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Article

A Note on Nearly Sasakian Manifolds

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Abstract: A class of nearly Sasakian manifolds is considered. We discussion geometric effects of some symmetries on such manifolds, and show, under a certain condition, that the class of Ricci-symmetric nearly Sasakian manifolds is a subclass of Einstein manifolds. We prove that a nearly Sasakian space form with Ricci tensor satisfying the Codazzi equation is either a Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} = 1$ or a 5-dimensional proper nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} > 1$. We also prove that the spectrum of the operator H^2 generated by the nearly Sasakian manifold is a set of simple eigenvalue 0 and an eigenvalue of multiplicity 4. We show that there exist integrable distributions on the same manifolds with totally geodesic leaves, and prove that there are no proper nearly Sasakian space forms with constant sectional curvature.

Keywords: nearly Sasakian space forms; k-nullity distribution; locally symmetric manifold; semi-symmetric manifold; Ricci-symmetric manifold

MSC: Primary 53C15; Secondary 53C25

1. Introduction

The concept of nearly Sasakian manifolds was introduced by Blair, Yano and Showers in [2] as an odd-dimensional counterpart of nearly Kähler manifolds. They proved that normal nearly Sasakian structure is Sasakian, and hence in particular is contact. Also, in the same paper, it is shown that a hypersurface of a nearly Kähler manifold is nearly Sasakian if and only if it is quasi-umbilical with respect to the (almost) contact form. This result was supported by an example starting that S^5 properly imbedded in S^6 inherits a nearly Sasakian structure which is not a Sasakian structure. That is why nearly Sasakian manifolds may be considered as an odd-dimensional analogue of nearly Kähler manifolds. But, it is more difficult than expected to find relationships between the two structures, like for the duo Sasakian and Kähler structures (see [4] for details).

Nearly Sasakian structures can also be seen as the vanishing of the symmetric part of Sasakian structures. Several authors have studied these structures [3], [4], [11], [12] and references therein. For instance, Olszak in [11] and [12] gave a good number of properties nearly Sasakian structures. He proved that if nearly Sasakian manifolds are not Sasakian, they are of dimension 5 and of constant curvature. Olszak also proved some equivalent conditions for non-Sasakian nearly Sasakian manifold to be of dimension 5 and showed that such manifolds are Einstein.

In [4], among other results, the authors proved that any nearly Sasakian manifold admits two types of integrable distributions with totally geodesic leaves which are, respectively, Sasakian and 5-dimensional nearly Sasakian manifolds.

In this paper, we consider the same nearly Sasakian structures by paying attention to some foliations and curvature properties. Some of these foliations are generated by the symmetry properties on curvature and Ricci tensors.

A Riemannian manifold is *locally symmetric* if its curvature tensor R is parallel, i.e., $\nabla R = 0$, where ∇ is the Levi-Civita connection on Riemannian manifold extended to act on tensors as a derivation and R is the corresponding curvature tensor. This class of manifolds contains one of manifolds of constant curvature. The integrability condition of $\nabla R = 0$ is $R \cdot R = 0$, where again R is extended to act on tensors as a derivation. Manifolds which satisfy the latter condition are called *semi-symmetric* and have been classified by Szabó ([14] and [15], for more details). A Riemannian manifold is called *Ricci semi-symmetric*, if $R \cdot \mathrm{Ric} = 0$. The set of all manifolds which are Ricci semi-symmetric contains the set of manifolds which are semi- symmetric. This means that every semi-symmetric manifold is Ricci semi-symmetric. The converse is not true in general.

The present paper studies the two foliations stated by Olszak in papers [11] and [12]. He proved that a locally symmetric proper nearly Sasakian manifold is of constant curvature and of dimension 5. These foliations were also investigated by Cappelletti-Montano *et al* in [3] and [4].

The paper is organized as follows. In Section 2, we recall the definition of a nearly Sasakian manifold and give some identities formulas of the underlying tensors, supported by two examples. In Section 3, we discuss the two foliations as stated in [4] and [11]. We establish the geometric effects of the semi-symmetry and Ricci-symmetry on nearly Sasakian manifolds. Under a certain condition, we show that the class of Ricci-symmetric nearly Sasakian manifolds is a subclass of Einstein manifolds. We prove that these foliations exist canonically in a locally symmetric nearly Sasakian manifold of constant curvature and k-space. Some examples are also established. In Section 4, we find an algebraic formula of the curvature tensor for nearly Sasakian manifolds (Proposition 4.3). Under a certain condition on the Ricci tensor, we prove that a nearly Sasakian space form is either a Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} = 1$ or a 5-dimensional proper nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} > 1$. In the same settings, we also prove that the spectrum of the operator H^2 has a simple eigenvalue 0 and an eigenvalue of multiplicity 4. We show that there exist integrable distributions with totally geodesic leaves (Theorem 4.7 and Theorem 4.8). Contrary to [11, Theorem 6.1], we prove that there are no proper nearly Sasakian space forms with constant sectional curvature (Theorem 4.9).

2. Preliminaries

Let M be a $(2n+1)$-dimensional manifold endowed with an almost contact structure (ϕ, ξ, η), i.e., ϕ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying [1]

$$\phi^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{2.1}$$

It follows that $\phi\xi = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. Then (ϕ, ξ, η, g) is called an almost contact metric structure on M if (ϕ, ξ, η) is an almost contact structure on M and g is a Riemannian metric on M such that [1]

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector field X, Y on M. It follows that the $(1,1)$-tensor field ϕ is skew-symmetric and $\eta(X) = g(\xi, X)$.

If, moreover,

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X, \tag{2.3}$$

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where ∇ is the Levi-Civita connection for the Riemannian metric g, we call M a *nearly Sasakian manifold*. From (2.3), we have

$$\nabla_X \xi = -\phi X - HX, \tag{2.4}$$

where

$$HX = \phi\left(\nabla_\xi \phi\right) X. \tag{2.5}$$

It is skew-symmetric and anticommutes with ϕ. The tensor field H is of type $(1,1)$ and satisfies $H\xi = 0$, $\eta \circ H = 0$ and

$$\nabla_\xi H = -\nabla_\xi \phi = \phi H = -\frac{1}{3}\mathcal{L}_\xi \phi, \tag{2.6}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ. The vanishing of H gives a necessary and sufficient condition for a nearly Sasakian manifold to be Sasakian (see [7] and references therein).

It is easy to see that

$$H^2 X = \left(\nabla_\xi \phi\right)^2 X. \tag{2.7}$$

The divergence of ξ is given by

$$\operatorname{div} \xi = 0. \tag{2.8}$$

Example 2.1. We consider the 5-dimensional manifold $M = \{(x_1, x_2, \cdots, x_5) \in \mathbb{R}^5 : x_2 \neq 0,\ x_5 \neq 0\}$ with standard coordinates $(x_1, x_2, \cdots, x_5)$. The vector fields,

$$\begin{aligned} X_1 &= 2\left(x_2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_1}\right), \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \xi = -\frac{\partial}{\partial x_3}, \\ X_4 &= 2\left(x_5 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right), \quad X_5 = \frac{\partial}{\partial x_5}, \end{aligned}$$

are linearly independent at each point of M. Let g be the Riemannian metric on M defined by $g(X_i, X_j) = \delta_{ij}$, for any $i, j = 1, 2, \cdots, 4$, where δ_{ij} is the Kronecker symbol, and $g(\xi, \xi) = 1$. Locally, the metric g is given by

$$g = \frac{1}{4}(1 - x_2^2)dx_1^2 + dx_2^2 + dx_3^2 + \frac{1}{4}(1 - x_5^2)dx_4^2 + dx_5^2.$$

Let η be the 1-form on M defined by $\eta = -dx_3$. Let ϕ be the $(1,1)$-tensor field defined by, $\phi X_1 = X_2$, $\phi X_2 = -X_1$, $\phi X_3 = 0$, $\phi X_4 = X_5$ and $\phi X_5 = -X_4$. By linearity of ϕ and g, the relations (2.1) and (2.2) are satisfied on $\mathbb{R}^5$. Thus, (ϕ, ξ, η, g) defines an almost contact metric structure on $\mathbb{R}^5$. Let ∇ be the Levi–Civita connection with respect to the metric g. Then, the non-vanishing Lie brackets are

$$[X_1, X_2] = [X_4, X_5] = 2\xi.$$

These lead to the following non-vanishing components of the covariant derivative

$$\begin{aligned} &\nabla_{X_1} X_2 = \xi, \ \ \nabla_{X_1} \xi = -X_2, \ \ \nabla_{X_2} X_1 = -\xi, \ \ \nabla_{X_2} \xi = X_1, \\ &\nabla_\xi X_1 = -X_2, \ \ \nabla_\xi X_2 = X_1, \ \ \nabla_\xi X_4 = -X_5, \ \ \nabla_\xi X_5 = X_4, \\ &\nabla_{X_4} \xi = -X_5, \ \ \nabla_{X_4} X_5 = \xi, \ \ \nabla_{X_5} \xi = X_4, \ \ \nabla_{X_5} X_4 = -\xi. \end{aligned}$$

Using these covariant derivatives, it is easy to see that the relation (2.3) is satisfied and therefore (ϕ, ξ, η, g) is a nearly Sasakian structure.

