motion in $\mathbb{R}^p$ with exponent D. If $0 < a_1, a_p < 1$ and $\det\mathrm{Cov}(X(1)) > 0$, then for any vector $\theta \in \Gamma$, all $0 < h < h_0$ and all $0 < t < h_0 - h$ with some $h_0 > 0$,*

$$\mathrm{Var}(\langle X(t+h), \theta\rangle \mid \langle X(s), \theta\rangle : 0 \le s \le t) \ge K_{3,1}\varphi^2(h). \tag{13}$$

Proof. From the representation (11) it easily follow that if $\{X(t)\}$ is an operator fractional Brownian motion with exponent D, then

$$\begin{aligned}
&\mathrm{Var}(\langle X(t+h), \theta\rangle \mid \langle X(s), \theta\rangle : 0 \le s \le t) \\
&\ge \mathrm{Var}\Big(\int_t^{t+h} \langle (t+h-x)^{D-\frac{1}{2}I}\mathbf{B}(dx), \theta\rangle\Big) \\
&= \mathrm{Var}\Big(\int_t^{t+h} \langle (t+h-x)^{D^*-\frac{1}{2}I}\theta, \mathbf{B}(dx)\rangle\Big) \\
&= \int_t^{t+h} \|(t+h-x)^{D^*-\frac{1}{2}I}\theta\|^2 dx \\
&= \int_0^{h} \|(h-x)^{D^*-\frac{1}{2}I}\theta\|^2 dx.
\end{aligned} \tag{14}$$

It follows from Facts v), vi) and vii) that $\varphi(h) \le \|U\|\|(h^D)^*\theta\|$ and $\|(h-x)^{D^*-\frac{1}{2}I}\theta\| \ge \|(h-x)^{D^*}\theta\|/\|(h-x)^{\frac{1}{2}I}\|$. Thus, by Lemma 1,

$$\begin{aligned}
&\int_0^h \varphi^{-2}(h)\|(h-x)^{D^*-\frac{1}{2}I}\theta\|^2 dx \\
&\ge \int_0^h \frac{\|(h-x)^{D^*}\theta\|^2}{\|U\|^2\|(h-x)^{\frac{1}{2}I}\|^2\|(h^D)^*\theta\|^2} dx \\
&\ge \int_0^h \frac{(1-x/h)^{2a_1+2\epsilon}}{\|U\|^2(h-x)} dx \\
&= \int_0^h \frac{(h-x)^{2a_1-1+2\epsilon}}{\|U\|^2 h^{2a_1+2\epsilon}} dx \\
&\ge K_{3,2}
\end{aligned} \tag{15}$$

Combining (14) and (15), we get (13). The proof is completed. □

2.3. Small ball probability

We establish the following estimation of small ball probability of spatial surfaces for operator fractional Brownian motion in any given direction $\theta \in \Gamma$.

Proposition 2. *Let $X = \{X(t), t \in \mathbb{R}\}$ be an operator fractional Brownian motion in $\mathbb{R}^p$ with exponent D. If $0 < a_1, a_d < 1$ and $\det\mathrm{Cov}(X(1)) > 0$, then for every compact set $T \subset \mathbb{R}$, any $t_0 \in T$ and any vector $\theta \in \Gamma$ and all $x \in (0,1)$,*

$$\exp\Big(-\frac{K_{3,3}h}{\Psi(x^2)}\Big) \le \mathbb{P}(M(t_0,h) \le x) \le \exp\Big(-\frac{K_{3,4}h}{\Psi(x^2)}\Big), \tag{16}$$

where $M(t_0,h) = M(t_0,h,\theta) = \sup_{s\in T, |s|\le h} |\langle X(t_0+s) - X(t_0), \theta\rangle|$ denotes the local modulus of continuity of $X(t)$ on t_0 in direction θ, $\Psi(x) = \inf\{y : \varphi(y) > x\}$ is the right-continuous inverse function of φ.

