# SHARP STABILITY FOR LSI

EMANUEL INDREI

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY

ABSTRACT. A fundamental tool in mathematical physics is the logarithmic Sobolev inequality. A quantitative version proven by Carlen with a remainder involving the Fourier-Wiener transform is equivalent to an entropic uncertainty principle more general than the Heisenberg uncertainty principle. In the stability, the remainder is in terms of an entropy, not a metric. Recently, a stability result for $H^1$ was obtained by Dolbeault, Esteban, Figalli, Frank, and Loss in terms of an $L^p$ norm. Afterwards, Brigati, Dolbeault, and Simonov discussed the stability problem involving a stronger norm. A full characterization with a necessary and sufficient condition to have $H^1$ convergence is identified in this paper. Moreover, an explicit $H^1$ bound via a moment assumption is shown. Also, the $L^p$ stability of Dolbeault, Esteban, Figalli, Frank, and Loss is proven to be sharp.

## 1. Introduction

The LSI appears in various branches of statistical mechanics, quantum field theory, and mathematical statistics. There exist many formulations of the classical Gaussian logarithmic Sobolev inequality which states that for smooth, positive, normalized functions

$$\int_{\mathbb{R}^n} f(x)\log f(x)d\gamma(x) \leq \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{f(x)}d\gamma(x), \tag{1.1}$$

where $d\gamma := (2\pi)^{-n/2}e^{-|x|^2/2}dx$ and is equivalent to Nelson's hypercontractive inequality. The integral term on the right-hand-side of the inequality is known as the Fisher information and is often denoted by $I(f)$ whereas the left-hand-side is the entropy and represented by $Ent(f)$. There are several proofs utilizing the central limit theorem, Ornstein–Uhlenbeck semigroup, Prekopka-Leindler inequality, optimal transport theory, and harmonic analysis. The historical evolution started with Stam [31] and Federbush [15]. Gross obtained the LSI in 1975 [18] utilizing probabilistic methods and Cordero-Erausquin discovered a very elegant and simple proof via optimal transport theory in 2002 [10].

An important application is via the stabilization of the Glauber-Langevin dynamic models for the Ising model [18, 30]. In addition, LSI appeared in Perelman's proof of

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*Contact Address:* 150 N. University Street, West Lafayette, IN 47907-2067, USA.

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the Poincare conjecture and also plays a fundamental role in the calculus of variations and partial differential equations [29].

Although simple-to-state, the LSI is delicate: a simple computation shows that equality holds if $f(x) = e^{a\cdot x+b_a}$; however, that these are the only functions achieving equality was solved by Eric Carlen [8] in two ways: first, if $g \in L^p(\mathbb{R}^{2n})$ is a product function in $(x, y)$ and $\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right)$, then $g$ as well as its factors are all Gaussian functions; subsequently, he proved a Minkowski-type inequality and derived the strict superadditivity of the Fisher information which combined with the factorization theorem.

The other method is based on the Beckner-Hirschman entropic uncertainty principle: let

$$U : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, dm)$$

be defined by $Uh(x) = 2^{-\frac{n}{4}}e^{\pi|x|^2}h(x)$ and let the Fourier transform be defined by

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int e^{-2\pi i\xi\cdot x}f(x)dx.$$

With this, the Fourier–Wiener transform is $\mathcal{W} = U\mathcal{F}U^*$, where $U^*$ is the adjoint of $U$. Supposing $dm = 2^{\frac{n}{2}}e^{-2\pi|x|^2}dx$ and $f \in L^2(dm)$ normalized, Carlen showed that the entropic uncertainty principle is equivalent to

(1.2)
$$\mathrm{Ent}_{dm}(|\mathcal{W}f|^2) := \int |\mathcal{W}f|^2 \log|\mathcal{W}f|^2 dm \leq \frac{1}{2\pi}\int |\nabla f|^2 dm - \int |f|^2\log|f|^2 dm =: \delta_c(f),$$

and if $f \geq 0$, $\mathrm{Ent}_{dm}(|\mathcal{W}f|^2) = 0$ only if $f_a(x) = e^{2\pi(a\cdot x-\frac{|a|^2}{2})}$ where $a \in \mathbb{R}^n$.