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Throughout this paper, manifolds are assumed to be connected and of class C^∞ and all tensor fields are of class C^∞, and $\Gamma(\Xi)$ will denote the $\mathcal{F}(M)$-module of smooth sections of a vector bundle Ξ.

A vector field V on M is said to be an *affine Killing vector field*, if it satisfies (see [9, page 51])

$$\mathcal{L}_V \nabla = 0. \tag{2.9}$$

The relation (2.9) is equivalent to

$$R(V,X)Y + \nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V = 0, \tag{2.10}$$

where the Riemannian curvature tensor R of M is a $(1,3)$ tensor field defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.11}$$

for any vector fields X, Y and Z on M.

The relation (2.9) is the integrability condition for the Killing vector field V (see [9] for more details). If M is nearly Sasakian, then using (2.4), it is easy to check that the characteristic vector field ξ is Killing. Hence, the vector field ξ is affine Killing. The converse is not true in general. In [9], it is proven that the converse holds when the underlying manifold is compact and without boundary.

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional nearly Sasakian manifold. By (2.10)

$$R(X,\xi)Y = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi. \tag{2.12}$$

Therefore, we have the following formulas [7]:

$$\begin{aligned} R(\xi,X)Y &= (\nabla_X \phi)\, Y + (\nabla_X H)\, Y \\ &= g(X - H^2X, Y)\xi - \eta(Y)(X - H^2X) \\ &= \{g(X,Y) - g(H^2X,Y)\}\xi - \eta(Y)X + \eta(Y)H^2X, \end{aligned} \tag{2.13}$$

$$\left(\nabla_X H^2\right) Y = \eta(Y)\,(\phi + H)\, H^2X + g((\phi + H)\, H^2X, Y)\xi, \tag{2.14}$$

$$g((\nabla_X \phi)Y, HZ) = -\eta(Y)g(H^2X, \phi Z) + \eta(X)g(H^2Y, \phi Z) + \eta(Y)g(HX, Z), \tag{2.15}$$

for any $X, Y, Z \in \Gamma(TM)$.

As proven in [7] and using the relations (2.13) - (2.15), we have

$$(\nabla_X \phi)Y = -\eta(X)\phi HY - \eta(Y)(X - \phi HX) + g(X - \phi HX, Y)\xi, \tag{2.16}$$

$$(\nabla_X H)\, Y = \eta(X)\phi HY + \eta(Y)(H^2X - \phi HX) - g(H^2X - \phi HX, Y)\xi, \tag{2.17}$$

$$\begin{aligned} (\nabla_X \phi H)Y &= \eta(Y)(\phi H^2X + HX) - \eta(X)(\phi H^2Y + HY) \\ &\quad - g(HX + \phi H^2X, Y)\xi. \end{aligned} \tag{2.18}$$

Now, for any vector fields X and Y on M,

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y + \eta(X)H^2Y - \eta(Y)H^2X. \tag{2.19}$$

Then, for any $X \in \Gamma(TM)$,

$$\mathrm{Ric}(X,\xi) = (2n - \mathrm{trace}\, H^2)\eta(X). \tag{2.20}$$

By (2.13), we have, for any $X, Y \in \Gamma(TM)$,

$$R(X,\xi)Y = -g(X,Y)\xi + \eta(Y)X - \eta(Y)H^2X + g(H^2Y,X)\xi. \tag{2.21}$$

3. Foliations of a nearly Sasakian manifold

In [4] and [11], for instance, the authors showed that there are two foliations in any nearly Sasakian manifold whose leaves are, respectively, Sasakian or 5-dimensional nearly Sasakian non-Sasakian manifolds. This fact is led by the square of skew symmetric operator H, i.e., H^2. The latter plays an important role as well as its spectrum.

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional nearly Sasakian manifold. Olszak in [11] showed that if M satisfies the condition

$$H^2 = \alpha\{\mathbb{I} - \eta \otimes \xi\}, \tag{3.1}$$

for some real number α, then $\dim M = 5$. The converse is true if the real number α is non-zero (see [12, Theorem 4.1] for more details).

We say that M is a *proper nearly Sasakian manifold* if it is a nearly Sasakian non-Sasakian manifold.

Let $D := \ker \eta$ be the contact distribution and $D^\perp$ be the distribution spanned the structure vector field ξ. Then, we have the following decomposition

$$TM = D \oplus D^\perp, \tag{3.2}$$

where $\oplus$ denotes the orthogonal direct sum. By the decomposition (3.2), any $X \in \Gamma(TM)$ is written as

$$X = QX + Q^\perp X, \tag{3.3}$$

where Q and $Q^\perp$ are the projection morphisms of TM into D and $D^\perp$, respectively. Here, it is easy to see that $Q^\perp X = \eta(X)\xi$ and $X = QX + \eta(X)\xi$.

If (3.1) is satisfied, then, for any non-zero vector field $X \in \Gamma(D)$,

$$-g(HX,HX) = \alpha g(X,X), \text{ i.e., } \alpha = -\frac{g(HX,HX)}{g(X,X)}. \tag{3.4}$$

This means that, there is $\lambda \in \mathbb{R}$ such that $\alpha = -\lambda^2 \leq 0$, and therefore, (3.1) becomes

$$H^2 = -\lambda^2\{\mathbb{I} - \eta \otimes \xi\}. \tag{3.5}$$

As examples for both Sasakian and proper nearly Sasakian manifolds, we have the following.

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Example 3.1. Let M be the 5-dimensional manifold defined in Example 2.1. It is easy to see the components of the tensor field H is of type $(1,1)$ are given

$$\begin{aligned}
H\xi &= \phi\left(\nabla_\xi \phi\right)\xi = 0, \\
HX_1 &= \phi\left(\nabla_\xi \phi\right)X_1 = \phi\nabla_\xi\phi X_1 - \phi^2\nabla_\xi X_1 \\
&= \phi\nabla_\xi X_2 - \phi^2\nabla_\xi X_1 = \phi X_1 + \phi^2 X_2 = X_2 - X_2 = 0, \\
HX_2 &= \phi\left(\nabla_\xi \phi\right)X_2 = \phi\nabla_\xi\phi X_2 - \phi^2\nabla_\xi X_2 \\
&= -\phi\nabla_\xi X_1 - \phi^2\nabla_\xi X_2 = -X_1 + X_1 = 0, \\
HX_4 &= \phi\left(\nabla_\xi \phi\right)X_4 = \phi\nabla_\xi\phi X_4 - \phi^2\nabla_\xi X_4 \\
&= \phi\nabla_\xi X_5 - \phi^2\nabla_\xi X_4 = X_5 - X_5 = 0 \\
HX_5 &= \phi\left(\nabla_\xi \phi\right)X_5 = \phi\nabla_\xi\phi X_5 - \phi^2\nabla_\xi X_5 \\
&= -\phi\nabla_\xi X_4 - \phi^2\nabla_\xi X_5 = -X_4 + X_4 = 0.
\end{aligned}$$

This means that H vanishes everywhere. Therefore, in this case, the structure in (2.3) reduces to $(\nabla_X\phi)Y = g(X,Y)\xi - \eta(X)Y$, for any vector fields X and Y on M, which shows that M is a Sasakian manifold.

In [2], the authors proved how to induce a nearly Sasakian structure on S^5. In order to do so, they look at S^5 as a hypersurface in S^6 equipped with its nearly Kähler structure.

Example 3.2. We recall a basic example of 5-dimensional nearly Sasakian manifolds [1,2,4]. First consider $\mathbb{R}^7$ as the imaginary part of the Cayley numbers $\mathbb{O}$, with the cross product $\times$ induced by the Cayley product. Let S^6 be the unit sphere in $\mathbb{R}^7$ and $N = \sum_{i=1}^{7} x_i \frac{\partial}{\partial x_i}$ the unit outer normal. We can define an almost complex structure J on S^6 by $JX = N \times X$, which implies, for any vector field X on S^6,

$$J^2 = N \times (N \times X) = -X.$$

This almost complex structure is nearly Kähler (non-Kähler) with respect to the induced Riemannian metric. As in [4], now we consider S^5 as a totally umbilical hypersurface of S^6 defined by $x_7 = \frac{\sqrt{2}}{2}$, with unit normal at each point x given by $\nu = x - \sqrt{2}\frac{\partial}{\partial x_7} = \sum_{i=1}^{6} x_i \frac{\partial}{\partial x_i} - \frac{\sqrt{2}}{2}\frac{\partial}{\partial x_7}$ so that the shape operator is $A = -\mathbb{I}$. Let (ϕ, ξ, η, g) be the induced almost contact metric structure, where

$$\xi = -J\nu = \sqrt{2}\left(x_1\frac{\partial}{\partial x_6} - x_2\frac{\partial}{\partial x_5} - x_3\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial x_2} - x_6\frac{\partial}{\partial x_1}\right),$$

and η given by the restriction of $\sqrt{2}\,(x_1dx_6 - x_6dx_1 + x_5dx_2 - x_2dx_5 + x_4dx_3 - x_3dx_4)$ to S^5. This structure is nearly Sasakian, but not Sasakian with constant sectional curvature 2. The latter means that, for any vector fields X and Y on S^5,

$$R(X,Y)\xi = 2\left\{\eta(Y)X - \eta(X)Y\right\},$$

which implies that $-\phi^2X - H^2X = 2\left\{X - \eta(X)\xi\right\}$, that is, $H^2X = -\left\{X - \eta(X)\xi\right\}$ with $\lambda^2 = 1$.