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Proof. Since $\mathrm{Cov}(X(1))$ is invertible, there exists a real invertible $p \times p$ matrix U such that $\mathrm{Cov}(X(1)) = UU^*$. By the operator self-similarity, for every $h \in \mathbb{R} \setminus \{0\}$,

$$\mathrm{Cov}(X(h)) = \mathrm{Cov}(|h|^D X(1)) = |h|^D U(|h|^D U)^*. \tag{17}$$

We denote the matrix $|h|^D U$ by U_h. Then, for $h \in \mathbb{R} \setminus \{0\}$, $U_h^{-1}X(h)$ is normal random variables in $\mathbb{R}^p$ with mean 0 and covariance matrix I_d. Thus, for all $x \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{P}(\varphi^{-1}(h)\langle X(t+h) - X(t), \theta\rangle \le x) \\ &= \mathbb{P}(\varphi^{-1}(h)\langle X(h), \theta\rangle \le x) \\ &= \mathbb{P}(\varphi^{-1}(h)\langle U_h U_h^{-1} X(h), \theta\rangle \le x) \\ &= \mathbb{P}(\varphi^{-1}(h)\langle U_h^{-1} X(h), \bar{\theta}_h\rangle \le x) \\ &= \mathbb{P}(\langle U_h^{-1} X(h), \tilde{\theta}_h\rangle \le x), \end{aligned} \tag{18}$$

where $\bar{\theta}_h = (U_h)^*\theta$ and $\tilde{\theta}_h = \|(U_h)^*\theta\|^{-1}(U_h)^*\theta$ is an unit vector in $\mathbb{R}^d$. Noting that $\langle U_h^{-1}X(h), \tilde{\theta}_h\rangle$ is a standard normal random variable, (18) implies that $\varphi^{-1}(h)\langle X(t+h) - X(t), \theta\rangle$ is a standard normal random variable. Thus,

$$\mathbb{E}[|\langle X(t+h) - X(t), \theta\rangle|^2] = \varphi^2(h). \tag{19}$$

Equip $S = [0, h]$ with the canonical metric

$$d(s,t) = \|\langle X(s) - X(t), \theta\rangle\|_2, \;\; s, t \in S, \tag{20}$$

and denote by $N_d(S, \epsilon)$ the smallest number of d-balls of radius $\epsilon > 0$ needed to cover S. Then it is easy to see that for all $\epsilon \in (0, 1)$,

$$N_d(S, \epsilon) \le \frac{K_{3,5}h}{\Psi(\epsilon^2)}. \tag{21}$$

Moreover, it follows from Lemma 1 that Ψ has the doubling property, i.e., $K_{3,6}\Psi(\epsilon) \le \Psi(\epsilon/2) \le K_{3,7}\Psi(\epsilon)$. Hence the lower bound in (16) follows from Lemma 3.

The proof of the upper bound in (16) is based on an argument in [30]. For any integer $n \ge 2$, we choose n points $t_{n,i} \in [0, 1]$, where $t_{n,i} = ih/n$, $i \in \{1, ..., n\}$. Then,

$$\mathbb{P}(M(t_0, h) \le x) \le \mathbb{P}\Big(\max_{1 \le i \le n} |\langle X(t_{n,i}), \theta\rangle| \le x\Big). \tag{22}$$

By Anderson's inequality for Gaussian measures and Lemma 5, we derive the following upper bound for the conditional probabilities

$$\mathbb{P}(X(t_{n,i}) \le x | X(t_{n,j}), 1 \le j \le i-1) \le \Phi\Big(\frac{K_{3,8}x}{\varphi(n^{-1}h)}\Big), \tag{23}$$

where $\Phi(x)$ is the distribution function of a standard normal random variable. It follows from (22) and (23) that

$$\mathbb{P}(M(t_0, h) \le x) \le \Big[\Phi\Big(\frac{K_{3,9}x}{\varphi(n^{-1}h)}\Big)\Big]^n. \tag{24}$$

By taking n to be the smallest integer $h[\Psi(x^2)]^{-1}$, we obtain the upper bound in (16). □

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3. Results

3.1. Zero-one laws for operator fractional Brownian motion

We establish the following zero-one laws for operator fractional Brownian motion to have Chung's law of the iterated logarithm, which may be of independent interest.

Lemma 6. *Let $X = \{X(t), t \in \mathbb{R}\}$ be an operator fractional Brownian motion in $\mathbb{R}^p$ with exponent D. If $0 < a_1, a_d < 1$ and $\det\mathrm{Cov}(X(1)) > 0$, then for every compact set $T \subset \mathbb{R}$, any $t_0 \in T$ and any vector $\theta \in \Gamma$, there exist a constant $0 \le C \le \infty$ such that*

$$\liminf_{h \to 0+} f_h M(t_0, h) = C \quad \text{a.s.}, \tag{25}$$

where

$$f_h = \frac{1}{\varphi(h/\log\log(1/h))}. \tag{26}$$

Proof. Let m be a scattered Gaussian random measure on $\mathbb{R}$ with Lebesgue measure l as its control measure; that is, $\{m(A), A \in \mathcal{E}\}$ is a centered Gaussian process on $\mathcal{E} = \{E \subset \mathbb{R} : l(E) < \infty\}$ with covariance function