The uncertainty principle, conjectured by Hirschman [19], eventually was proven by Beckner [2] and states that the sum of the entropies of a function and its Fourier transform is bounded below by $n(1 - \log 2)$. A rescaling of (1.2) improves (1.1). Moreover, Carlen's quantitative LSI in the previously-stated form improves upon Heisenberg's uncertainty principle via the Beckner-Hirschman inequality.

In the quantitative inequality, the remainder is via an entropy, not metric. The first metric result was shown by myself and Marcon [24]: for $\epsilon > 0$ and $M > 0$, consider the family of functions

$$\mathcal{F}(\epsilon, M) := \{e^{-h} : (-1+\epsilon) \leq D^2h \leq M\}$$

(where the inequalities are in the sense of the constants times the identity matrix); there exists $C = C(\epsilon, M) > 0$ so that for all $f \in \mathcal{F}(\epsilon, M)$ with unit mass and zero barycenter,

$$W_2(fd\gamma, d\gamma) \leq C\delta(f)^{\frac{1}{2}}.$$

Also, we constructed examples which show that the $\frac{1}{2}$-exponent is sharp and obtained new bounds on the entropy which show that $W_2$-stability is not true without extra assumptions.

The following quantitative LSI was proven in 2022 [12]: there exists a dimensionless $\kappa > 0$ such that assuming $u \in H^1(e^{-\pi|x|^2}dx)$,

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(1.3)

$$\pi\delta_*(u) := \int |\nabla u|^2 e^{-\pi|x|^2}dx - \pi\int |u|^2 \ln\Big(\frac{|u|^2}{||u||_{L^2}^2}\Big)e^{-\pi|x|^2}dx \geq \kappa \inf_{a,c}\int |u - ce^{a\cdot x}|^2 e^{-\pi|x|^2}dx.$$

In [7, p. 5] the stability problem relative to a stronger norm is stated:

"a stability on the Gaussian logarithmic Sobolev inequality is shown in [23], although the distance is measured only by an $L^2(\mathbb{R}^n, d\gamma)$ norm. Whether a stronger estimate can be obtained in the limiting case $p = 2$, eventually under some restriction, is therefore so far an open question."

The authors suspected that unlike the $L^2$-stability (1.3), a quantitative stability in a stronger norm may not hold for all of $H^1$. This is actually the case. The optimal condition to have the $H^1$ convergence is identified in Theorem 1.1. Moreover, there exists an explicit $H^1$ bound via a moment assumption; the moment assumption is sharp: the inequality is not true if $\alpha = \infty$. Also, (1.3) is sharp via Theorem 1.2.

**Theorem 1.1.** *1. Let $\{u_k\}$ be normalized and centered in $L^2(e^{-\pi|x|^2}dx)$ and suppose $\delta_*(u_k) \to 0$ as $k \to \infty$, then*

$$|u_k| \to 1$$

*in $H^1(e^{-\pi|x|^2}dx)$ if and only if*

$$m_2(u_k) := \int |x|^2|u_k(x)|^2 e^{-\pi|x|^2}dx \to \int |x|^2 e^{-\pi|x|^2}dx = m_2(1).$$

*2. If $u$ is normalized and centered in $L^2(e^{-\pi|x|^2}dx)$ &*

$$m_4(u) := \int |x|^4|u(x)|^2 e^{-\pi|x|^2}dx \leq \alpha < \infty,$$

*then*

$$\big|\big||u| - 1\big|\big|_{H^1(e^{-\pi|x|^2}dx)} \leq a_\alpha\Big(\delta_*^{\frac{1}{2}}(u) + \delta_*(u)\Big)^{\frac{1}{2}},$$

*$\infty > a_\alpha > 0$.*

*3. There are densities $\{u_k\}$ normalized and centered in $L^2(e^{-\pi|x|^2}dx)$, $\delta_*(u_k) \to 0$, and*

$$||u_k||_{H^1(e^{-\pi|x|^2}dx)} \to \infty.$$

**Theorem 1.2.** (1.3) *has the optimal rate.*

A simple version of (1.3) appears in the next lemma. Note that thanks to this reduction, one may without loss of generality assume the functions to be centered and normalized in Theorem 1.1.