Next, we present some classes of nearly Sasakian manifolds in which the condition (3.5) is satisfied.

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Suppose M is a semi-symmetric nearly Sasakian manifold, that is, $R(X,Y)\cdot R = 0$, for any vector fields X and Y on M, where $R(X,Y)$ operates on R as a derivation of then tensor algebra at each point. Now, let X and Y be vector fields in D such that $g(X,Y) = 0$. Then, using (2.19) and (2.21), one has,

$$\begin{aligned}(R(X,\xi)\cdot R)(X,Y)Y &= R(X,\xi)R(X,Y)Y - R(X,Y)R(X,\xi)Y - R(R(X,\xi)X,Y)Y\\ &- R(X,R(X,\xi)Y)Y\\ &= -g(X,R(X,Y)Y)\xi + \eta(R(X,Y)Y)X - \eta(R(X,Y)Y)H^2X\\ &+ g(H^2X,R(X,Y)Y)\xi + \{g(X,X) - g(H^2X,X)\}\{g(Y,Y) - g(H^2Y,Y)\}\xi\\ &- g(X,H^2Y)g(X,H^2Y)\xi. \end{aligned} \tag{3.6}$$

Hence,

$$\begin{aligned}&- g(X,R(X,Y)Y)\xi + \eta(R(X,Y)Y)X - \eta(R(X,Y)Y)H^2X\\ &+ g(H^2X,R(X,Y)Y)\xi + \{g(X,X) - g(H^2X,X)\}\{g(Y,Y) - g(H^2Y,Y)\}\xi\\ &- g(X,H^2Y)g(X,H^2Y)\xi = 0. \end{aligned} \tag{3.7}$$

Thus, considering ξ-component of (3.7), we get

$$\begin{aligned}g(R(X,Y)Y,X) &= g(H^2X,R(X,Y)Y) + g(X,X)g(Y,Y) + g(X,X)g(HY,HY)\\ &+ g(Y,Y)g(HX,HX) + g(HX,HX)g(HY,HY)\\ &- g(HX,HY)g(HX,HY). \end{aligned} \tag{3.8}$$

If the condition (3.5) is satisfied, then, from the relation (3.8), one obtains,

$$(1+\lambda^2)g(R(X,Y)Y,X) = (1+2\lambda^2+\lambda^4)g(X,X)g(Y,Y). \tag{3.9}$$

Therefore,

$$g(R(X,Y)Y,X) = (1+\lambda^2)g(X,X)g(Y,Y). \tag{3.10}$$

Therefore, we have

Theorem 3.3. *Let (M,ϕ,ξ,η,g) be a nearly Sasakian manifold satisfying the Nomizu's condition, i.e., $R(X,Y)\cdot R = 0$, for any vector fields X and Y on M. If*

$$H^2 = -\lambda^2\{\mathbb{I} - \eta\otimes\xi\},$$

for some real number λ, then M is of constant curvature $1+\lambda^2$. Moreover, M is non-Sasakian, for any $\lambda \neq 0$.

Now, we define the following. Let κ be a real constant. The κ-nullity distribution $N(\kappa)$ on M is defined as the assignment $M \ni p \longmapsto N_p(\kappa)$, where $N_p(\kappa)$ is the κ-nullity space at p defined as

$$N_p(\kappa) = \{Z \in T_pM : R(X,Y)Z = \kappa\left(g(Y,Z)X - g(X,Z)Y\right),\ \ \forall X,Y \in T_pM\},$$

where T_pM is the tangent space at $p \in M$ (see [5] and [6] for more details and reference therein). If the characteristic vector field $\xi \in N(\kappa)$, then M will be called *κ-space*.

Therefore, we have the following result.

Theorem 3.4. *Let (M,ϕ,ξ,η,g) be a nearly Sasakian manifold. Then M satisfies the condition (3.5) if and only if M is a $(1+\lambda^2)$-space.*

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Proof. If the condition (3.5) is satisfied, then for any vector vector fields X and Y on M,

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y - \lambda^2\eta(X)\{Y - \eta(Y)\xi\} + \lambda^2\eta(Y)\{X - \eta(X)\xi\}$$
$$= (1+\lambda^2)\{\eta(Y)X - \eta(X)Y\}.$$

The converse is straightforward and this completes the proof. □

If M is Ricci semi-symmetric nearly Sasakian manifold, then

$$(R(X,Y)\cdot \mathrm{Ric})\,(Z,W) = -\mathrm{Ric}(R(X,Y)Z,W) - \mathrm{Ric}(Z,R(X,Y)W)$$
$$= 0, \quad \forall\, X,\ \ Y,\ \ Z,\ \ W \in \Gamma(TM). \tag{3.11}$$

Now, using (2.19) and (2.20), we have

$$(R(X,Y)\cdot \mathrm{Ric})\,(\xi,Z) = -\mathrm{Ric}(R(X,Y)\xi,Z) - \mathrm{Ric}(\xi,R(X,Y)Z)$$
$$= -\eta(Y)\mathrm{Ric}(X,Z) + \eta(X)\mathrm{Ric}(Y,Z) - \eta(X)\mathrm{Ric}(H^2Y,Z)$$
$$+ \eta(Y)\mathrm{Ric}(H^2X,Z) - (2n - \mathrm{trace}\ H^2)\eta(R(X,Y)Z). \tag{3.12}$$

Now, by the relation (2.20), we have,

$$(R(\xi,X)\cdot \mathrm{Ric})(Y,\xi) = -\mathrm{Ric}(R(\xi,X)Y,\xi) - \mathrm{Ric}(Y,R(\xi,X)\xi)$$
$$= -(2n - \mathrm{trace}\,H^2)g(X,Y) + (2n - \mathrm{trace}\,H^2)g(H^2X,Y)$$
$$+ \mathrm{Ric}(X,Y) - \mathrm{Ric}(H^2X,Y). \tag{3.13}$$

If the condition (3.5) is satisfied on M, then (3.13) becomes

$$(R(\xi,X)\cdot \mathrm{Ric})(Y,\xi) = -2n(1+\lambda^2)^2 g(X,Y) + (1+\lambda^2)\mathrm{Ric}(X,Y). \tag{3.14}$$

Therefore, we have the following.

Theorem 3.5. *A Ricci-symmetric nearly Sasakian manifold satisfying (3.5) is Einstein.*

Proof. Let M be a Ricci-symmetric nearly Sasakian manifold satisfying (3.5). Then, for any vector fields X and Y on M, and using (3.14), the Ricci tensor is given by $\mathrm{Ric}(X,Y) = 2n(1+\lambda^2)g(X,Y)$, which completes the proof. □

In [11], Olszak proved that if a nearly Sasakian non-Sasakian manifold is locally symmetric, then it is of constant curvature and of dimension 5.

Now, if we assume that the nearly Sasakian manifold M is of constant curvature κ, then the curvature tensor R of M satisfies the equation [16]:

$$R(X,Y)Z = \kappa\{g(Y,Z)X - g(X,Z)Y\}, \tag{3.15}$$

for any vector fields X , Y and Z on M. then putting $Z = \xi$ into (3.15) and using (2.19), we have

$$\eta(Y)\{(\kappa-1)X + H^2X\} = \eta(X)\{(\kappa-1)Y + H^2Y\}. \tag{3.16}$$

This implies that

$$H^2X = -(\kappa-1)\{X - \eta(X)\xi\}, \tag{3.17}$$

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for any vector field X on M. Therefore, we have:

Theorem 3.6. *Let (M, ϕ, ξ, η, g) be a nearly Sasakian manifold. If M is of constant curvature κ, then M is either Sasakian or satisfies the condition (3.5) with $\kappa = 1 + \lambda^2$ with $\lambda \neq 0$, and a $(1 + \lambda^2)$-space.*

A nearly Sasakian manifold M is locally symmetric if

$$(\nabla_W R)(X, Y)Z = 0,$$

for any vector fields X, Y, Z and W on M.

