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Estimates for generalized parabolic Marcinkiewicz integrals with rough kernels on product domains

Mohammed Ali [1] and **Hussain Al-Qassem** [2,*]

[1] Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan; myali@just.edu.jo
[2] Mathematics Program, Department of Mathematics, Statistics and Physics, College of Arts and Sciences, Qatar University, 2713, Doha, Qatar
* Correspondence: husseink@qu.edu.qa

Abstract: We prove L^p estimates of a class of generalized Marcinkiewicz integral operators with mixed homogeneity on product domains. By using these estimates along with an extrapolation argument we obtain the boundedness of our operators under very weak conditions on the kernel functions. Our results in this paper improve and extend several known results on both generalized Marcinkiewicz integrals and parabolic Marcinkiewicz integrals on product domains.

Keywords: Triebel-Lizorkin space, rough kernel; parabolic Marcinkiewicz integral; product domains; extrapolation.

1. Introduction

Throughout this article, s ($s = \kappa$ or η) is assumed to be an integer greater than or equal 2. Also, $\mathbb{S}^{s-1}$ is assumed to be the unit sphere in the Euclidean space $\mathbb{R}^s$ which is equipped with the normalized Lebesgue surface measure $d\sigma_s(\cdot) \equiv d\sigma$.

For fixed $\beta_{s,k} \geq 1$ ($k \in \{1, 2, \cdots, s\}$), we define the mapping $\Theta : \mathbb{R}^+ \times \mathbb{R}^s \to \mathbb{R}$ by $\Theta(\tau_s, v) = \sum_{k=1}^{s} v_k^2 \tau_s^{-2\beta_{s,k}}$ with $v = (v_1, v_2, \ldots, v_s) \in \mathbb{R}^s$. For a fixed $v \in \mathbb{R}^s$, the unique solution to the equation $\Theta(\tau_s, v) = 1$ is denoted by $\tau_s \equiv \tau_s(v)$. The metric space $(\mathbb{R}^s, \tau_s)$ is known by the mixed homogeneity space associated to $\{\beta_{s,k}\}_{k=1}^{s}$. Let D_{τ_s} be the diagonal $s \times s$ matrix

$$D_{\tau_s} = \begin{bmatrix} \tau_s^{\beta_{s,1}} & & 0 \\ & \ddots & \\ 0 & & \tau_s^{\beta_{s,s}} \end{bmatrix}.$$

The following transformation presents the change of variables concerning the space $(\mathbb{R}^s, \tau_s)$:

$$\begin{aligned} v_1 &= \tau_s^{\beta_{s,1}} \cos x_1 \cdots \cos x_{s-2} \cos x_{s-1}, \\ v_2 &= \tau_s^{\beta_{s,2}} \cos x_1 \cdots \cos x_{s-2} \sin x_{s-1}, \\ &\vdots \\ v_{s-1} &= \tau_s^{\beta_{s,s-1}} \cos x_1 \sin x_2, \\ v_s &= \tau_s^{\beta_{s,s}} \sin x_1. \end{aligned}$$

Hence, $dv = \tau_s^{\beta_s - 1} J_s(v') d\tau_s d\sigma(v')$, where

$$\beta_s = \sum_{k=1}^{s} \beta_{s,k}, \quad J_s(v') = \sum_{k=1}^{s} \beta_{s,k} (v'_k)^2, \quad v' = D_{\tau_s^{-1}} v \in \mathbb{S}^{s-1},$$

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and $\tau_s^{\beta_s-1}J_s(v')$ is the Jacobian of the transformation.

Fabes and Riviére showed in [1] that $J_s \in \mathcal{C}^\infty(\mathbb{S}^{s-1})$ and that there is a constant $A \geq 1$ satisfying

$$1 \leq J_s(v') \leq A.$$

For $\rho_1 = a_1 + ia_2, \rho_2 = b_1 + ib_2$ *with* $a_1, b_1 \in (0,\infty)$ *and* $a_2, b_2 \in (-\infty,\infty)$, we assume that

$$\mathcal{K}_{\mho,h}(v,\omega) = \frac{\mho(v,\omega)h(\tau_\kappa(v),\tau_\eta(\omega))}{(\tau_\kappa(v))^{\beta_\kappa-\rho_1}(\tau_\eta(\omega))^{\beta_\eta-\rho_2}},$$

where h is a measurable function defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and $\mho$ is a measurable function defined on $\mathbb{R}^\kappa \times \mathbb{R}^\eta$ which is integrable over $\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}$ and satisfies the following properties:

$$\mho(D_{\tau_\kappa}v, D_{\tau_\eta}\omega) = \mho(v,\omega), \ \forall \tau_\kappa, \tau_\eta > 0 \tag{1}$$

and

$$\int_{\mathbb{S}^{\kappa-1}} \mho(v,.)J_\kappa(v)d\sigma(v) = \int_{\mathbb{S}^{\eta-1}} \mho(.,\omega)J_\eta(\omega)d\sigma(\omega) = 0. \tag{2}$$

For $g \in \mathcal{S}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$, we define the generalized parabolic Marcinkiewicz integral $\mathcal{G}^{(\mu)}_{\mho,h}$ on product domains by

$$\mathcal{G}^{(\mu)}_{\mho,h}(g)(x,y) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} |T_{s,r}(g)(x,y)|^\mu \frac{dsdr}{sr} \right)^{1/\mu},$$

where

$$T_{s,r}(g)(x,y) = \frac{1}{s^{\rho_1}r^{\rho_2}} \int_{\tau_\kappa(v)\leq s} \int_{\tau_\eta(\omega)\leq r} g(x-v, y-\omega)\mathcal{K}_{\mho,h}(v,\omega)dvd\omega$$

and $1 < \mu < \infty$.