**Lemma 1.3.** (1.3) *is equivalent to*

$$\int |\nabla w|^2 e^{-\pi|x|^2}dx - \pi\int |w|^2 \ln|w|^2 e^{-\pi|x|^2}dx \geq \kappa\int \big|w - 1\big|^2 e^{-\pi|x|^2}dx$$

*in the space of non-negative functions which satisfy*

$$||w||_{L^2(e^{-\pi|x|^2}dx)} = 1$$

$$\int x|w|^2 e^{-\pi|x|^2} dx = 0.$$

In particular, a completely equivalent version of (1.3) with a moment assumption and modulus $\omega$ was already proven in [23] utilizing a combination of optimal transport theory and Fourier analysis. Observe also that the non-negativity assumption appeared in Carlen's proof of the equality cases [8]. Suppose without loss of generality that

$$||u||_{L^2(e^{-\pi|x|^2}dx)} = 1$$

$$\int x|u|^2 e^{-\pi|x|^2} dx = 0.$$

Now set $dm = 2^{\frac{n}{2}} e^{-2\pi|x|^2} dx$, $d\gamma = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx$,

$$w(x) = u(\sqrt{2}x)$$

$$f(x) = |u|^2\left(\frac{x}{\sqrt{2\pi}}\right)$$

& observe

$$\int |w|^2 dm = \int |u|^2 e^{-\pi|x|^2} dx = \int f d\gamma$$

$$\int x f d\gamma = 0 = \int x|w|^2 dm.$$

Suppose

$$\int |x|^2 |u|^2 e^{-\pi|x|^2} dx \leq M_\alpha,$$

$M_\alpha \geq \frac{n}{2\pi}$. Note that $|\nabla|w|| = |\nabla w|$ a.e., therefore assume $w \geq 0$. An application of [23, Corollary 1.21] then implies that there exists a modulus $\omega$ such that

$$\int |w-1|^2 dm \leq a\omega(\delta_c(w)) = a\omega(\delta_*(u))$$

where $a = a(M_\alpha) > 0$,

$$\delta_c(w) := \frac{1}{2\pi} \int |\nabla w|^2 dm - \int |w|^2 \ln |w|^2 dm.$$

Thanks to

$$\int |w-1|^2 dm = \int |u-1|^2 e^{-\pi|x|^2} dx,$$

$$\omega\Big(\int |\nabla u|^2 e^{-\pi|x|^2} dx - \pi \int |u|^2 \ln |u|^2 e^{-\pi|x|^2} dx\Big) \geq \frac{1}{\bar{a}} \int |u-1|^2 e^{-\pi|x|^2} dx,$$

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$\bar{a} > 0$. Moreover if $n = 1$,

$$\delta(f) := \frac{1}{2}\mathrm{I}(f) - \mathrm{H}(f) = \frac{1}{2}\int \frac{|\nabla f|^2}{f} d\gamma - \int f \ln f d\gamma,$$

[23, Theorem 1.1] yields

$$\int |f-1| d\gamma \leq \overline{a}_1 \delta^{\frac{1}{4}}(f),$$

with $\overline{a}_1 = \overline{a}_1(M_\alpha) > 0$. Therefore assuming $u \geq 0$,

$$\begin{aligned}\int |u-1|^2 e^{-\pi|x|^2} dx &= \int |\sqrt{f}-1|^2 d\gamma \\ &\leq \int |f-1| d\gamma \\ &\leq \overline{a}_1 \delta^{\frac{1}{4}}(f) = \overline{a}_1 \delta_*^{\frac{1}{4}}(u).\end{aligned}$$

In addition, higher dimensional quantitative inequalities that also included an explicit modulus appeared in [14, 23, 24].