We know that the covariant derivative of R, namely, ∇R,

$$\begin{aligned}(\nabla_Z R)(X, Y, W) = \nabla_Z R(X, Y)W - R(\nabla_Z X, Y)W - R(X, \nabla_Z Y)W \\ - R(X, Y)\nabla_Z W.\end{aligned} \tag{3.18}$$

By putting $W = \xi$ into (3.18), one has

$$\begin{aligned}(\nabla_Z R)(X, Y, \xi) = \{g(\phi Z, X) + g(HZ, X)\}Y - \{g(\phi Z, Y) + g(HZ, Y)\}X - \{g(\phi Z, X) \\ + g(HZ, X)\}H^2 Y + \{g(\phi Z, Y) + g(HZ, Y)\}H^2 X + \eta(X)(\nabla_Z H^2)Y \\ - \eta(Y)(\nabla_Z H^2)X + R(X, Y)\phi Z + R(X, Y)HZ.\end{aligned} \tag{3.19}$$

Using (2.14), the term $\eta(X)(\nabla_Z H^2)Y - \eta(Y)(\nabla_Z H^2)X$ becomes

$$\begin{aligned}\eta(X)(\nabla_Z H^2)Y - \eta(Y)(\nabla_Z H^2)X = \eta(X)g(\phi H^2 Z, Y)\xi + \eta(X)g(H^3 Z, Y)\xi \\ - \eta(Y)g(\phi H^2 Z, X)\xi - \eta(Y)g(H^3 Z, X)\xi.\end{aligned} \tag{3.20}$$

Therefore,

$$\begin{aligned}(\nabla_Z R)(X, Y, \xi) = \{g(\phi Z, X) + g(HZ, X)\}Y - \{g(\phi Z, Y) + g(HZ, Y)\}X \\ - \{g(\phi Z, X) + g(HZ, X)\}H^2 Y + \{g(\phi Z, Y) + g(HZ, Y)\}H^2 X \\ + \eta(X)g(\phi H^2 Z, Y)\xi + \eta(X)g(H^3 Z, Y)\xi - \eta(Y)g(\phi H^2 Z, X)\xi \\ - \eta(Y)g(H^3 Z, X)\xi + R(X, Y)\phi Z + R(X, Y)HZ.\end{aligned} \tag{3.21}$$

If a nearly Sasakian manifold M is locally symmetric, then (3.21) leads to,

$$\begin{aligned}0 = \{g(\phi Z, X) + g(HZ, X)\}g(Y, W) - \{g(\phi Z, Y) + g(HZ, Y)\}g(X, W) \\ - \{g(\phi Z, X) + g(HZ, X)\}g(H^2 Y, W) + \{g(\phi Z, Y) + g(HZ, Y)\}g(H^2 X, W) \\ + \eta(X)g(\phi H^2 Z, Y)\eta(W) + \eta(X)g(H^3 Z, Y)\eta(W) - \eta(Y)g(\phi H^2 Z, X)\eta(W) \\ - \eta(Y)g(H^3 Z, X)\eta(W) + g(R(X, Y)\phi Z, W) + g(R(X, Y)HZ, W),\end{aligned} \tag{3.22}$$

for any vector field X, Y, Z and W on M. Since

$$\begin{aligned}g(R(X, Y)\phi Z, W) + g(R(X, Y)HZ, W) = -g(R(X, Y)W, \phi Z) - g(R(X, Y)W, HZ) \\ = -g(R(X, Y)W, \phi Z + HZ).\end{aligned} \tag{3.23}$$

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The relation (3.22) becomes

$$\begin{aligned}0 &= g(Y,W)g(\phi Z+HZ,X) - g(X,W)g(\phi Z+HZ,Y)\\ &- g(H^2Y,W)g(\phi Z+HZ,X) + g(H^2X,W)g(\phi Z+HZ,Y)\\ &+ \eta(X)\eta(W)g(\phi Z+HZ,H^2Y) - \eta(Y)\eta(W)g(\phi Z+HZ,H^2X)\\ &- g(R(X,Y)W,\phi Z+HZ). \end{aligned} \tag{3.24}$$

Thus,

$$\begin{aligned}R(X,Y)W &= g(Y,W)X - g(X,W)Y - g(H^2Y,W)X + g(H^2X,W)Y\\ &+ \eta(X)\eta(W)H^2Y - \eta(Y)\eta(W)H^2X\\ &= \{g(Y,W) - g(H^2Y,W)\}X - \{g(X,W) - g(H^2X,W)\}Y\\ &+ \eta(W)\{\eta(X)H^2Y - \eta(Y)H^2X\}. \end{aligned} \tag{3.25}$$

Therefore, we have the following.

Theorem 3.7. *Let (M,ϕ,ξ,η,g) be a nearly Sasakian manifold. If M is locally symmetric, then the curvature tensor R of M is given by, for any vector fields X, Y and Z on M,*

$$\begin{aligned}R(X,Y)Z &= g(Y - H^2Y,Z)X - g(X - H^2X,Z)Y\\ &+ \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X\}. \end{aligned} \tag{3.26}$$

Moreover, the Ricci tensor Ric *and scalar curvature* Scal *are given, respectively, by*

$$\mathrm{Ric}(X,Y) = 2ng(X - H^2X,Y) - \eta(X)\eta(Y)\,\mathrm{trace}\,H^2, \tag{3.27}$$

$$\textit{and}\quad \mathrm{Scal} = (2n+1)\left\{2n - \mathrm{trace}\,H^2\right\}. \tag{3.28}$$

Proof. Let $\{E_i\}_{1\le i\le 2n+1}$ be an orthonormal frame with respect to g. Then the scalar curvature is given by

$$\mathrm{Scal} = \sum_{i=1}^{2n+1}\mathrm{Ric}(E_i,E_i) = (2n+1)\left\{2n - \mathrm{trace}\,H^2\right\},$$

which completes the proof. □

Note that the geometric information of the relations (3.26), (3.27) and (3.28) depends on the one of the operator H^2. Let M be a locally symmetric nearly Sasakian manifold. Then the curvature tensor R of M satisfies the equation (3.26). In addition, if M is of constant curvature κ, then, by comparing both (3.15) and (3.26), one has,

$$\begin{aligned}(\kappa-1)\{g(Y,Z)X - g(X,Z)Y\} &= -g(H^2Y,Z)X + g(H^2X,Z)Y\\ &+ \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X\}.\end{aligned}$$

Letting $Y = Z = \xi$, this equation reduces to $H^2X = -(\kappa-1)\{X - \eta(X)\xi\}$. This means that M is either Sasakian (when $\kappa = 1$) or non-Sasakian (when $\kappa \neq 1$) satisfying $H^2X = -\lambda^2\{X - \eta(X)\xi\}$, with $\kappa = 1+\lambda^2$ and $\lambda \neq 0$. The converse is straightforward, that is, if $H^2X = -(\kappa-1)\{X - \eta(X)\xi\}$, then, using (3.26), the curvature tensor R satisfies $R(X,Y)Z = \kappa\{g(Y,Z)X - g(X,Z)Y\}$, i.e., M is of constant curvature κ. By [12, Theorem 4.1], we have the following.

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Theorem 3.8. *Let (M, ϕ, ξ, η, g) be a locally symmetric nearly Sasakian manifold. Then M is of constant curvature κ if and only if M is either a Sasakian or a 5-dimensional proper nearly Sasakian manifold.*

Contrary to the result given in [12, Corollary 4.2], Theorem 3.8 shows that, in general, a locally symmetric nearly Sasakian manifold M is not of constant curvature.

As a consequence to this theorem, we have the following result.

Corollary 1. *There exist no locally symmetric nearly Sasakian manifolds of constant curvature such that $H^2 \neq -\lambda^2\{\mathbb{I} - \eta \otimes \xi\}$, for some real number λ.*

4. Curvature tensor properties

First of all, we shall prove the following propositions.

Proposition 4.1. *Let (M, ϕ, ξ, η, g) be a nearly Sasakian manifold and let R be the Riemannian curvature tensor of M. Then,*

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z &= 2\{g(\phi X, Y) + g(HX, Y)\}\phi HZ - \eta(Z)\{\eta(X)(\phi H^2 Y + HY) \\ &- \eta(Y)(\phi H^2 X + HX)\} - g(Y - \phi HY, Z)\{\phi X + HX\} + g(X - \phi HX, Z)\{\phi Y + HY\} \\ &- \{g(\phi Y, Z) + g(HY, Z)\}\{X - \phi HX\} + \{g(\phi X, Z) + g(HX, Z)\}\{Y - \phi HY\} \\ &+ \{\eta(X) g(HY + \phi H^2 Y, Z) - \eta(Y) g(HX + \phi H^2 X, Z)\}\xi, \end{aligned} \tag{4.1}$$

for any vector fields X, Y and Z on M.

Proof. The proof follows from straightforward calculations. □

From (4.2), we have the following

$$\begin{aligned} g(R(X,Y)\phi Z, \phi W) - g(R(X,Y)Z, W) &= -\eta(W) g(R(Z,\xi)X, Y) + 2g(\phi X, Y) g(HZ, W) \\ &+ 2g(HX,Y)g(HZ,W) - \eta(X)\eta(Z)g(H^2Y, W) - \eta(X)\eta(Z)g(HY, \phi W) \\ &+ \eta(Y)\eta(Z)g(H^2X, W) + \eta(Y)\eta(Z)g(HX, \phi W) - g(\phi Y, Z)g(X, \phi W) \\ &+ g(\phi Y, Z)g(HX, W) - g(HY, Z)g(X, \phi W) + g(HY, Z)g(HX, W) + g(\phi X, Z)g(Y, \phi W) \\ &- g(\phi X, Z)g(HY, W) + g(HX, Z)g(Y, \phi W) - g(HX, Z)g(HY, W) - g(Y, Z)g(X, W) \\ &+ \eta(X)\eta(W)g(Y, Z) - g(Y, Z)g(HX, \phi W) + g(\phi HY, Z)g(X, W) - \eta(X)\eta(W)g(\phi HY, Z) \\ &+ g(\phi HY, Z)g(HX, \phi W) + g(X, Z)g(Y, W) - \eta(Y)\eta(W)g(X, Z) + g(X, Z)g(HY, \phi W) \\ &- g(\phi HX, Z)g(Y, W) + \eta(Y)\eta(W)g(\phi HX, Z) - g(\phi HX, Z)g(HY, \phi W). \end{aligned} \tag{4.2}$$