We notice that if $\beta_{\kappa,1} = \beta_{\kappa,2} = \cdots = \beta_{\kappa,\kappa} = 1$ and $\beta_{\eta,1} = \beta_{\eta,2} = \cdots = \beta_{\eta,\eta} = 1$, then we have $\beta_\kappa = \kappa$, $\tau_\kappa(v) = |v|$, $\beta_\eta = \eta$, $\tau_\eta(\omega) = |\omega|$, and $(\mathbb{R}^\kappa \times \mathbb{R}^\eta, \tau_\kappa, \tau_\eta) = (\mathbb{R}^\kappa \times \mathbb{R}^\eta, |\cdot|, |\cdot|)$. In this case, we denote the operator $\mathcal{G}^{(\mu)}_{\mho,h}$ by $\mathcal{M}^{(\mu)}_{\mho,h}$. In addition, when $\mu = 2$, $h \equiv 1$ and $\rho_1 = 1 = \rho_2$, we denote $\mathcal{M}^{(\mu)}_{\mho,h}$ by $\mathcal{M}_\mho$ which is the classical Marcinkiewicz integral on product domains. The investigation of the boundedness of $\mathcal{M}_\mho$ began in [2] in which the author proved the L^2 boundedness of $\mathcal{M}_\mho$ under the condition $\mho \in L(\log L)^2(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$. Subsequently, the investigation of the L^p boundedness of $\mathcal{M}_\mho$ was considered by many authors (see for instance [3–6]).

On the other hand, the investigation of the L^p boundedness of the operator $\mathcal{G}^{(\mu)}_{\mho,h}$ was considered by many authors. For example, Al-Salman introduced $\mathcal{G}^{(\mu)}_{\mho,h}$ in [7] in which he proved that $\mathcal{G}^{(2)}_{\mho,1}$ is bounded on $L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ for all $p \in (1,\infty)$ provided that $\mho \in L(\log L)(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$. Later on, the authors of [8] improved the results in [7]. In fact, they proved the L^p boundedness of $\mathcal{G}^{(2)}_{\mho,h}$ for all $|1/2 - 1/p| < \min\{1/2, 1/\ell'\}$ whenever $\mho$ in $B_q^{(0,0)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ with $q > 1$ or $\mho$ in $L(\log L)(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$, and $h \in \Delta_\ell(\mathbb{R}_+ \times \mathbb{R}_+)$ with $\ell > 1$. Here, $\Delta_\ell(\mathbb{R}_+ \times \mathbb{R}_+)$ (for $\ell > 1$) refers to the set of all measurable functions h such that

$$\|h\|_{\Delta_\ell(\mathbb{R}_+\times\mathbb{R}_+)} = \sup_{k,j\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} |h(\tau_\kappa,\tau_\eta)|^\ell \frac{d\tau_\kappa d\tau_\eta}{\tau_\kappa\tau_\eta} \right)^{1/\ell} < \infty.$$

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Let us now recall the definition of Triebel-Lizorkin spaces on product domains. Let $1 < \mu, p < \infty$ and $\overrightarrow{c} = (c_1, c_2) \in \mathbb{R} \times \mathbb{R}$. The homogeneous Triebel-Lizorkin space $\dot{F}_p^{\overrightarrow{c},\mu}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ is defined to be the set of all tempered distributions g on $\mathbb{R}^\kappa \times \mathbb{R}^\eta$ satisfying

$$\|g\|_{\dot{F}_p^{\overrightarrow{c},\mu}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} = \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{kc_1\mu} 2^{jc_2\mu} \left| (\psi_{k,\kappa} \otimes \psi_{j,\eta}) * g \right|^\mu \right)^{1/\mu} \right\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} < \infty,$$

where for $s \in \{\kappa, \eta\}$ and $x \in \mathbb{R}^s$, $\widehat{\psi_{j,s}}(x) = 2^{-js} D_s(2^{-j}x)$ and $D_s \in C_0^\infty(\mathbb{R}^s)$ is radial function satisfies the following:

(1) $D_s \in [0,1]$,

(2) $supp\,(D_s) \subset \left\{ x \in \mathbb{R}^s : |x| \in [\frac{1}{2}, 2] \right\}$,

(3) $D_s(x) \geq A > 0$ if $|x| \in [\frac{3}{5}, \frac{5}{3}]$ for some constant A,

(4) $\sum_{j\in\mathbb{Z}} D_s(2^{-j}x) = 1$ with $x \neq 0$.

The authors of [9] proved that the space $\dot{F}_p^{\overrightarrow{c},\mu}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ satisfies the following properties:

(i) For $p \in (1, \infty)$, we have $\dot{F}_p^{0,\overrightarrow{2}}(\mathbb{R}^\kappa \times \mathbb{R}^\eta) = L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$,

(ii) If $\mu_1 \leq \mu_2$, then $\dot{F}_p^{\overrightarrow{c},\mu_1}(\mathbb{R}^\kappa \times \mathbb{R}^\eta) \subseteq \dot{F}_p^{\overrightarrow{c},\mu_2}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$,

(iii) $\dot{F}_{p'}^{-\overrightarrow{c},\mu'}(\mathbb{R}^\kappa \times \mathbb{R}^\eta) = \left(\dot{F}_p^{\overrightarrow{c},\mu}(\mathbb{R}^\kappa \times \mathbb{R}^\eta) \right)^*$, where p' is the exponent conjugate to p,

(iv) The Schwartz space $\mathcal{S}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ is dense in $\dot{F}_p^{\overrightarrow{c},\mu}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$.