The optimal inequality via Theorem 1.2 is

$$||u-1||_{L^2(e^{-\pi|x|^2}dx)} \lesssim \delta_*^{\frac{1}{2}}(u),$$

and the more general stability

$$||\nabla u||_{L^2(e^{-\pi|x|^2}dx)} \leq a_1 \delta_*^{\frac{1}{2}}(u) \tag{1.4}$$

with the sharp exponent $\frac{1}{2}$ was proved in [24] for probability measures which are absolutely continuous with respect to the Gaussian measure $d\gamma$ and the density $f(x) = |u|^2(\frac{x}{\sqrt{2\pi}})$ satisfies a log-$C^{1,1}$ assumption. This was achieved via optimal transport theory [24, Theorem 1.1, Remark 4.3]: with the assumptions, there exists $\alpha \in (0, \frac{1}{2})$ such that

$$\int f \ln f d\gamma \leq \alpha \int \frac{|\nabla f|^2}{f} d\gamma.$$

Hence

$$\delta(f) = \frac{1}{2}\int \frac{|\nabla f|^2}{f} d\gamma - \int f \ln f d\gamma \geq (\frac{1}{2} - \alpha)\int \frac{|\nabla f|^2}{f} d\gamma,$$

thus proving (1.4) with a simple change of variables. A surprising Ornstein-Uhlenbeck semigroup proof enables the explicit calculation of the sharp $a_1$ with a Poincaré assumption on $f d\gamma$, $f : \mathbb{R}^n \to \mathbb{R}_+$ [14]. Observe that the Poincaré assumption implies the log-$C^{1,1}$ assumption where one of the inequalities for the log-$C^{1,1}$ assumption is precluded. In particular, it is one of the rare inequalities that highlights the sharp exponent and constant of proportionality. However, the $4^{th}$ moment assumption in Theorem 1.1 includes the Poincaré assumption and approximates the optimal moment assumption. There exists a sequence $\{u_k\}$ where

$$\delta_*(u_k) \to 0$$

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$$\int |x|^2 |u_k(x)|^2 e^{-\pi|x|^2} dx \to \infty.$$

A more general stability inequality for probability measures which are absolutely continuous with respect to the Gaussian measure was obtained with a combination of a Wasserstein metric and entropy. The techniques in [14, 16, 23, 24] involve optimal transport, semigroup theory, Fourier analysis, and probability. The recent proof of (1.3) in [12] is a fundamental achievement. One interesting feature is the lack of additional assumptions for (1.3) via the Bianchi-Egnell method. Since the logarithmic Sobolev inequality has appeared in different fields, e.g. optimal transport theory, probability, statistical mechanics, quantum field theory, Riemannian geometry, thermodynamics, and information theory, a large collection of articles recently investigated various stability formulations of similar inequalities: see for instance [1, 3–6, 8, 9, 11, 13, 14, 16, 17, 20–22, 24–28].

## 2. Proofs

*Proof of Lemma 1.3.* Assume

$$||u||_{L^2(e^{-\pi|x|^2}dx)} = 1$$

$$\int x|u|^2 e^{-\pi|x|^2} dx = 0;$$

set

$$u = ce^{a\cdot x};$$

then note that $a = 0$, $c = 1$ (via $u \geq 0$). In particular, $u = 1$ is the only normalized & centered minimizer. Also, assuming the analog

$$\int |\nabla w|^2 e^{-\pi|x|^2} dx - \pi \int |w|^2 \ln |w|^2 e^{-\pi|x|^2} dx \geq \kappa \int |w-1|^2 e^{-\pi|x|^2} dx, \tag{2.1}$$

if $w \geq 0$,

$$||w||_{L^2(e^{-\pi|x|^2}dx)} = 1,$$

$$\int x|w|^2 e^{-\pi|x|^2} dx = 0,$$

one also obtains (1.3): let

$$||u|| = ||u||_{L^2(e^{-\pi|x|^2}dx)}$$

$$\alpha := \int x|u|^2 e^{-\pi|x|^2} dx$$

$$w(x) := \frac{u(x + \frac{\alpha}{||u||^2}) e^{-\frac{\pi}{||u||^2}(\alpha\cdot x + \frac{|\alpha|^2}{2||u||^2})}}{||u||}$$