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Using the identity, $g(R(X,Y)\phi Z,\phi W) = g(R(\phi Z,\phi W)X,Y)$, (4.2) becomes

$$\begin{aligned}
&g(R(\phi Z,\phi W)X,Y) - g(R(Z,W)X,Y) = -\eta(W)g(R(Z,\xi)X,Y) + 2g(\phi X,Y)g(HZ,W)\\
&+2g(HX,Y)g(HZ,W) - \eta(X)\eta(Z)g(H^2Y,W) - \eta(X)\eta(Z)g(HY,\phi W)\\
&+\eta(Y)\eta(Z)g(H^2X,W) + \eta(Y)\eta(Z)g(HX,\phi W) - g(\phi Y,Z)g(X,\phi W)\\
&+g(\phi Y,Z)g(HX,W) - g(HY,Z)g(X,\phi W) + g(HY,Z)g(HX,W) + g(\phi X,Z)g(Y,\phi W)\\
&-g(\phi X,Z)g(HY,W) + g(HX,Z)g(Y,\phi W) - g(HX,Z)g(HY,W) - g(Y,Z)g(X,W)\\
&+\eta(X)\eta(W)g(Y,Z) - g(Y,Z)g(HX,\phi W) + g(\phi HY,Z)g(X,W) - \eta(X)\eta(W)g(\phi HY,Z)\\
&+g(\phi HY,Z)g(HX,\phi W) + g(X,Z)g(Y,W) - \eta(Y)\eta(W)g(X,Z) + g(X,Z)g(HY,\phi W)\\
&-g(\phi HX,Z)g(Y,W) + \eta(Y)\eta(W)g(\phi HX,Z) - g(\phi HX,Z)g(HY,\phi W). \qquad (4.3)
\end{aligned}$$

Therefore, we have the following.

Proposition 4.2. *Let (M,ϕ,ξ,η,g) be a nearly Sasakian manifold and let R be the Riemannian curvature tensor of M. Then,*

$$\begin{aligned}
&R(\phi X,\phi Y)Z - R(X,Y)Z = -\eta(Y)g(H^2X,Z)\xi + \eta(Y)\eta(Z)H^2X + 2g(HX,Y)\phi Z\\
&+2g(HX,Y)HZ - \eta(Z)\eta(X)H^2Y - \eta(Z)\eta(X)\phi HY + \eta(X)g(H^2Z,Y)\xi\\
&+\eta(X)g(HZ,\phi Y)\xi + g(Z,\phi Y)\phi X - g(HZ,Y)\phi X + g(Z,\phi Y)HX - g(HZ,Y)HX\\
&+g(\phi Z,X)\phi Y + g(\phi Z,Y)HY + g(HZ,X)\phi Y + g(HZ,X)HY - g(Z,Y)X\\
&-g(HZ,\phi Y)X - g(Z,Y)\phi HX + \eta(Z)\eta(Y)\phi HX - g(HZ,\phi Y)\phi HX + g(Z,X)Y\\
&+g(Z,X)\phi HY - g(\phi HZ,X)Y + \eta(Y)g(\phi HZ,X)\xi - g(\phi HZ,X)\phi HY, \qquad (4.4)
\end{aligned}$$

for any vector fields X, Y and Z on M.

A plane section in TM is called a ϕ-section if there exists a vector $X \in TM$ orthogonal to ξ such that $\{X,\phi X\}$ span the section. The sectional curvature $K(X,\phi X)$, denoted $\mathcal{H}(X)$, is called *ϕ-sectional curvature*. For any vector fields X and $Y \in D = \ker\eta$, we have

$$g(R(X,\phi X)X,\phi X) = -\mathcal{H}g(X,X)^2. \qquad (4.5)$$

Taking the inner product g with respect to ϕW of (4.2) and for any X, Y and Z in D, one has

$$\begin{aligned}
g(R(X,Y)\phi Z,\phi W) = {}& g(R(X,Y)Z,W) + 2\{g(\phi X,Y) + g(HX,Y)\}g(HZ,W)\\
&-\{g(\phi Y,Z) + g(HY,Z)\}g(X-\phi HX,\phi W) + \{g(\phi X,Z)\\
&+g(HX,Z)\}g(Y-\phi HY,\phi W) - g(Y-\phi HY,Z)g(\phi X+HX,\phi W)\\
&+g(X-\phi HX,Z)g(\phi Y+HY,\phi W). \qquad (4.6)
\end{aligned}$$

Letting $Y = \phi Y$, $Z = \phi X$ and $W = Y$ in the relation (4.6), we obtain

$$\begin{aligned}
g(R(X,\phi Y)X,\phi Y) = {}& g(R(X,\phi Y)Y,\phi X) + g(X,Y)^2 - g(HX,\phi Y)^2 + g(X,\phi Y)^2\\
&-g(HX,Y)^2 - g(X,X)g(Y,Y). \qquad (4.7)
\end{aligned}$$

Likewise, we have, for any $X, Y \in \Gamma(D)$,

$$g(R(X,\phi X)Y,\phi X) = g(R(X,\phi X)X,\phi Y). \qquad (4.8)$$

Substituting $X+Y$ in (4.5), using (4.8), the left-hand side of (4.5) becomes,

$$\begin{aligned} g(R(X+Y,\phi X+\phi Y)(X+Y),\phi X+\phi Y) &= g(R(X,\phi X)X,\phi X)+g(R(Y,\phi Y)Y,\phi Y) \\ &+g(R(Y,\phi X)X,\phi X)+g(R(X,\phi Y)X,\phi X)+g(R(Y,\phi Y)X,\phi X)+g(R(X,\phi X)Y,\phi X) \\ &+g(R(Y,\phi X)Y,\phi X)+g(R(X,\phi Y)Y,\phi X)+g(R(Y,\phi Y)Y,\phi X)+g(R(X,\phi X)X,\phi Y) \\ &+g(R(Y,\phi X)X,\phi Y)+g(R(X,\phi Y)X,\phi Y)+g(R(Y,\phi Y)X,\phi Y)+g(R(X,\phi X)Y,\phi Y) \\ &+g(R(Y,\phi X)Y,\phi Y)+g(R(X,\phi Y)Y,\phi Y). \end{aligned} \tag{4.9}$$

Using (4.7) and (4.8), the Bianchi Identity, i.e., $g(R(Y,\phi Y)X,\phi X) = g(R(X,\phi Y)Y,\phi X) + g(R(\phi X,\phi Y)X,Y)$ and $g(R(X,\phi X)Y,\phi Y) = g(R(X,\phi Y)Y,\phi X) + g(R(\phi X,\phi Y)X,Y)$, one has,

$$\begin{aligned} &g(R(X+Y,\phi X+\phi Y)(X+Y),\phi X+\phi Y) \\ &\quad = 4g(R(X,\phi Y)Y,\phi X)+4g(R(Y,\phi Y)Y,\phi X)+4g(R(X,\phi X)X,\phi Y) \\ &\quad +2g(R(\phi X,\phi Y)X,Y)+g(R(Y,\phi X)Y,\phi X)+g(R(X,\phi Y)X,\phi Y) \\ &\quad +g(R(X,\phi X)X,\phi X)+g(R(Y,\phi Y)Y,\phi Y) \end{aligned} \tag{4.10}$$

Now, using (4.5), the relation (4.10) becomes, for any X and Y in D,

$$\begin{aligned} &g(R(X+Y,\phi X+\phi Y)(X+Y),\phi X+\phi Y) \\ &\quad = 4g(R(X,\phi Y)Y,\phi X)+4g(R(Y,\phi Y)Y,\phi X)+4g(R(X,\phi X)X,\phi Y) \\ &\quad +2g(R(\phi X,\phi Y)X,Y)+g(R(Y,\phi X)Y,\phi X)+g(R(X,\phi Y)X,\phi Y) \\ &\quad -\mathcal{H}g(X,X)^2-\mathcal{H}g(Y,Y)^2. \end{aligned} \tag{4.11}$$

Substituting $X+Y$ in (4.5), the right-hand side of (4.5) gives

$$\begin{aligned} -\mathcal{H}g(X,X)^2 = -\mathcal{H}\{&g(X,X)^2+4g(X,X)g(X,Y)+2g(X,X)g(Y,Y) \\ &+4g(X,Y)^2+4g(X,Y)g(Y,Y)+g(Y,Y)^2\}. \end{aligned} \tag{4.12}$$

The equality of (4.11) and (4.12), we get

$$\begin{aligned} &\frac{1}{2}\{4g(R(X,\phi Y)Y,\phi X)+4g(R(Y,\phi Y)Y,\phi X)+4g(R(X,\phi X)X,\phi Y) \\ &\quad +2g(R(\phi X,\phi Y)X,Y)+g(R(Y,\phi X)Y,\phi X)+g(R(X,\phi Y)X,\phi Y)\} \\ &\quad = -\mathcal{H}\{2g(X,Y)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y) \\ &\quad +g(X,X)g(Y,Y)\}. \end{aligned} \tag{4.13}$$