$$\mathbb{E}[m(E)m(F)] = l(E \cap F).$$

Let $m_1, ..., m_d$ be d independent copies of m, and define

$$\mathbf{m}(A) = (m_1(A), ..., m_d(A)).$$

Then, we consider a version of operator fractional Brownian motion

$$X(t) = \int_{\mathbb{R}} (1 - \cos(tx)) \left(\frac{1}{|x|}\right)^{D+I/2} d\mathbf{m}(x) + \int_{\mathbb{R}} \sin(tx) \left(\frac{1}{|x|}\right)^{D+I/2} d\mathbf{m}'(x), \tag{27}$$

where $\mathbf{m}'$ is an independent copy of $\mathbf{m}$. This stochastic integral representation of operator fractional Brownian motion is given in [24].

Let $\Omega_1 := O(0,1) \subset \mathbb{R}$ and for $n \ge 2$, $\Omega_n := O(0,n) \setminus O(0,n-1) \subset \mathbb{R}$ such that $\Omega_1, \Omega_2, ...,$ are mutually disjoint, where the following notation is used: $O(x,r) = \{y \in \mathbb{R} : |x-y| \le r\}$. For $n \ge 1$ and $t \in \mathbb{R}$, let

$$Z_n(t) := \int_{\Omega_n} (1 - \cos(tx)) \left(\frac{1}{|x|}\right)^{D+I/2} d\mathbf{m}(x) + \int_{\Omega_n} \sin(tx) \left(\frac{1}{|x|}\right)^{D+I/2} d\mathbf{m}'(x), \tag{28}$$

Then $Z_n = \{Z_n(t), t \in \mathbb{R}\}, n = 1, 2, ...,$ are independent Gaussian fields. By (28), we express

$$X(t) = \sum_{n=1}^{\infty} Z_n(t), \quad t \in \mathbb{R}.$$

Equip $S = [0,1]$ with the canonical metric

$$d_{Z_n}(s,t) = d_{Z_n}(s,t,\theta) = \|\langle Z_n(s) - Z_n(t), \theta\rangle\|_2, \qquad s, t \in T,$$

and denote by $N(d_{Z_n}, S, \epsilon)$ the smallest number of d_{Z_n}-balls of radius $\epsilon > 0$ needed to cover S.

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It follows from (28) that

$$\langle Z_n(s)-Z_n(t),\theta\rangle = \int_{\Omega_n}(\cos(tx)-\cos(sx))\Big\langle\Big(\frac{1}{|x|}\Big)^{D^*+I/2}\theta, d\mathbf{m}(x)\Big\rangle + \int_{\Omega_n}(\sin(sx)-\sin(tx))\Big\langle\Big(\frac{1}{|x|}\Big)^{D^*+I/2}\theta, d\mathbf{m}'(x)\Big\rangle.$$

Thus,

$$\begin{aligned} d_{Z_n}(s,t) &= \Big(2\int_{\Omega_n}(1-\cos((t-s)x))\Big\|\Big(\frac{1}{|x|}\Big)^{D^*+I/2}\theta\Big\|^2 dx\Big)^{1/2} \\ &\le |t-s|\Big(\int_{\Omega_n}|x|^2\Big\|\Big(\frac{1}{|x|}\Big)^{D^*+I/2}\theta\Big\|^2 dx\Big)^{1/2} \\ &=: |t-s|K_n, \qquad s,t\in\mathbb{R}. \end{aligned} \tag{29}$$

To obtain the last inequality, in the integral we bound $1-\cos(tx)$ by $|t|^2|x|^2/2$. Then, by (29) and Theorem 4.1 in [27], we have

$$\limsup_{h\to 0+}\sup_{t,t+s\in T:|s|\le h}\frac{|\langle Z_n(t+s)-Z_n(t),\theta\rangle|}{\tau(0,h)} \le K_{4,1} \quad \text{a.s.}, \tag{30}$$

where $\tau(0,h)=|h|\sqrt{\log(1/h)}$. Put

$$X_M(t)=\sum_{n=1}^{M} Z_n(t), \quad t\in T.$$

It follows from Facts i)-vii) and $0<a_1<1$ that

$$h\sqrt{\log(1/h)}f_h \le K_{4,2}h^{1-a_1+\epsilon}\to 0$$

as h tends to zero. This, together with (30), yields that

$$\lim_{h\to 0+}\sup_{s\in T:|s|\le h} f_h|\langle X_M(t_0+s)-X_M(t_0),\theta\rangle| = 0 \ \text{ a.s.}$$