& observe

$$||w|| = ||w||_{L^2(e^{-\pi|x|^2}dx)} = 1$$

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$$\int x|w|^2 e^{-\pi|x|^2}dx = 0$$

$$\begin{aligned}
\int |w-1|^2 e^{-\pi|x|^2}dx &= \frac{1}{||u||^2}\int \Big|u(x+\frac{\alpha}{||u||^2})e^{-(\frac{\pi\alpha}{||u||^2}\cdot x+\frac{\pi|\alpha|^2}{2||u||^4})} - ||u||\Big|^2 e^{-\pi|x|^2}dx \\
&= \frac{1}{||u||^2}\int \Big|u(x+\frac{\alpha}{||u||^2}) - e^{(\frac{\pi\alpha}{||u||^2}\cdot x+\frac{\pi|\alpha|^2}{2||u||^4}+\ln||u||)}\Big|^2 e^{-2(\frac{\pi\alpha}{||u||^2}\cdot x+\frac{\pi|\alpha|^2}{2||u||^4})}e^{-\pi|x|^2}dx \\
&= \frac{1}{||u||^2}\int \Big|u(x+\frac{\alpha}{||u||^2}) - e^{(\frac{\pi\alpha}{||u||^2}\cdot x+\frac{\pi|\alpha|^2}{2||u||^4}+\ln||u||)}\Big|^2 e^{-\pi\left|x+\frac{\alpha}{||u||^2}\right|^2}dx \\
&= \frac{1}{||u||^2}\int \Big|u(x) - e^{(\frac{\pi\alpha}{||u||^2}\cdot x-\frac{\pi|\alpha|^2}{2||u||^4}+\ln||u||)}\Big|^2 e^{-\pi|x|^2}dx. \qquad (2.2)
\end{aligned}$$

(2.3)

$$\begin{aligned}
&\int |\nabla w|^2 e^{-\pi|x|^2}dx \\
&= \frac{1}{||u||^2}\int \Big|\nabla u(x+\frac{\alpha}{||u||^2})e^{-\frac{\pi}{||u||^2}(\alpha\cdot x+\frac{|\alpha|^2}{2||u||^2})} - \frac{\pi\alpha}{||u||^2}u(x+\frac{\alpha}{||u||^2})e^{-\frac{\pi}{||u||^2}(\alpha\cdot x+\frac{|\alpha|^2}{2||u||^2})}\Big|^2 e^{-\pi|x|^2}dx \\
&= \frac{1}{||u||^2}\Big(\int \Big|\nabla u(x+\frac{\alpha}{||u||^2})\Big|^2 e^{-\pi\left|x+\frac{\alpha}{||u||^2}\right|^2}dx - \Big\langle\frac{2\pi\alpha}{||u||^2}, \int \nabla u(x+\frac{\alpha}{||u||^2})u(x+\frac{\alpha}{||u||^2})e^{-\pi\left|x+\frac{\alpha}{||u||^2}\right|^2}dx\Big\rangle \\
&+ \frac{\pi^2|\alpha|^2}{||u||^4}\int \Big|u(x+\frac{\alpha}{||u||^2})\Big|^2 e^{-\pi\left|x+\frac{\alpha}{||u||^2}\right|^2}dx\Big) \\
&= \frac{1}{||u||^2}\Big(\int |\nabla u(x)|^2 e^{-\pi|x|^2}dx - \Big\langle\frac{2\pi\alpha}{||u||^2}, \int \nabla u(x)u(x)e^{-\pi|x|^2}dx\Big\rangle + \frac{\pi^2|\alpha|^2}{||u||^4}\int |u(x)|^2 e^{-\pi|x|^2}dx\Big).
\end{aligned}$$

Moreover,

$$\int \nabla u(x)u(x)e^{-\pi|x|^2}dx = -\int \nabla u(x)u(x)e^{-\pi|x|^2}dx + 2\pi\int x|u(x)|^2 e^{-\pi|x|^2}dx$$

yields

$$\int \nabla u(x)u(x)e^{-\pi|x|^2}dx = \pi\int x|u(x)|^2 e^{-\pi|x|^2}dx = \pi\alpha.$$