Putting $X=\phi Y$, $Y=X$ and $Z=Y$ in (4.2), we have

$$\begin{aligned} -g(R(Y,\phi X)Y,\phi X)-g(R(\phi Y,X)Y,\phi X) &= g(HY,\phi X)^2-g(Y,\phi X)^2 \\ &+g(HY,X)^2+g(Y,Y)g(X,X)-g(X,Y)^2. \end{aligned} \tag{4.14}$$

Therefore, we have

$$\begin{aligned} g(R(Y,\phi X)Y,\phi X) &= g(R(X,\phi Y)Y,\phi X)+g(X,Y)^2-g(HY,\phi X)^2 \\ &+g(Y,\phi X)^2-g(HY,X)^2-g(X,X)g(Y,Y). \end{aligned} \tag{4.15}$$

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Adding (4.7) and (4.15), one obtains

$$\begin{aligned} g(R(X,\phi Y)X,\phi Y)+g(R(Y,\phi X)Y,\phi X) &= 2g(R(X,\phi Y)Y,\phi X) \\ &+2g(X,Y)^2-2g(HX,\phi Y)^2+2g(X,\phi Y)^2-2g(HX,Y)^2 \\ &-2g(X,X)g(Y,Y). \end{aligned} \tag{4.16}$$

Putting (4.16) into (4.13), we have

$$\begin{aligned} 3g(R(X,\phi Y)Y,\phi X) &+2g(R(Y,\phi Y)Y,\phi X)+2g(R(X,\phi X)X,\phi Y) \\ &+g(R(\phi X,\phi Y)X,Y)+g(X,Y)^2-g(HX,\phi Y)^2+g(X,\phi Y)^2 \\ &-g(HX,Y)^2-g(X,X)g(Y,Y) \\ &=-\mathcal{H}\{2g(X,Y)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y) \\ &+g(X,X)g(Y,Y)\}. \end{aligned} \tag{4.17}$$

Since

$$\begin{aligned} g(R(\phi X,\phi Y)X,Y) &= g(R(X,Y)X,Y)+g(HX,Y)^2-g(X,\phi Y)^2 \\ &-g(X,Y)^2+g(HX,\phi Y)^2+g(X,X)g(Y,Y), \end{aligned} \tag{4.18}$$

the relation (4.17) becomes

$$\begin{aligned} 3g(R(X,\phi Y)Y,\phi X)+2g(R(Y,\phi Y)Y,\phi X)+2g(R(X,\phi X)X,\phi Y)+g(R(X,Y)X,Y) \\ =-\mathcal{H}\{2g(X,Y)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y) \\ +g(X,X)g(Y,Y)\}. \end{aligned} \tag{4.19}$$

Replacing Y by $-Y$ in (4.19), one obtains

$$\begin{aligned} 3g(R(X,\phi Y)Y,\phi X)-2g(R(Y,\phi Y)Y,\phi X)-2g(R(X,\phi X)X,\phi Y) \\ +g(R(X,Y)X,Y)=-\mathcal{H}\{2g(X,Y)^2-2g(X,X)g(X,Y) \\ -2g(X,Y)g(Y,Y)+g(X,X)g(Y,Y)\}. \end{aligned} \tag{4.20}$$

Now summing (4.19) and (4.20),

$$3g(R(X,\phi Y)Y,\phi X)+g(R(X,Y)X,Y)=-\mathcal{H}\{2g(X,Y)^2+g(X,X)g(Y,Y)\}. \tag{4.21}$$

Replacing Y by ϕY in (4.21) and using curvature identities, we have the following,

$$3g(R(\phi Y,\phi X)Y,X)+g(R(X,\phi Y)X,\phi Y)=-\mathcal{H}\{2g(X,\phi Y)^2+g(X,X)g(Y,Y)\}. \tag{4.22}$$

By the relations (4.7) and (4.18), the left hand side of (4.22) becomes

$$\begin{aligned} 3g(R(\phi Y,\phi X)Y,X)+g(R(X,\phi Y)X,\phi Y) &= 3g(R(X,Y)X,Y)+g(R(X,\phi Y)Y,\phi X) \\ &+2g(HX,Y)^2-2g(X,\phi Y)^2-2g(X,Y)^2+2g(HX,\phi Y)^2 \\ &+2g(X,X)g(Y,Y). \end{aligned} \tag{4.23}$$

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Putting the pieces (4.22) and (4.23) together leads to

$$\begin{aligned} g(R(X,\phi Y)Y,\phi X) = &-3g(R(X,Y)X,Y) - \mathcal{H}\{2g(X,\phi Y)^2 + g(X,X)g(Y,Y)\} \\ &- 2g(HX,Y)^2 + 2g(X,\phi Y)^2 + 2g(X,Y)^2 - 2g(HX,\phi Y)^2 \\ &- 2g(X,X)g(Y,Y). \end{aligned} \tag{4.24}$$

Substituting (4.24) into (4.21) gives

$$\begin{aligned} -8g(R(X,Y)X,Y) - 3\mathcal{H}\{2g(X,\phi Y)^2 + g(X,X)g(Y,Y)\} - 6g(HX,Y)^2 \\ + 6g(X,\phi Y)^2 + 6g(X,Y)^2 - 6g(HX,\phi Y)^2 \\ - 6g(X,X)g(Y,Y) = -\mathcal{H}\{2g(X,Y)^2 + g(X,X)g(Y,Y)\}. \end{aligned} \tag{4.25}$$

Therefore, we have

$$\begin{aligned} 4g(R(X,Y)X,Y) = &(\mathcal{H}+3)\{g(X,Y)^2 - g(X,X)g(Y,Y)\} \\ &- 3(\mathcal{H}-1)g(X,\phi Y)^2 - 3g(HX,Y)^2 - 3g(HX,\phi Y)^2. \end{aligned} \tag{4.26}$$

Replacing X by $X+Z$ and Y by $Y+W$ in both side of (4.26), one has,

$$\begin{aligned} 8g(R(X,Y)Z,W) + 8g(R(Z,Y)X,W) = &-4(\mathcal{H}+3)g(X,Z)g(Y,W) \\ &+ 2(\mathcal{H}+3)\{g(X,Y)g(Z,W) + g(X,W)g(Z,Y)\} \\ &- 6(\mathcal{H}-1)\{g(X,\phi Y)g(Z,\phi W) + g(X,\phi W)g(Z,\phi Y)\} \\ &- 6\{g(HX,Y)g(HZ,W) + g(HX,W)g(HZ,Y) \\ &+ g(HX,\phi Y)g(HZ,\phi W) + g(HX,\phi W)g(HZ,\phi Y)\}. \end{aligned} \tag{4.27}$$

Replacing Y by Z and Z by Y in (4.27), and multiplying both sides by -1, we have

$$\begin{aligned} &- 8g(R(X,Z)Y,W) - 8g(R(Y,Z)X,W) = 4(\mathcal{H}+3)g(X,Y)g(Z,W) \\ &- 2(\mathcal{H}+3)\{g(X,Z)g(Y,W) + g(X,W)g(Z,Y)\} + 6(\mathcal{H}-1)\{g(X,\phi Z)g(Y,\phi W) \\ &+ g(X,\phi W)g(Y,\phi Z)\} + 6\{g(HX,Z)g(HY,W) + g(HX,W)g(HY,Z) \\ &+ g(HX,\phi Z)g(HY,\phi W) + g(HX,\phi W)g(HY,\phi Z)\}. \end{aligned} \tag{4.28}$$

Adding (4.27) and (4.28),

$$\begin{aligned} &8g(R(X,Y)Z,W) + 16g(R(Z,Y)X,W) - 8g(R(X,Z)Y,W) \\ &= 6(\mathcal{H}+3)\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\} - 6(\mathcal{H}-1)\{g(X,\phi Y)g(Z,\phi W) \\ &- g(X,\phi Z)g(Y,\phi W) + 2g(X,\phi W)g(Z,\phi Y)\} - 6\{g(HX,Y)g(HZ,W) \\ &- g(HX,Z)g(HY,W) + 2g(HX,W)g(HZ,Y) + g(HX,\phi Y)g(HZ,\phi W) \\ &- g(HX,\phi Z)g(HY,\phi W) + 2g(HX,\phi W)g(HZ,\phi Y)\}. \end{aligned} \tag{4.29}$$

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By virtue of Bianchi identity, $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ and $g(R(Z,Y)X,W) = g(R(X,W)Z,Y)$, the relation ((4.29),

$$\begin{aligned}
&24g(R(X,W)Z,Y) = 6(\mathcal{H}+3)\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\} \\
&- 6(\mathcal{H}-1)\{g(X,\phi Y)g(Z,\phi W) - g(X,\phi Z)g(Y,\phi W) + 2g(X,\phi W)g(Z,\phi Y)\} \\
&- 6\{g(HX,Y)g(HZ,W) - g(HX,Z)g(HY,W) + 2g(HX,W)g(HZ,Y) \\
&+ g(HX,\phi Y)g(HZ,\phi W) - g(HX,\phi Z)g(HY,\phi W) + 2g(HX,\phi W)g(HZ,\phi Y)\}.
\end{aligned} \tag{4.30}$$