Recently, the authors of [10] employed the extrapolation argument of Yano [11] to prove that whenever Ω lies in the space $L(\log L)^{2/\mu}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ or in the space $B_q^{(0,\frac{2}{\mu}-1)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$, then for all $p \in (1, \infty)$,

$$\left\| \mathcal{M}_{\mho,1}^{(\mu)}(g) \right\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} \leq A_p \|g\|_{\dot{F}_p^{\overrightarrow{0},\mu}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)}, \tag{3}$$

where $B_q^{(0,\alpha)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ ($\alpha > -1$, $q > 1$) refers to a special class of block spaces introduced in [12]. Very recently, the result in [10] was improved in [13] in which the authors proved that if $\mho \in L(\log L)^{2/\mu}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}) \cup B_q^{(0,\frac{2}{\mu}-1)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ with $q > 1$ and $h \in \Delta_\ell(\mathbb{R}_+ \times \mathbb{R}_+)$, then $\mathcal{M}_{\mho,h}^{(\mu)}$ is bounded on $L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ for $p \in (\ell', \infty)$ with $\mu \geq \ell'$ and for $p \in (1, \mu)$ with $\mu \leq \ell'$ if $2 < \ell < \infty$; and also for $\ell' < p < \infty$ with $\mu \geq \ell'$ and for $p \in (\frac{\mu\ell'}{\mu+\ell'-1}, \frac{\mu'\ell}{\mu'-\ell})$ with $\mu \leq \ell'$ if $1 < \ell \leq 2$.

In the view of the results in [8] regarding the boundedness of the parabolic Marcinkiewicz operator $\mathcal{G}_{\mho,h}^{(2)}$ and the results in [13] regarding the boundedness of the generalized parametric Marcinkiewicz operator $\mathcal{M}_{\mho,h}^{(\mu)}$, we have the following natural question: Is the integral operator $\mathcal{G}_{\mho,h}^{(\mu)}$ bounded under the same conditions on h and $\mho$ that was assumed in [13] ?

In this article, we shall answer the above question in the affirmative. In fact, we prove the following:

Theorem 1. *Let $\mho \in L^q\left(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}\right)$ for some $q \in (1,2]$ and $h \in \Delta_\ell(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\ell \in (1,2]$. Then there exists a real number $A_p > 0$ such that*

$$\left\| \mathcal{G}_{\mho,h}^{(\mu)}(g) \right\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} \leq A_{p,\mho,h} \left((q-1)(\ell-1) \right)^{-2/\mu} \|g\|_{\dot{F}_p^{\overrightarrow{0},\mu}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)}$$

for $p \in (\frac{\mu\ell'}{\mu+\ell'-1}, \frac{\mu'\ell}{\mu'-\ell})$ *if* $\mu \leq \ell'$, *and for* $\ell' < p < \infty$ *if* $\mu \geq \ell'$; *where* $A_{p,\mho,h} = A_p \|\mho\|_{L^q(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})} \|h\|_{\Delta_\ell(\mathbb{R}_+\times\mathbb{R}_+)}$ *and* A_p *is independent of* $\mho, h, q, \ell$

Theorem 2. *Let* $\mho \in L^q\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}\right)$ *with* $q \in (1,2]$ *and* $h \in \Delta_\ell(\mathbb{R}_+\times\mathbb{R}_+)$ *for some* $\ell \in (2,\infty)$. *Then*

$$\left\|\mathcal{G}^{(\mu)}_{\mho,h}(g)\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq A_{p,\mho,h}\left(\frac{\ell}{q-1}\right)^{2/\mu} \|f\|_{\dot{F}_p^{\vec{0},\mu}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}$$

for all $p \in (\ell',\infty)$ *if* $\mu \geq \ell'$ *and for all* $p \in (1,\mu)$ *if* $\mu \leq \ell'$.

By using the extrapolation argument in [11,14]) and the estimates in Theorems 1 - 2, we obtain the following results.

Theorem 3. *Assume that h is given as in Theorem 1.*
(i) If $\mho \in B_q^{(0,\frac{2}{\mu}-1)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$ *with* $q > 1$, *then the inequality*

$$\left\|\mathcal{G}^{(\mu)}_{\mho,h}(g)\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq A_p\left(\|\mho\|_{B_q^{(0,\frac{2}{\mu}-1)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}+1\right)\|h\|_{\Delta_\ell(\mathbb{R}_+\times\mathbb{R}_+)}\|g\|_{\dot{F}_p^{\vec{0},\mu}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}$$

holds for $\ell' < p < \infty$ *if* $\mu \geq \ell'$, *and for* $p \in (\frac{\mu\ell'}{\mu+\ell'-1}, \frac{\mu'\ell}{\mu'-\ell})$ *if* $\mu \leq \ell'$.
(ii) If $\mho \in L(\log L)^{2/\mu}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$, *then the inequality*

$$\left\|\mathcal{G}^{(\mu)}_{\mho,h}(g)\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq A_p\left(\|\mho\|_{L(logL)^{2/\mu}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}+1\right)\|h\|_{\Delta_\ell(\mathbb{R}_+\times\mathbb{R}_+)}\|g\|_{\dot{F}_p^{\vec{0},\mu}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}$$

holds for $\ell' < p < \infty$ *if* $\mu \geq \ell'$, *and for* $p \in (\frac{\mu\ell'}{\mu+\ell'-1}, \frac{\mu'\ell}{\mu'-\ell})$ *if* $\mu \leq \ell'$.