Therefore, the random variable

$$\liminf_{h\to 0+} f_h M(t_0,h)$$

is measurable with respect to the tail field of $\{Z_n\}_{n=1}^{\infty}$ and hence is constant almost surely. This implies that (25) holds. The proof is completed. □

3.2. Chung's law of the iterated logarithm for spatial surfaces

We shall establish the following Chung's law of the iterated logarithm for operator fractional Brownian motion.

Theorem 1. *Let $X=\{X(t),t\in\mathbb{R}\}$ be an operator fractional Brownian motion in $\mathbb{R}^p$ with exponent D. If $0<a_1,a_d<1$ and $\det\mathrm{Cov}(X(1))>0$, then for every compact set $T\subset\mathbb{R}$, any $t_0\in T$ and any vector $\theta\in\Gamma$,*

$$\liminf_{h\to 0+} f_h M(t_0,h) = K_{4,3} \quad \text{a.s.} \tag{31}$$

Denote the standard basis of $\mathbb{R}^p$ by $(e_1,...,e_p)$. By choosing $\theta=e_i$ and using Theorem 1 we obtain the following result about Chung's law of the iterated logarithm for the components $X_i (i=1,...,p)$ of X.

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Corollary 1. *Let $X = \{X(t), t \in \mathbb{R}\}$ be an operator fractional Brownian motion in $\mathbb{R}^p$ with exponent D. If $0 < a_1, a_d < 1$ and $\det\mathrm{Cov}(X(1)) > 0$, then for every compact set $T \subset \mathbb{R}$, $t_0 \in T$ and every $i = 1, ..., p$,*

$$\liminf_{h\to 0+} f_{i,h} M_i(t_0, h) = K_{4,4} \quad \text{a.s.}, \tag{32}$$

where $M_i(t_0, h) = M_i(t_0, h, \theta) = \sup_{s\in T, |s|\le h} |X_i(t_0 + s) - X_i(t_0)|$ denotes the uniform modulus of continuity of the i-th component of $X(t)$ on t_0, and

$$f_{i,h} = \frac{1}{\|((h/\log\log(1/h))^D U)^* e_i\|}.$$

Remark 2. *By making use of (4.6) and (4.8) in [24], (32) implies that there exists a constant $p_i \ge 1$ such that for every compact set $T \subset \mathbb{R}$, $t_0 \in T$ and every $i = 1, ..., p$,*

$$\liminf_{h\to 0+} \frac{(\log\log(1/h))^{a_i}}{h^{a_i}(\log 1/h)^{p_i - 1}} M_i(t_0, h) \le K_{4,5} \quad \text{a.s.}, \tag{33}$$

where a_i is defined in Section 2.

By choosing $D = \mathrm{diag}\{H_1, ..., H_p\}$ in Corollary 1, where $H_i \in (0, 1)$ for all $1 \le i \le p$, as an immediate consequence of Corollary 1, we have the following Chung's law of the iterated logarithm for the multivariate fractional Brownian motion, which may be of independent interest.

Proposition 3. *Let $X = \{X(t), t \in \mathbb{R}\}$ be a multivariate fractional Brownian motion in $\mathbb{R}^p$ with exponent $D = \mathrm{diag}\{H_1, ..., H_p\}$, and $U = (u_{ij})$ be a real invertible $p \times p$ matrix U such that $\mathrm{Cov}(X(1)) = UU^*$. Then, for every compact set $T \subset \mathbb{R}$, any $t_0 \in T$ and any vector $\theta \in \Gamma$,*

$$\liminf_{h\to 0+} \frac{(\log\log(1/h))^{H_k}}{h^{H_k}} M(t_0, h) = K_{4,3}\theta_k \sqrt{u_{k1}^2 + \cdots + u_{kp}^2} \quad \text{a.s.}, \tag{34}$$

where $k = \arg\min\{H_1, ..., H_p\}$ and $K_{4,3}$ is given as in (31), and for for every compact set $T \subset \mathbb{R}$, $t_0 \in T$ and every $i = 1, ..., d$,

$$\liminf_{h\to 0+} \frac{(\log\log(1/h))^{H_i}}{h^{H_i}} M_i(t_0, h) = K_{4,4} \sqrt{u_{i1}^2 + \cdots + u_{ip}^2} \quad \text{a.s.}, \tag{35}$$

where $K_{4,4}$ is given as in (32).