By exchanging W and Y in (4.30), one obtains

$$\begin{aligned}
&24g(R(X,Y)Z,W) = 6(\mathcal{H}+3)\{g(X,W)g(Z,Y) - g(X,Z)g(Y,W)\} \\
&- 6(\mathcal{H}-1)\{g(X,\phi W)g(Z,\phi Y) - g(X,\phi Z)g(W,\phi Y) + 2g(X,\phi Y)g(Z,\phi W)\} \\
&- 6\{g(HX,W)g(HZ,Y) - g(HX,Z)g(HW,Y) + 2g(HX,Y)g(HZ,W) \\
&+ g(HX,\phi W)g(HZ,\phi Y) - g(HX,\phi Z)g(HW,\phi Y) + 2g(HX,\phi Y)g(HZ,\phi W)\},
\end{aligned} \tag{4.31}$$

for any X, Y, Z and $W \in \Gamma(D)$. Now, for any vector field X on M, we may write $X = QX + \eta(X)\xi$, where Q is the projection onto D. Thus, for any X, Y , Z and $W \in \Gamma(TM)$,

$$\begin{aligned}
&g(R(QX,QY)QZ,QW) = g(R(X,Y)Z,W) - \eta(X)\eta(W)\{g(Y,Z) - g(H^2Z,Y)\} \\
&+ \eta(X)\eta(Z)\{g(W,Y) - g(H^2W,Y)\} - \eta(Y)\eta(Z)\{g(W,X) - g(H^2W,X)\} \\
&+ \eta(Y)\eta(W)\{g(Z,X) - g(H^2Z,X)\}.
\end{aligned} \tag{4.32}$$

From (4.31), and using (4.32), we have the following

$$\begin{aligned}
&24g(R(X,Y)Z,W) = 6(\mathcal{H}+3)\{g(X,W)g(Z,Y) - \eta(Z)\eta(Y)g(X,W) \\
&- \eta(X)\eta(W)g(Z,Y) - g(X,Z)g(Y,W) + \eta(Y)\eta(W)g(X,Z) \\
&+ \eta(X)\eta(Z)g(Y,W)\} - 6(\mathcal{H}-1)\{g(X,\phi W)g(Z,\phi Y) - g(X,\phi Z)g(W,\phi Y) \\
&+ 2g(X,\phi Y)g(Z,\phi W)\} - 6\{g(HX,W)g(HZ,Y) - g(HX,Z)g(HW,Y) \\
&+ 2g(HX,Y)g(HZ,W) + g(HX,\phi W)g(HZ,\phi Y) - g(HX,\phi Z)g(HW,\phi Y) \\
&+ 2g(HX,\phi Y)g(HZ,\phi W)\} + 24\eta(X)\eta(W)\{g(Y,Z) - g(H^2Z,Y)\} \\
&- 24\eta(X)\eta(Z)\{g(W,Y) - g(H^2W,Y)\} + 24\eta(Y)\eta(Z)\{g(W,X) - g(H^2W,X)\} \\
&- 24\eta(Y)\eta(W)\{g(Z,X) - g(H^2Z,X)\}.
\end{aligned} \tag{4.33}$$

Therefore, we have the following.

Proposition 4.3. *Let (M, ϕ, ξ, η, g) be a dimensional nearly Sasakian manifold. The necessary and sufficient condition for M to have pointwise constant ϕ-holomorphic sectional curvature H is*

$$\begin{aligned} R(X,Y)Z = {} & \frac{\mathcal{H}+3}{4}\{g(Z,Y)X - g(X,Z)Y\} + \frac{\mathcal{H}-1}{4}\{\eta(X)\eta(Z)Y \\ & - \eta(Z)\eta(Y)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Z,Y)\xi + g(Z,\phi Y)\phi X \\ & + g(X,\phi Z)\phi Y + 2g(X,\phi Y)\phi Z\} - \frac{1}{4}\{g(HZ,Y)HX + g(HX,Z)HY \\ & + 2g(HX,Y)HZ - g(HZ,\phi Y)\phi HX - g(HX,\phi Z)\phi HY \\ & - 2g(HX,\phi Y)\phi HZ\} + \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X\} \\ & + \{\eta(Y)g(H^2Z,X) - \eta(X)g(H^2Z,Y)\}\xi, \end{aligned} \tag{4.34}$$

for all vector fields X, Y and Z on M.

From (4.34), the Ricci tensor Ric of the Riemannian metric g gives

$$\begin{aligned} \mathrm{Ric}(X,Y) = {} & \frac{n(\mathcal{H}+3)+\mathcal{H}-1}{2}g(X,Y) - \frac{(n+1)(\mathcal{H}-1)}{2}\eta(X)\eta(Y) \\ & - \frac{5}{2}g(X,H^2Y) - \eta(X)\eta(Y)\mathrm{trace}\, H^2. \end{aligned} \tag{4.35}$$

Moreover, the Ricci curvature satisfies

$$\mathrm{Ric}(\phi X, \phi Y) = \mathrm{Ric}(X,Y) - (2n - \mathrm{trace}\, H^2)\eta(X)\eta(Y). \tag{4.36}$$

The scalar curvature τ are given by

$$\tau = \frac{1}{2}\{n(2n+1)(\mathcal{H}+3) + n(\mathcal{H}-1)\} - \frac{7}{2}\mathrm{trace}\, H^2. \tag{4.37}$$

Lemma 4.4. *The eigenvalues of the operator H^2 are constant.*

Proof. The proof follows from a straightforward calculation using (2.14). □

For any vector field X, Y, Z and W, one has,

$$\begin{aligned} (\nabla_X \mathrm{Ric})(Y,Z) = {} & \frac{n+1}{2}X(\mathcal{H})g(Y,Z) - \left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \mathrm{trace}\, H^2\right\}\{\eta(Z)(\nabla_X\eta)Y \\ & + \eta(Y)(\nabla_X\eta)Z\} + \frac{5}{2}\{g((\nabla_X H)Y, HZ) + g(HY, (\nabla_X H)Z)\} \\ = {} & \frac{n+1}{2}X(\mathcal{H})g(Y,Z) + \left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \mathrm{trace}\, H^2\right\}\{\eta(Z)g(\phi X, Y) \\ & + \eta(Z)g(HX,Y) + \eta(Y)g(\phi X, Z) + \eta(Y)g(HX,Z)\} \\ & + \frac{5}{2}\left\{\eta(Y)g(H^2X, HZ) - \eta(Y)g(\phi HX, HZ) + \eta(Z)g(H^2X, HY) \right. \\ & \left. - \eta(Z)g(\phi HX, HY)\right\}. \end{aligned} \tag{4.38}$$

Let $\{E_i\}_{1 \le i \le 2n+1}$ be an arbitrary local orthonormal frame field on M. Then

$$\nabla_X \tau = 2\sum_{i=1}^{2n+1}(\nabla_{E_i} Ric)(X, E_i) = (n+1)X(\mathcal{H}). \tag{4.39}$$

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On the other hand, using (4.37) and Lemma 4.4, one obtains

$$\nabla_X\tau = \frac{1}{2}\{n(2n+1)X(\mathcal{H}) + nX(\mathcal{H})\} - \frac{1}{2}X(\text{trace } H^2) = n(n+1)X(\mathcal{H}). \tag{4.40}$$

From (4.39) and (4.40), we have, for any vector field Xon M,

$$n(n+1)X(\mathcal{H}) = (n+1)X(\mathcal{H}). \tag{4.41}$$

This leads to $(n-1)X(\mathcal{H}) = 0$, for any vector field X on M. If $n > 1$ and M is connected, then H is constant on M. Therefore, using Ogiue's ideas given [10], we have the following theorem.

Theorem 4.5. *Let M be a $(2n+1)$-dimensional nearly Sasakian manifold ($n > 1$). If the ϕ-holomorphic sectional curvature at any point of M is independent of the choice of ϕ-holomorphic section, then it is constant on M and the curvature tensor is given by*

$$\begin{aligned}
R(X,Y)Z &= \frac{\mathcal{H}+3}{4}\{g(Z,Y)X - g(X,Z)Y\} + \frac{\mathcal{H}-1}{4}\{\eta(X)\eta(Z)Y - \eta(Z)\eta(Y)X \\
&+ \eta(Y)g(X,Z)\xi - \eta(X)g(Z,Y)\xi + g(Z,\phi Y)\phi X + g(X,\phi Z)\phi Y + 2g(X,\phi Y)\phi Z\} \\
&- \frac{1}{4}\{g(HZ,Y)HX + g(HX,Z)HY + 2g(HX,Y)HZ - g(HZ,\phi Y)\phi HX \\
&- g(HX,\phi Z)\phi HY - 2g(HX,\phi Y)\phi HZ\} + \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X\} \\
&+ \{\eta(Y)g(H^2Z,X) - \eta(X)g(H^2Z,Y)\}\xi,
\end{aligned} \tag{4.42}$$

for any vector fields X, Y and Z on M.

A complete and simply connected nearly Sasakian manifold with constant ϕ-holomorphic sectional curvature is said to be a *nearly Sasakian space form*. So, we have the following result.