Theorem 4. *Suppose that* $\mho \in L(\log L)^{2/\mu}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}) \cup B_q^{(0,\frac{2}{\mu}-1)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$ *with* $q > 1$ *and* $h \in \Delta_\ell(\mathbb{R}_+\times\mathbb{R}_+)$ *with* $2 < \ell < \infty$. *Then the integral operator* $\mathcal{G}^{(\mu)}_{\mho,h}$ *is bounded on* $L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ *for* $p \in (\ell',\infty)$ *if* $\mu \geq \ell'$, *and for* $p \in (1,\mu)$ *if* $\mu \leq \ell'$.

Remarks

(i) For the special case $h \equiv 1$ and $\mu = 2$, the authors of [5] showed that $\mathcal{M}^{(2)}_{\mho,1}$ is bounded on $L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ for all $p \in (1,\infty)$ under the condition $\Omega \in L(\log L)(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$. Also, they found that this condition is the weakest possible condition so that the boundedness of $\mathcal{M}^{(2)}_{\mho,1}$ holds. On the other hand, the L^p $(1 < p < \infty)$ boundedness of $\mathcal{M}^{(2)}_{\mho,1}$ was proved in [6] if $\Omega \in B_q^{(0,0)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$ with $q > 1$. Also, the optimality of the condition $\Omega \in B_q^{(0,0)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$ is established. Therefore, the conditions on $\mho$ in Theorem 3 as well as Theorem 4 are the weakest known conditions in their respective classes for the case $\mu = 2$ and $h \equiv 1$.

(ii) In Theorem 4, when we consider the special case $h \equiv 1$, we get that $\mathcal{G}^{(\mu)}_{\mho,1}$ is bounded on $L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ for all $p \in (1,\infty)$ if $\mho \in L(\log L)^{2/\mu}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}) \cup B_q^{(0,\frac{2}{\mu}-1)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$, which improves the results in [7,10].

(iii) When $\mu = \ell'$ with $2 < \ell < \infty$, Theorem 4 gives the boundedness of $\mathcal{G}^{(\mu)}_{\mho,h}$ for all $p \in (1,\infty)$.

(iv) For the case $\mu = 2$ and $\ell \in (1,2]$, the range of p in Theorem 3 is better than the range obtained in Theorem 1.2 in [8] in which the authors proved the L^p boundedness of $\mathcal{G}^{(2)}_{\mho,h}$ only for $p \in (\frac{2\ell'}{\ell'-2}, \frac{2\ell}{2-\ell})$.

(v) For $s \in \{\kappa,\eta\}$ with $\beta_{s,1} = \beta_{s,2} = \cdots = \beta_{s,s} = 1$, our results are the same as that obtained in [13].

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Throughout the rest of the paper, the letter A stands for a positive constant which is independent of the essential variables and its value not necessary the same at each occurrence.

2. Auxiliary Lemmas

In this section we need to introduce some notations and establish some lemmas. For $\gamma \geq 2$, consider the family of measures $\{\sigma_{\mathcal{K}_{\mho,h},s,r} := \sigma_{s,r} : s, r \in \mathbb{R}_+\}$ and its concerning maximal operators σ_h^* and $M_{h,\gamma}$ on $\mathbb{R}^\kappa \times \mathbb{R}^\eta$ given by

$$\iint_{\mathbb{R}^\kappa \times \mathbb{R}^\eta} g\, d\sigma_{s,r} = \frac{1}{s^{\rho_1} r^{\rho_2}} \int_{1/2s \leq \tau_\kappa(v) \leq s} \int_{1/2r \leq \tau_\eta(\omega) \leq r} \mathcal{K}_{\mho,h}(v,\omega) g(v,\omega) dv d\omega,$$

$$\sigma_h^*(g)(v,\omega) = \sup_{s,r \in \mathbb{R}_+} ||\sigma_{s,r}| * g(v,\omega)|,$$

and

$$M_{h,\gamma}(g)(v,\omega) = \sup_{j,k \in \mathbb{Z}} \int_{\gamma^j}^{\gamma^{j+1}} \int_{\gamma^k}^{\gamma^{k+1}} ||\sigma_{s,r}| * g(v,\omega)| \frac{dsdr}{sr},$$

where $|\sigma_{s,r}|$ is defined in the same way as $\sigma_{s,r}$ except that $h\mho$ is replaced by $|h\mho|$.

We shall need the following two lemmas from [8].

Lemma 1. *Let $\mho \in L^q\left(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}\right)$ and $h \in \Delta_\ell\left(\mathbb{R}_+ \times \mathbb{R}_+\right)$ for some $q, \ell > 1$. Then there exists $A_{h,\mho} > 0$ such that*

$$\|\sigma_{s,r}\| \leq A_{h,\mho}, \tag{4}$$

$$\int_{\gamma^j}^{\gamma^{j+1}} \int_{\gamma^k}^{\gamma^{k+1}} |\hat{\sigma}_{s,r}(\zeta,\xi)|^2 \frac{dsdr}{sr} \leq A_{h,\mho}^2 \ln^2(\gamma) \left|D_{\gamma^k}\zeta\right|^{\pm\frac{2\delta}{n_1 \ln(\gamma)}} \left|D_{\gamma^j}\xi\right|^{\pm\frac{2\delta}{n_2 \ln(\gamma)}}, \tag{5}$$

where $\|\sigma_{s,r}\|$ is the total variation of $\sigma_{s,r}$, $0 < \delta < \min\{\frac{1}{2}, \frac{n_1}{2q'}, \frac{n_2}{2q'}, \frac{n_1}{\beta_\kappa}, \frac{n_2}{\beta_\eta}\}$ and n_1, n_2 denote the distinct numbers of $\{\beta_{\kappa,k}\}$, $\{\beta_{\eta,j}\}$, respectively.