Remark 3. (34) implies that the minimum growth rate of multivariate fractional Brownian motion in any given direction θ is determined by the minimum of $\{H_1, ..., H_p\}$. In addition, $\arg\min(H_1, ...H_p)$ determines the constant on the right hand side of (34). Although as stated in Section 1, the i-th component of the multivariate fractional Brownian motion is a fractional Brownian motion with exponent $H_i \in (0, 1)$, $1 \le i \le p$, (35) implies that the minimum growth rate of the i-th coordinate direction of multivariate fractional Brownian motion depends on corresponding covariance matrix and hence the interrelationship between all directions.

Proof of Theorem 1. Throughout, it is sufficient to consider h-values which make the iterated logarithm positive and $t_0 = 0$. Put $M(h) = M(0, h)$. We first show that

$$\liminf_{h\to 0+} f_h M(h) \ge K_{4,6} \quad \text{a.s.} \tag{36}$$

Let $\epsilon > 0$ and $\gamma > 1$, and for $k = 1, 2, ...$ put $h_k = \gamma^{-k}$ and $\beta_k = \varphi(K_{3,4}(1+\epsilon)^{-1} h_k / \log\log(1/h_k))$. Then, by (16),

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$$\sum^{\infty}\mathbb{P}(M(h_k)\le\beta_k)\le K\sum^{\infty}(\log\gamma^k)^{-(1+\epsilon)}<\infty$$

where the sums are over all k large enough to make $k\log\gamma>1$ and $\beta_k<1$. Hence, by the Borel-Cantelli lemma, $M(h_k)\ge\beta_k$ for all k greater than some $k_0=k_0(\omega)$. Further, for $k\ge k_0$ and $h_k\le h<h_{k-1}$,

$$M(h)\ge M(h_k)\ge\beta_k=f_h^{-1}(f_h\beta_k).$$

Hence, by Lemma 1, (36) holds.

Next, we prove that

$$\liminf_{h\to0+}f_hM(h)\le K_{4,7}\quad\text{a.s.}\tag{37}$$

Let $\epsilon\in(0,1)$ and $q>1$ be arbitrary real number. This time we choose

$$h_k=e^{-k^q},\;\;J_k=ke^{k^q},\;\;\gamma_k=\varphi(K_{3,3}(1-\epsilon)^{-1}h_k/\log\log(1/h_k)).$$

Define the process $Y_k(t):=Y(t,J_{k-1},J_k)$ by

$$\begin{aligned}Y(t,J_{k-1},J_k)=&\int_{|x|\in(J_{k-1},J_k)}(1-\cos(tx))\Big(\frac{1}{|x|}\Big)^{D+I/2}d\mathbf{m}(x)\\&+\int_{|x|\in(J_{k-1},J_k)}\sin(tx)\Big(\frac{1}{|x|}\Big)^{D+I/2}d\mathbf{m}'(x),\end{aligned}\tag{38}$$

and denote $\widetilde{Y}_k(t):=X(t)-Y_k(t)$. Clearly, by (27), $X(t)=Y_k(t)+\widetilde{Y}_k(t)$, and for every $k=1,2,...$, $Y_k(\cdot)$ has stationary increments and $Y_k(\cdot)$, $k=1,2,...$ are independent due to the virtue of independent increments of $\mathbf{m}$.

For simplify notation, put $\overline{M}(h)=M(Y;0,h,\theta)$ and $\widetilde{M}(h)=M(\widetilde{Y};0,h,\theta)$, where M is defined in Lemma 6. For any $\epsilon\in(0,1)$, put

$$G_k=\{M(h_k)\le\gamma_k\},\;\;\overline{G}_k=\{\overline{M}(h_k)\le(1-\epsilon)\gamma_k\}\text{ and }\widetilde{G}_k=\{\widetilde{M}(h_k)\ge\epsilon\gamma_k\}.$$