Theorem 4.6. *Let M be a $(2n+1)$-dimensional complete and simply connected nearly Sasakian manifold ($n > 1$). Then M is a nearly Sasakian space form if and only if the curvature tensor R is given by is (4.42).*

Now, if the Ricci tensor satisfies the Codazzi equation, that is,

$$(\nabla_X\text{Ric})(Y,Z) = (\nabla_Y\text{Ric})(X,Z), \;\; \forall\, X,\; Y,\; Z \in \Gamma(TM), \tag{4.43}$$

then, one has

$$\begin{aligned}
&\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \text{trace } H^2\right\}\{2\eta(Z)g(\phi X, Y + 2\eta(Z)g(HX,Y) \\
&+\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z) + \eta(Y)g(HX,Z) - \eta(X)g(HY,Z)\} \\
&+ \frac{5}{2}\Big\{\eta(Y)g(H^2X,HZ) - \eta(X)g(H^2Y,HZ) + \eta(X)g(\phi HY,HZ) \\
&-\eta(Y)g(\phi HX,HZ) + 2\eta(Z)g(H^2X,HY) - 2\eta(Z)g(\phi HX,HY)\Big\} = 0.
\end{aligned} \tag{4.44}$$

Letting $X = \xi$ into this equation gives

$$\begin{aligned}
&\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \text{trace } H^2\right\}\{-g(\phi Y,Z) - g(HY,Z)\} \\
&+ \frac{5}{2}\Big\{-g(H^2Y,HZ) + g(\phi HY,HZ)\Big\} = 0.
\end{aligned} \tag{4.45}$$

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If $Z = \phi Y$, then this equation becomes

$$\frac{2}{5}\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \text{trace}\, H^2\right\}\{g(Y,Y) - \eta(Y)\eta(Y)\} + g(HY,HY) = 0. \tag{4.46}$$

Setting

$$\mu = \frac{2}{5}\left\{\frac{(n+1)(\mathcal{H}-1)}{2} + \text{trace}\, H^2\right\}. \tag{4.47}$$

Then, the relation (4.46) leads to

$$H^2 = \mu\{\mathbb{I} - \eta \otimes \xi\}. \tag{4.48}$$

By Lemma 4.4 and Theorem 4.6, the function μ defined in (4.47) on a Sasakian space form is a constant. Taking into account the reasoning that led to (3.5), $\mu = -\lambda^2$. This means that μ is non-positive. By Theorem 4.1 in [12], we have the following result.

Theorem 4.7. *A nearly Sasakian space form with the Ricci tensor satisfying the Codazzi equation (4.43) is either a Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} = 1$ or a 5-dimensional proper nearly Sasakian manifold with a constant ϕ-holomorphic sectional curvature $\mathcal{H} < 1$.*

Proof. The last assertion follows from (4.47), (4.48) and the sign of μ. □

Note that $H\xi = 0$, i.e., 0 is an eigenvalue of H^2. Also, being H skew-symmetric, the non-vanishing eigenvalues of H^2 are negative as proven by the relation (3.5). So, the spectrum of H^2 is of type

$$\text{Spec}(H^2) = \{0, -\lambda^2, \cdots, -\lambda^2\}, \;\; \lambda \neq 0.$$

Let denote by $\mathbb{R}\xi$ the 1-dimensional distribution generated by ξ, and by $D(0)$ and $D(-\lambda^2)$ the distributions of the eigenvectors with eigenvalues 0 and $-\lambda^2$, respectively.

Let us consider an eigenvector X with eigenvalue $-\lambda^2$. From (2.4), we have

$$H^2\nabla_X\xi = -(\nabla_X H^2)\xi = -(\phi + H)H^2X = -\lambda^2\nabla_X\xi. \tag{4.49}$$

This means that $\nabla_X\xi$ is an eigenvector with eigenvalue $-\lambda^2$. On the other hand, (2.14) implies $\nabla_\xi H^2 = 0$. Using this, we have

$$H^2\nabla_\xi X = \nabla_\xi H^2 X = -\xi(\lambda^2)X - \lambda^2\nabla_\xi X = -\lambda^2\nabla_\xi X. \tag{4.50}$$

So, $\nabla_\xi X$ is also an eigenvector with eigenvalue $-\lambda^2$. Now, if X and Y are eigenvectors with eigenvalue $-\lambda^2$, orthogonal to ξ, from (2.14), we get

$$\begin{aligned} H^2(\nabla_X Y) &= \nabla_X H^2 Y - (\nabla_X H^2)Y \\ &= -\lambda^2\nabla_X Y + \lambda^2 g(\phi X + HX, Y)\xi. \end{aligned} \tag{4.51}$$

If $\lambda = 0$, we have $\nabla_X Y \in D(0)$. If $\lambda \neq 0$, one obtains

$$H^2(\phi^2\nabla_X Y) = \phi^2(H^2\nabla_X Y) = -\lambda^2\phi^2(\nabla_X Y), \tag{4.52}$$

and thus,

$$\nabla_X Y = -\phi^2\nabla_X Y + \eta(\nabla_X Y)\xi,$$

belongs to the distribution $\mathbb{R}\xi \oplus D(-\lambda^2)$.

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Note that, if X is an eigenvector of H^2 with eigenvalue $-\lambda^2$, then $X, \phi X, HX, H\phi X$ are orthogonal eigenvectors of H^2 with eigenvalue $-\lambda^2$. Using Theorem 4.7, 0 is a simple eigenvalue and the multiplicity of the eigenvalue $-\lambda^2$ is 4. Therefore, we have the following result.

Theorem 4.8. *Let M be a nearly Sasakian space form with Ricci tensor satisfying the Codazzi equation (4.43). Then the spectrum of H^2 is of type*

$$\mathrm{Spec}(H^2) = \{0, -\lambda^2,\ -\lambda^2,\ -\lambda^2,\ -\lambda^2\},\ \ \lambda \neq 0,$$

where 0 is a simple eigenvalue and $-\lambda^2$ is an eigenvalue of multiplicity 4. Moreover, the distributions $D(0)$ and $\mathbb{R}\xi \oplus D(-\lambda^2)$ are integrable with totally geodesic leaves.

In [11, Theorem 6.1], Olszak proved, under the condition (3.1), that a proper nearly Sasakian space form is 5-dimensional manifold of constant sectional curvature. Next, we prove otherwise using the notion of projectively flat. First of all, we note that the class of manifolds with Ricci tensor satisfying the Codazzi equation (4.43) is a subclass of projectively flat manifolds (see [8, Proposition 5] for more details). The concept of "projectively flat" is defined using the so-called projective curvature tensor which is an important tensor from the differential geometric point of view. In case, there exists a one-to-one correspondence between each coordinate neighbourhood of a manifold M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be *locally projectively flat*. It is known that the Levi-Civita connection of a non-degenerate metric g is locally projectively flat if and only if g has constant sectional curvature [8, p. 411].

For $n \geq 1$, the nearly Sasakian manifold M is locally projectively flat if and only if the well-known projective curvature tensor $\mathcal{P}$ vanishes, where $\mathcal{P}$ is defined as

$$\mathcal{P}(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{Ric(Y,Z)X - Ric(X,Z)Y\}, \tag{4.53}$$

for any vector fields X, Y and Z on M.

Theorem 4.9. *A proper nearly Sasakian space form is not of constant sectional curvature.*

Proof. Let M be a proper nearly Sasakian space form. If M is of constant sectional curvature, then M is locally projectively flat, that is, the relation projective curvature tensor $\mathcal{P}$ in (4.53) vanishes. A direct calculation of (4.53) leads to

$$\begin{aligned}
&2nR(X,Y)Z - \{Ric(Y,Z)X - Ric(X,Z)Y\} = -\frac{\mathcal{H}-1}{2}\{g(Z,Y)X - g(X,Z)Y\}\\
&+\left\{\mathrm{trace}\, H^2 + \frac{\mathcal{H}-1}{2}\right\}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} + \frac{n(\mathcal{H}-1)}{2}\{\eta(Y)g(X,Z)\xi\\
&-\eta(X)g(Z,Y)\xi + g(Z,\phi Y)\phi X + g(X,\phi Z)\phi Y + 2g(X,\phi Y)\phi Z\}\\
&-\frac{n}{2}\{g(HZ,Y)HX + g(HX,Z)HY + 2g(HX,Y)HZ - g(HZ,\phi Y)\phi HX\\
&-g(HX,\phi Z)\phi HY - 2g(HX,\phi Y)\phi HZ\} + \eta(Z)\{\eta(X)H^2Y - \eta(Y)H^2X\}\\
&+2n\{\eta(Y)g(H^2Z,X) - \eta(X)g(H^2Z,Y)\}\xi - \frac{5}{2}g(HY,HZ)X + \frac{5}{2}g(HX,HZ)Y.
\end{aligned} \tag{4.54}$$

Now, putting $Y = Z \in \Gamma(D)$ into (4.54) and considering $X \in \Gamma(D)$ such that $g(X,Y) = 0$, we have,

$$2ng(\mathcal{P}(X,Y)Y,Y) = \frac{5}{2}g(HX,HY)g(Y,Y). \tag{4.55}$$

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Since M is locally projectively flat, then (4.55) vanishes, that is,

$$0 = g(HX, HY)g(Y, Y), \quad \forall\, X \in \Gamma(D).$$

This implies that $H^2Y = 0$, as $g(Y, Y) \neq 0$, for any $Y \in \Gamma(D)$. Since $H\xi = 0$, so $H^2 = 0$ on M, a contradiction as M is non-Sasakian. This completes the proof. □

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