Lemma 2. *Let $\mho \in L^1\left(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}\right)$ and $h \in \Delta_\ell\left(\mathbb{R}_+ \times \mathbb{R}_+\right)$ for some $\ell > 1$. Then we have that*

$$\|\sigma_h^*(g)\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} \leq \widetilde{A}_{p,h,\mho} \|g\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} \tag{6}$$

for all $p \in (\ell', \infty)$, where $\widetilde{A}_{p,h,\mho} = A_p \|h\|_{\Delta_\ell(\mathbb{R}_+ \times \mathbb{R}_+)} \|\mho\|_{L^1(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})}$.

By using Lemma 2, it is easy to show that

$$\|M_{h,\gamma}(g)\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} \leq \widetilde{A}_{p,h,\mho} \ln^2(\gamma) \|g\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} \tag{7}$$

for all $p \in (\ell', \infty)$.

Now we need to prove the following result:

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Lemma 3. *Let* $\mho \in L^q\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}\right)$, $h \in \Delta_\ell\left(\mathbb{R}_+\times\mathbb{R}_+\right)$ *with* $1<\ell,q\leq 2$ *and* $\gamma = 2^{\ell' q'}$. *Then for all* $p \in (\frac{\mu\ell'}{\mu+\ell'-1}, \frac{\mu'\ell}{\mu'-\ell})$ *with* $\mu \in (1,\ell']$, *we have*

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\gamma^j}^{\gamma^{j+1}}\int_{\gamma^k}^{\gamma^{k+1}}\left|\sigma_{s,r}*\mathcal{F}_{j,k}\right|^{\mu}\frac{dsdr}{sr}\right)^{1/\mu}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq A_{h,\mho}\left(\frac{1}{(q-1)(\ell-1)}\right)^{2/\mu}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{F}_{j,k}\right|^{\mu}\right)^{1/\mu}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)},$$

where $\{\mathcal{F}_{j,k}(\cdot,\cdot),\ j,k\in\mathbb{Z}\}$ *is any class of functions defined on* $\mathbb{R}^\kappa\times\mathbb{R}^\eta$.

Proof. Let us start with the case $p \in (\mu, \frac{\mu'\ell}{\mu'-\ell})$. It is clear that

$$\begin{aligned}\left|\sigma_{s,r}*\mathcal{F}_{j,k}(v,\omega)\right|^{\mu} &\leq A\,\|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}^{(\mu/\mu')}\,\|h\|_{\Delta_\ell(\mathbb{R}_+\times\mathbb{R}_+)}^{(\mu/\mu')}\int_{r/2}^{r}\int_{s/2}^{s}\iint_{\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}}|J_\kappa(v)|\,|J_\eta(\omega)|\\ &\times\left|\mathcal{F}_{j,k}(v-D_{\tau_\kappa}x,\omega-D_{\tau_\eta}y)\right|^{\mu}|\mho(x,y)|\,d\sigma(x)d\sigma(y)\,|h(\tau_\kappa,\tau_\eta)|^{\mu-\frac{\mu\ell}{\mu'}}\frac{d\tau_\kappa d\tau_\eta}{\tau_\kappa\tau_\eta}.\end{aligned} \tag{8}$$

By duality there exists a non-negative function $\varphi \in L^{(p/\mu)'}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ such that $\|\varphi\|_{L^{(p/\mu)'}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq 1$ and

$$\begin{aligned}&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\gamma^j}^{\gamma^{j+1}}\int_{\gamma^k}^{\gamma^{k+1}}\left|\sigma_{s,r}*\mathcal{F}_{j,k}\right|^{\mu}\frac{dsdr}{sr}\right)^{1/\mu}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}^{\mu}\\ &= \iint_{\mathbb{R}^\kappa\times\mathbb{R}^\eta}\sum_{j,k\in\mathbb{Z}}\int_{\gamma^j}^{\gamma^{j+1}}\int_{\gamma^k}^{\gamma^{k+1}}\left|\sigma_{s,r}*\mathcal{F}_{j,k}(v,\omega)\right|^{\mu}\frac{dsdr}{sr}\varphi(v,\omega)dvd\omega.\end{aligned} \tag{9}$$

Thus, by the last two inequalities amd Hölder's inequality, we obtain that

$$\begin{aligned}&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\gamma^j}^{\gamma^{j+1}}\int_{\gamma^k}^{\gamma^{k+1}}\left|\sigma_{s,r}*\mathcal{F}_{j,k}\right|^{\mu}\frac{dsdr}{sr}\right)^{1/\mu}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}^{\mu} \leq A\,\|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}^{(\mu/\mu')}\,\|h\|_{\Delta_1(\mathbb{R}_+\times\mathbb{R}_+)}^{(\mu/\mu')}\\ &\times\iint_{\mathbb{R}^\kappa\times\mathbb{R}^\eta}\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{F}_{j,k}(v,\omega)\right|^{\mu}\right)M_{|h|^{\mu-\frac{\mu\ell}{\mu'}},\gamma}(\overline{\varphi})(-v,-\omega)dvd\omega\\ &\leq A\,\|h\|_{\Delta_1(\mathbb{R}_+\times\mathbb{R}_+)}^{(\mu/\mu')}\,\|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}^{(\mu/\mu')}\left\|\sum_{j,k\in\mathbb{Z}}\left|\mathcal{F}_{j,k}\right|^{\mu}\right\|_{L^{(p/\mu)}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}\left\|M_{|h|^{\frac{\mu(\mu'-\ell)}{\mu'}},\gamma}(\overline{\varphi})\right\|_{L^{(p/\mu)'}(\mathbb{R}^n\times\mathbb{R}^m)},\end{aligned}$$

where $\overline{\varphi}(v,\omega)=\varphi(-v,-\omega)$. As $|h|^{\frac{\mu(\mu'-\ell)}{\mu'}}$ belongs to the space $\Delta_{\frac{\mu'\ell}{\mu(\mu'-\ell)}}(\mathbb{R}_+\times\mathbb{R}_+)$, then by employing (7), we obtain that

$$\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\gamma^j}^{\gamma^{j+1}}\int_{\gamma^k}^{\gamma^{k+1}}\left|\sigma_{s,r}*\mathcal{F}_{j,k}\right|^{\mu}\frac{dsdr}{sr}\right)^{1/\mu}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq A_{\mho,h}\ln^{2/\mu}(\gamma)\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{F}_{j,k}\right|^{\mu}\right)^{1/\mu}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \tag{10}$$

for all $p \in (\mu, \frac{\mu'\ell}{\mu'-\ell})$.