It follows from (38) that

$$\begin{aligned}\langle\widetilde{Y}_k(t),\theta\rangle=&\int_{|x|\notin(J_{k-1},J_k)}(1-\cos(tx))\Big\langle\Big(\frac{1}{|x|}\Big)^{D+I/2}d\mathbf{m}(x),\theta\Big\rangle\\&+\int_{|x|\notin(J_{k-1},J_k)}\sin(tx)\Big\langle\Big(\frac{1}{|x|}\Big)^{D+I/2}d\mathbf{m}'(x),\theta\Big\rangle\\=&\int_{|x|\notin(J_{k-1},J_k)}(1-\cos(tx))\Big\langle\Big(\frac{1}{|x|}\Big)^{D^*+I/2}\theta,d\mathbf{m}(x)\Big\rangle\\&+\int_{|x|\notin(J_{k-1},J_k)}\sin(tx)\Big\langle\Big(\frac{1}{|x|}\Big)^{D^*+I/2}\theta,d\mathbf{m}'(x)\Big\rangle.\end{aligned}$$

By Lemmas 1 and Facts i)-vii) we have

$$\begin{aligned}&\gamma_k^{-1}\Big\|\Big(\frac{1}{|x|}\Big)^{D^*+I/2}\theta\Big\|\\&=\|U^*((K_{3,3}(1-\epsilon)^{-1}h_k/\log\log(1/h_k))^D)^*\theta\|^{-1}\cdot\Big\|\Big(\frac{1}{|x|}\Big)^{I/2}\Big(\frac{1}{|x|}\Big)^{D^*}\theta\Big\|\\&\le K\|U\|\Big(\frac{1}{|x|}\Big)^{1/2}\|((h_k/\log\log(1/h_k))^D)^*\theta\|^{-1}\cdot\Big\|\Big(\frac{1}{|x|}\Big)^{D^*}\theta\Big\|\\&\le\begin{cases}K\|U\|\Big(\frac{1}{|x|}\Big)^{1/2}\Big(\frac{\log\log(1/h_k)}{h_k|x|}\Big)^{a_1-\varepsilon} & \text{if }|x|\ge J_k,\\K\|U\|\Big(\frac{1}{|x|}\Big)^{1/2}\Big(\frac{\log\log(1/h_k)}{h_k|x|}\Big)^{\frac{1}{\lambda(\theta)}+\varepsilon} & \text{if }|x|\le J_{k-1}.\end{cases}\end{aligned}\tag{39}$$

Thus,

$$\begin{aligned}
&\mathbb{E}[|\gamma_k^{-1}\langle\widetilde{Y}_k(h_k),\theta\rangle|^2] \\
&= \int_{|x|\notin(J_{k-1},J_k)}(1-\cos(h_kx))\Big(\gamma_k^{-1}\Big\|\Big(\frac{1}{|x|}\Big)^{D^*+I/2}\theta\Big\|\Big)^2dx \\
&\le K\|U\|^2h_k^{-\frac{2}{\lambda(\theta)}-2\varepsilon}(\log\log(1/h_k))^{\frac{2}{\lambda(\theta)}+2\varepsilon}\int_{|x|\le J_{k-1}}(1-\cos(h_kx))\frac{dx}{|x|^{\frac{2}{\lambda(\theta)}+1+2\epsilon}} \\
&\quad + K\|U\|^2h_k^{-2a_1+2\varepsilon}(\log\log(1/h_k))^{2a_1-2\varepsilon}\int_{|x|\ge J_k}(1-\cos(h_kx))\frac{dx}{|x|^{2a_1+1-2\epsilon}} \\
&\le K_{4,8}((h_kJ_{k-1})^{2-\frac{2}{\lambda(\theta)}-2\epsilon}(\log k)^{\frac{2}{\lambda(\theta)}+2\varepsilon}+(h_kJ_k)^{-2a_1+2\epsilon}(\log k)^{2a_1-2\varepsilon}).
\end{aligned} \tag{40}$$

To get the second inequality from bottom, in the first integral we bound $1-\cos(tx)$ by $|t|^2|x|^2$, and the second one by 2 to get the required bound. Thus, since $\lambda(\theta)\ge a_d^{-1}>1$,

$$h_kJ_{k-1}\le(k-1)e^{-q(k-1)^{q-1}},\quad h_kJ_k=k,$$

we have

$$\mathbb{E}[|\gamma_k^{-1}\langle\widetilde{Y}_k(h_k),\theta\rangle|^2]\le K_{4,9}k^{-2a_1+\epsilon} \tag{41}$$

for all large k.