Let us consider the case $p = \mu$, by Hölder's inequality and (8), we get

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\gamma^j}^{\gamma^{j+1}} \int_{\gamma^k}^{\gamma^{k+1}} \left| \sigma_{s,r} * \mathcal{F}_{j,k} \right|^{\mu} \frac{dsdr}{sr} \right)^{1/\mu} \right\|^{\mu}_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} \leq A \left\| \mho \right\|^{(\mu/\mu')}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})} \left\| h \right\|^{(\mu/\mu')}_{\Delta_1(\mathbb{R}_+\times\mathbb{R}_+)}$$

$$\times \sum_{j,k\in\mathbb{Z}} \int_{\mathbb{R}^\kappa\times\mathbb{R}^\eta} \int_{\gamma^j}^{\gamma^{j+1}} \int_{\gamma^k}^{\gamma^{k+1}} \int_{r/2}^{r} \int_{s/2}^{s} \int_{\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}} \left| \mathcal{F}_{j,k}(v - D_{\tau_\kappa} x, \omega - D_{\tau_\eta} y) \right|^{\mu}$$

$$\times \left| \mho(x,y) \right| \left| h(\tau_\kappa, \tau_\eta) \right|^{\frac{\mu(\mu'-\ell)}{\mu'}} d\sigma(x) d\sigma(y) \frac{d\tau_\kappa d\tau_\eta}{\tau_\kappa \tau_\eta} \frac{dsdr}{sr} dv d\omega$$

$$\leq A \left(\frac{1}{(q-1)(\ell-1)} \right)^2 \left\| \mho \right\|^{(\mu/\mu')+1}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})} \left\| h \right\|^{(\mu/\mu')+1}_{\Delta_1(\mathbb{R}_+\times\mathbb{R}_+)} \int_{\mathbb{R}^\kappa\times\mathbb{R}^\eta} \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k}(v,\omega) \right|^{\mu} \right) dv d\omega. \quad (11)$$

Finally we prove the lemma for the case $p \in (\frac{\mu\ell'}{\mu+\ell'-1}, \mu)$. Let $\mathcal{L}$ be the linear operator defined on any function $\mathcal{F} = \mathcal{F}_{j,k}(x,y)$ by $\mathcal{L}(\mathcal{F}) = \sigma_{\gamma^k s, \gamma^j r} * \mathcal{F}_{j,k}(x,y)$. It is easy to see that

$$\left\| \left\| \left\| \mathcal{L}(\mathcal{F}) \right\|_{L^1([1,\gamma)\times[1,\gamma)), \frac{dsdr}{sr}} \right\|_{l^1(\mathbb{Z}\times\mathbb{Z})} \right\|_{L^1(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq A \ln^2(\gamma) \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k} \right| \right) \right\|_{L^1(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}. \quad (12)$$

Also, by the inequality (6) we get

$$\begin{aligned} \left\| \sup_{j,k\in\mathbb{Z}} \sup_{(s,r)\in[1,\gamma]\times[1,\gamma]} \left| \sigma_{\gamma^k s, \gamma^j r} * \mathcal{F}_{j,k} \right| \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} &\leq \left\| \sigma_h^* \left(\sup_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k} \right| \right) \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \\ &\leq A_{h,\mho} \left\| \sup_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k} \right| \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \end{aligned}$$

for all $p \in (\ell', \infty)$, which in turn implies that

$$\left\| \left\| \left\| \sigma_{\gamma^k s, \gamma^j r} * \mathcal{F}_{j,k} \right\|_{L^\infty([1,\gamma]\times[1,\gamma], \frac{dsdr}{sr})} \right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})} \right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq A_{h,\mho} \left\| \left\| \mathcal{F}_{j,k} \right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}. \quad (13)$$

Consequently, the proof of the lemma is finished in the case $p \in (\frac{\mu\ell'}{\mu+\ell'-1}, \mu)$ if we interpolate (12) with (13). □

Lemma 4. *Let $\mho$ and $\{\mathcal{F}_{j,k}(\cdot,\cdot),\ j,k \in \mathbb{Z}\}$ be given as in Lemma 3. Suppose that $h \in \Delta_\ell(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\ell \in (1,\infty)$. Then there exists a positive constant $A_{h,\mho}$ such that*

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\gamma^j}^{\gamma^{j+1}} \int_{\gamma^k}^{\gamma^{k+1}} \left| \sigma_{s,r} * \mathcal{F}_{j,k} \right|^{\mu} \frac{dsdr}{sr} \right)^{1/\mu} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq A_{h,\mho} \left(\frac{\ell}{q-1} \right)^{2/\mu} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k} \right|^{\mu} \right)^{1/\mu} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \quad (14)$$

for all $p \in (1,\mu)$ if $\mu \leq \ell'$ and $\gamma \geq 2$; and

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\gamma^j}^{\gamma^{j+1}} \int_{\gamma^k}^{\gamma^{k+1}} \left| \sigma_{s,r} * \mathcal{F}_{j,k} \right|^{\mu} \frac{dsdr}{sr} \right)^{1/\mu} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq A_{h,\mho} \left(\frac{\ell}{(q-1)(\ell-1)} \right)^{2/\mu} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k} \right|^{\mu} \right)^{1/\mu} \right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \quad (15)$$

for all $p \in (\ell', \infty)$ if $\mu \geq \ell'$.

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A proof of this Lemma can be constructed by following a similar argument as that employed in the proof of Lemma 3 and following similar argument as that used in the proofs of Theorems 4-5 in [13] (with minor modifications). We omit the details.

3. Proof of the main results

Proof of Theorem 1. Suppose that $\mho \in L^q\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}\right)$ and $h \in \Delta_\ell\left(\mathbb{R}_+\times\mathbb{R}_+\right)$ for some $q,\ell \in (1,2]$, and that $\mu > 1$. By Minkowski's inequality we get

$$\begin{aligned}
\mathcal{G}^{(\mu)}_{\mho,h}(g)(x,y) &= \Bigg(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\Bigg|\sum_{j,k=0}^{\infty}\frac{1}{s^{\rho_1}r^{\rho_2}}\int_{2^{-j-1}s<\rho_\tau(v)\le 2^{-j}s}\int_{2^{-k-1}r<\rho_\kappa(\omega)\le 2^{-k}r}\mathcal{K}_{\mho,h}(v,\omega)\\
&\times\ g(x-v,y-\omega)dvd\omega\Bigg|^{\mu}\frac{dsdr}{sr}\Bigg)^{1/\mu}\\
&\le \sum_{j,k=0}^{\infty}\Bigg(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\Bigg|\frac{1}{s^{\rho_1}r^{\rho_2}}\int_{2^{-j-1}s<\rho_\tau(v)\le 2^{-j}s}\int_{2^{-k-1}r<\rho_\kappa(\omega)\le 2^{-k}r}\mathcal{K}_{\mho,h}(v,\omega)\\
&\times\ g(x-v,y-\omega)dvd\omega\Bigg|^{\mu}\frac{dsdr}{sr}\Bigg)^{1/\mu}\\
&\le \frac{2^{a_1+b_1}}{(2^{a_1}-1)(2^{b_1}-1)}\Bigg(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\sigma_{s,r}*g(x,y)|^{\mu}\frac{dsdr}{sr}\Bigg)^{1/\mu}. \qquad (16)
\end{aligned}$$

Let $\gamma = 2^{\ell' q'}$. For $k\in\mathbb{Z}$, choose a collection of smooth functions $\{\psi_k\}$ defined on $\mathbb{R}^+$ satisfying the following properties:

$$\begin{aligned}
\psi_k &\subset [0,1],\quad \sum_{k\in\mathbb{Z}}\psi_k(s)=1,\\
\text{supp}\ (\psi_k) &\subseteq [\gamma^{-1-k},\gamma^{1-k}],\ \ and\ \left|\frac{d^t\psi_k(s)}{ds^t}\right|\le\frac{C_t}{s^t},
\end{aligned}$$

where C_t does not depend γ. For $(\zeta,\xi)\in\mathbb{R}^\kappa\times\mathbb{R}^\eta$, define the operators $(\widehat{\Psi_k}(\zeta)) = \psi_k(\tau_\kappa(\zeta))$ and $(\widehat{\Psi_j}(\xi)) = \psi_j(\tau_\eta(\xi))$. Hence, for any $g\in\mathcal{S}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$,

$$\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\sigma_{s,r}*g(x,y)|^{\mu}\frac{dsdr}{sr}\right)^{1/\mu}\le A\sum_{n,m\in\mathbb{Z}}\mathcal{H}_{n,m}(g)(x,y), \qquad (17)$$

where

$$\mathcal{H}_{n,m}(g)(x,y) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\mathcal{V}_{n,m}(g)(x,y,s,r)|^{\mu}\frac{dsdr}{sr}\right)^{1/\mu}$$

and

$$\mathcal{V}_{n,m}(g)(x,y,s,r) = \sum_{j,k\in\mathbb{Z}}\sigma_{s,r}*\left(\Psi_{k+m}\otimes\Psi_{j+n}\right)*g(x,y)\chi_{[\gamma^k,\gamma^{k+1})\times[\gamma^j,\gamma^{j+1})}(s,r).$$

Thus, to finish the proof of Theorem 1, it is enough to show that there exists a positive constant ε such that

$$\|\mathcal{H}_{n,m}(g)\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}\le A_{p,h,\mho}\left(\frac{1}{(q-1)(\ell-1)}\right)^{2/\mu}2^{-\frac{\varepsilon}{2}(|n|+|m|)}\|f\|_{\dot{F}_p^{\vec{0},\mu}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \qquad (18)$$

for all $\ell' < p < \infty$ with $\ell' \le \mu$ and for all $p\in\left(\frac{\mu\ell'}{\mu+\ell'-1},\frac{\mu'\ell}{\mu'-\ell}\right)$ with $\ell'\ge\mu$.