By (44) and Corollary 3.2 in [17], p.59, we get we get

$$\sum^{\infty}\mathbb{P}(\widetilde{G}_k)\le K\sum^{\infty}\exp(-K_{4,10}\epsilon^2k^{2a_1-\epsilon})<\infty. \tag{42}$$

This implies that

$$\limsup_{k\to\infty}\gamma_k^{-1}\widetilde{M}(h_k)\le\epsilon\quad\text{a.s.} \tag{43}$$

It follows from (16) that

$$\sum^{\infty}\mathbb{P}(M(h_k)\le\gamma_k)\ge K\sum^{\infty}k^{-q(1-\epsilon)}=\infty \tag{44}$$

by choosing $q>1$ small enough such that $q(1-\epsilon)<1$, where the sums are over all k large enough to make $\gamma_k<1$.

It follows easily that

$$\mathbb{P}(\overline{G}_k)\ge\mathbb{P}(G_k)-\mathbb{P}(\widetilde{G}_k),$$

which, together with (42) and (44), yields

$$\sum_k\mathbb{P}(\overline{G}_k)=\infty.$$

Since $Y_k(\cdot),k=1,2,...$ are independent, by the Borel-Cantelli lemma,

$$\limsup_{k\to\infty}\gamma_k^{-1}\overline{M}_k\le1-\epsilon\quad\text{a.s.} \tag{45}$$

From Lemma 1, we have $f_{h_k}^{-1}\gamma_k^{-1}\le K_{4,11}$. It follows from (43) and (45) that

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$$
\begin{aligned}
&\liminf_{k\to\infty} f_{h_k} M(h_k) \\
&\leq K_{4,11} \liminf_{k\to\infty} \gamma_k^{-1} M(h_k) \\
&\leq K_{4,11} \Big(\liminf_{k\to\infty} \gamma_k^{-1} \overline{M}(h_k) + \limsup_{k\to\infty} \gamma_k^{-1} \widetilde{M}(h_k) \Big) \\
&\leq K_{4,11} \quad \text{a.s.}
\end{aligned}
$$

This yields that (37) holds.

We have thus established that

$$
K_{4,6} \leq \liminf_{h\to 0+} f_h M(h) \leq K_{4,7} \quad \text{a.s.}
$$

Lemma 6 guarantees that the liminf is constant. The proof is completed. □

4. Conclusions

Applying techniques developed in [30], in this article we obtain the estimations of small ball probability for spatial surfaces of operator fractional Brownian motion, including multivariate fractional Brownian motion. We obtain the strongly locally nondeterministic for spatial surfaces of operator fractional Brownian motion in any given direction θ. Applying these estimates we obtain Chung's laws of the iterated logarithm for spatial surfaces of operator fractional Brownian motion in any given direction θ. By combining our results and the Jordan decomposition theorem applied to the exponent D, it is possible to analyze the rates of change of spatial surfaces by the real parts of the eigenvalues of the exponent D.

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References

1. Amblard, P.-O.; Coeurjolly, J.-F.; Lavancier, F.; Philippe, A. Basic properties of the multivariate fractional Brownian motion. *ESAIM Probab. Stat.* **2010**, DOI: 10.1051/ps/2012011.
2. Chung, C.-F. Sample means, sample autocovariances, and linear regression of stationary multivariate long memory processes. *Econometric Theory* **2002**, *18*, 51–78.
3. Coeurjolly, J.-F.; Amblard, P.O.; Achard, S. Wavelet analysis of the multivariate fractional Brownian motion. *ESAIM Probab. Stat.* **2013**, *17*, 592–604
4. Cohen, S.; Meerschaert, M.M.; Rosiński, J. Modeling and simulation with operator scaling. *Stochastic Processes and Their Applications* **2010**, *120*, 2390–2411.
5. Davidson, J.; de Jong, R. The functional Central Limit Theorem and weak convergence to stochastic integrals II: Fractionally integrated processes. *Econometric Theory* **2000**, *16*, 643–666.
6. Davidson, J.; Hashimzade, N. Alternative frequency and time domain versions of fractional Brownian motion. *Econometric Theory* **2008**, *24*, 256–293.
7. Delgado, R. A reflected fBm limit for fluid models with ON/OFF sources under heavy traffic. *Stoch. Process. Appl.* **2007**, *117*, 188–201.
8. Didier, G.; Pipiras, V. Integral representations of operator fractional Brownian motion. *Bernoulli* **2011**, *17*, 1–33.
9. Dolado, J.; Marmol, F. Asymptotic inference results for multivariate long-memory processes. *Econometrics J.* **2004**, *7*, 168–190.
10. Hirsch, M.W.; Smale, S. Differential Equations, Dynamical Systems, and Linear Algebra. In *Pure and Applied Mathematics*, *60*, Academic Press: New York, London, 1974.

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11. Hudson, W.N.; Mason, J.D. Operator-self-similar processes in a finite-dimensional space. *Trans. Amer. Math. Soc.* **1982**, *273*, 281–297.
12. Kechagias, S.; Pipiras, V. Definitions and representations of multivariate long-rangle dependent time serires. *J. Time Ser. Anal.* **2015**, *36*, 1–25.
13. Konstantopoulos, T.; Lin, S.J. Fractional Brownian approximations of queueing networks. In *Stochastic Networks, Lecture Notes in Statistics*, *117*, 257–273, Springer: New York, 1996.
14. Laha, R.G.; Rohatgi, V.K. Operator self-similar stochastic processes in Rd. *Stoch. Process. Appl.* **1981**, *12*, 73–84.
15. Lavancier, F.; Philippe, A.; Surgailis, D. A two-sample test for comparison of long memory parameters. *J. Multivariate Anal.* **2010**, *101*, 2118–2136.
16. Lavancier, F.; Philippe, A.; Surgailis, D. Covariance function of vector self-similar process. *Stat. Probab. Lett.* **2009**, *79*, 2415–2421.
17. Ledoux, M.; Talagrand, M. *Probabilities in Banach Spaces*. Springer-Verlag: New York, 1991.
18. Li, W.V.; Shao, Q.-M. Gaussian processes: inequalities, small ball probabilities and applications. In *Stochastic Processes: Theory and Methods, Handbook of Statistics* (C.R. Rao and D. Shanbhag, editors) North-Holland **2001**, *19*, 533–597.
19. Maejima, M.; Mason, J.D. Operator-self-similar stable processes. *Stoch. Process. Appl.* **1994**, *54*, 139–163.
20. Majewski, K. Large deviations for multi-dimensional reflected fractional Brownian motion. *Stochastics* **2003**, *75*, 233–257.
21. Majewski, K. Fractional Brownian heavy traffic approximations of multiclass feedforward queueing networks. *Queueing Systems* **2005**, *50*, 199–230.
22. Marinucci, D.; Robinson, P. Weak convergence of multivariate fractional processes. *Stoch. Process. Appl.* **2000**, *86*, 103–120.
23. Mason, J.D.; Xiao, Y. Sample path properties of operator-self-similar Gaussian random fields. *teor.veroyatnost.i primenen* **2001**, *46*, 94–116.
24. Mason, J.D.; Xiao, Y. Sample path properties of operator-self-similiar Gaussian random fields, *Th. Probab. Appl.* **2002**, *46*, 58–78.
25. Meerschaert, M.M. Regular variation in $\mathbb{R}^k$. *Proc. Amer. Math. Soc.* **1988**, *102*, 341–348.
26. Meerschaert, M.M. Spectral decomposition for generalized domains of attraction. *Ann. Probab.* **1991**, *19*, 875–892.
27. Meerschaert, M.M.; Wang, W.; Xiao, Y. Fernique type inequality and moduli of continuity for anisotropic Gaussian random fields. *Tran. Amer. Math. Soc.* **2013**, *365*, 1081–1107.
28. Meerschaert, M.M.; Scheffler, H.P. Multivariable regular variation of functions and measures. *J. Appl. Anal.* **1999**, *5*, 125–146.
29. Meerschaert, M.M.; Scheffler, H.P. *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*. Wiley Interscience: New York, 2001.
30. Monrad, D.; Rootzén, H. Small values of Gaussian processes and functional laws of the iterated logarithm. *Probab. Th. Rel. Fields* **1995**, *101*, 173–192.
31. Sato, K. Self-similar processes with independent increments. *Probab. Th. Related Fields* **1991**, *89*, 285–300.
32. Seneta, E. Regularly Varying Functions. In *Lecture Notes in Mathematics*, *508*, Springer: Berlin, 1976.
33. Stoev, S.A.; Taqqu, M.S. How rich is the class of multifractional Brownian motions. *Stoch. Process. Appl.* **2006**, *116*, 200–221.
34. Talagrand, M. New Gaussian estimates for enlarged balls. *Geo. Func. Anal.* **1993**, *3*, 502–526.
35. Talagrand, M. Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. Probab.* **1995**, *23*, 767–775.
36. Wang, W. Invariance principles for generalized domains of semistable attraction. *Stoch. Process. Appl.* **2014**, *124*, 1–17.