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First, we estimate the norm of $\mathcal{H}_{n,m}(g)$ for the case $p=\mu=2$. By using Fubini's theorem along with Plancherel's theorem and the inequality (5) we get

$$
\begin{aligned}
&\|\mathcal{H}_{n,m}(g)\|^2_{L^2(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}\\
\leq\;& \sum_{j,k\in\mathbb{Z}}\iint_{B_{n+j,m+k}}\left(\int_{\gamma^j}^{\gamma^{j+1}}\int_{\gamma^k}^{\gamma^{k+1}}|\hat{\sigma}_{s,r}(\zeta,\xi)|^2\frac{dsdr}{sr}\right)|\hat{g}(\zeta,\xi)|^2\,d\zeta d\xi\\
\leq\;& A_p\left(\frac{1}{(q-1)(\ell-1)}\right)^2 A^2_{h,\mho}\sum_{j,k\in\mathbb{Z}}\iint_{B_{n+j,m+k}}\left|D_{\gamma^k}\zeta\right|^{\pm\frac{2\delta}{n_1\ln(\gamma)}}\left|D_{\gamma^j}\xi\right|^{\pm\frac{2\delta}{n_2\ln(\gamma)}}|\hat{g}(\zeta,\xi)|^2\,d\zeta d\xi\\
\leq\;& A_p\left(\frac{1}{(q-1)(\ell-1)}\right)^2 2^{-\varepsilon(|n|+|m|)}A^2_{h,\mho}\sum_{j,k\in\mathbb{Z}}\iint_{B_{n+j,m+k}}|\hat{g}(\zeta,\xi)|^2\,d\zeta d\xi\\
\leq\;& A_p\left(\frac{1}{(q-1)(\ell-1)}\right)^2 2^{-\varepsilon(|n|+|m|)}A^2_{h,\mho}\|g\|^2_{L^2(\mathbb{R}^\kappa\times\mathbb{R}^\eta)},
\end{aligned}
\tag{19}
$$

where $B_{j,k}=\left\{(\zeta,\xi)\in\mathbb{R}^\kappa\times\mathbb{R}^\eta:(|\zeta|,|\xi|)\in[\gamma^{-1-k},\gamma^{1-k}]\times[\gamma^{-1-j},\gamma^{1-j}]\right\}$ and $\varepsilon\in(0,1)$.

Now, let us estimate the L^p-norm of $\mathcal{H}_{n,m}(g)$. By using Lemma 3, Lemma 2.3 in [10] and (15) together with Littlewood-Paley theory, we get

$$
\begin{aligned}
&\|\mathcal{H}_{m,n}(g)\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}\\
\leq\;& A\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\gamma^j}^{\gamma^{j+1}}\int_{\gamma^k}^{\gamma^{k+1}}|\sigma_{s,r}*(\Psi_{m+k}\otimes\Psi_{n+j})*g|^\mu\frac{dsdr}{sr}\right)^{1/\mu}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}\\
\leq\;& A_{h,\mho}\frac{1}{[(q-1)(\ell-1)]^{2/\mu}}\left\|\left(\sum_{j,k\in\mathbb{Z}}|(\Psi_{m+k}\otimes\Psi_{n+j})*g|^\mu\right)^{1/\mu}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}\\
\leq\;& A_p\frac{1}{[(q-1)(\ell-1)]^{2/\mu}}A_{h,\mho}\|g\|_{\dot{F}_p^{\vec{0},\mu}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}
\end{aligned}
\tag{20}
$$

for $\ell'<p<\infty$ with $\mu\geq\ell'$, and also for $p\in(\frac{\mu\ell'}{\mu+\ell'-1},\frac{\mu'\ell}{\mu'-\ell})$ with $\mu\leq\ell'$. Therefore, by interpolating (19) with (20), we obtain (18). The proof of Theorem 1 is complete.

Proof of Theorem 2. A proof can be constructed by following a similar approach as that used in the proof of Theorem 1 except that we employ Lemma 4 instead of Lemma 3. We omit the details.

References

1. Fabes, E.; Riviére, N. Singular integrals with mixed homogeneity. *Studia Math.* **1966**, 27, 19-38.
2. Ding, Y. L^2-boundedness of Marcinkiewicz integral with rough kernel. *Hokk. Math. J.* **1998**, 27, 105–115.
3. Choi, Y. Marcinkiewicz integrals with rough homogeneous kernel of degree zero in product domains. *J. Math. Anal. Appl.* **2001**, 261, 53–60.
4. Chen, J.; Fan, D.; Ying, Y. Rough Marcinkiewicz integrals with $L(\log L)^2$ kernels. *Adv. Math.* **2001**, 30, 179–181.
5. Al-Qassem, A.; Al-Salman, A.; Cheng, L.; Pan, Y. Marcinkiewicz integrals on product spaces. *Studia Math.* **2005**, 167, 227–234.
6. Al-Qassem, H. Rough Marcinkiewicz integral operators on product spaces. *Collec. Math.* **2005**, 36, 275–297.
7. Al-Salman, A. Parabolic Marcinkiewicz Integrals along Surfaces on Product Domains. *Acta Math. Sin. Engl. Ser.* **2011**, 27, 1–18.
8. Ali, M; Aldolat, M. Parabolic Marcinkiewicz integrals on product spaces and extrapolation. *Open Math.* **2016**, 14, 649–660

9. Fan, D.; Wu, H. On the generalized Marcinkiewicz integral operators with rough kernels. *Canad. Math. Bull.* **2011**, 54, 100–112.
10. Al-Qassem, H.; Cheng, L.; Pan, Y. Generalized Littlewood-Paley functions on product spaces. *Turk. J. Math.* **2021**, 45, 319–345.
11. Yano, S. Notes on Fourier analysis. XXIX. An extrapolation theorem, *J. Math. Soc. Japan* **1951**, 3, 296–305.
12. Jiang, Y.; Lu, S. A class of singular integral operators with rough kernel on product domains. *Hokkaido Math. J.* **1995**, 24, 1–7.
13. Ali, M.; Al-Qassem, H. Class of Rough Generalized Marcinkiewicz Integrals on Product Domains. *Symmetry.* **2023**, 15, 823.https://doi.org/10.3390/sym15040823.
14. Sato, S. Estimates for singular integrals and extrapolation, *Studia Math.* **2009**, 192 (3), 219–233.