Observe

$$\Big\langle\frac{2\pi\alpha}{||u||^2}, \int \nabla u(x)u(x)e^{-\pi|x|^2}dx\Big\rangle = \frac{2\pi^2|\alpha|^2}{||u||^2},$$

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therefore (2.3) implies

$$\frac{1}{||u||^2}\Big(\int |\nabla u(x)|^2 e^{-\pi|x|^2}dx - \Big\langle \frac{2\pi\alpha}{||u||^2}, \int \nabla u(x)u(x)e^{-\pi|x|^2}dx\Big\rangle + \frac{\pi^2|\alpha|^2}{||u||^4}\int |u(x)|^2 e^{-\pi|x|^2}dx\Big)$$
$$= \frac{1}{||u||^2}\Big(\int |\nabla u(x)|^2 e^{-\pi|x|^2}dx - \frac{\pi^2|\alpha|^2}{||u||^2}\Big) = \int |\nabla w|^2 e^{-\pi|x|^2}dx. \tag{2.4}$$

Analogously,

$$\int |w|^2 \ln|w|^2 e^{-\pi|x|^2}dx = \frac{1}{||u||^2}\Big(\int \Big|u(x+\frac{\alpha}{||u||^2})\Big|^2 \ln\Big(\frac{|u(x+\frac{\alpha}{||u||^2})|^2}{||u||^2}\Big)e^{-\pi\left|x+\frac{\alpha}{||u||^2}\right|^2}dx$$
$$-\Big\langle \frac{2\pi\alpha}{||u||^2}, \int x\Big|u(x+\frac{\alpha}{||u||^2})\Big|^2 e^{-\pi\left|x+\frac{\alpha}{||u||^2}\right|^2}dx\Big\rangle - \frac{\pi|\alpha|^2}{||u||^2}\Big);$$

$$\int x\Big|u(x+\frac{\alpha}{||u||^2})\Big|^2 e^{-\pi\left|x+\frac{\alpha}{||u||^2}\right|^2}dx = \int \Big(x-\frac{\alpha}{||u||^2}\Big)|u(x)|^2 e^{-\pi|x|^2}dx = 0;$$

thus

$$\int |w|^2 \ln|w|^2 e^{-\pi|x|^2}dx = \frac{1}{||u||^2}\Big(\int |u(x)|^2 \ln\Big(\frac{|u(x)|^2}{||u||^2}\Big)e^{-\pi|x|^2}dx - \frac{\pi|\alpha|^2}{||u||^2}\Big). \tag{2.5}$$

Observe that (2.5) and (2.4) imply

$$\int |\nabla w|^2 e^{-\pi|x|^2}dx - \pi\int |w|^2\ln|w|^2 e^{-\pi|x|^2}dx = \frac{1}{||u||^2}\Big(\int |\nabla u|^2 e^{-\pi|x|^2}dx - \pi\int |u|^2\ln\Big(\frac{|u|^2}{||u||^2}\Big)e^{-\pi|x|^2}dx\Big) \tag{2.6}$$

and therefore (2.6) and (2.2) combine with (2.1):

$$\int |\nabla u|^2 e^{-\pi|x|^2}dx - \pi\int |u|^2 \ln\Big(\frac{|u|^2}{||u||^2_{L^2}}\Big)e^{-\pi|x|^2}dx \geq \kappa \inf_{a,c}\int |u-ce^{a\cdot x}|^2 e^{-\pi|x|^2}dx.$$

□

*Proof of Theorem 1.1.* 1. Observe that with $dm = 2^{\frac{n}{2}}e^{-2\pi|x|^2}dx$ and $f \in L^2(dm)$ normalized, setting

$$f = u(\sqrt{2}x),$$

$$\int |f|^2 dm = \int |u|^2 e^{-\pi|x|^2}dx = 1$$

&

$$\frac{1}{2\pi}\int |\nabla f|^2 dm - \int |f|^2\ln|f|^2 dm = \frac{1}{\pi}\int |\nabla u|^2 e^{-\pi|x|^2}dx - \int |u|^2\ln|u|^2 e^{-\pi|x|^2}dx.$$

Thus [23, Theorem 1.17] implies

$$
\begin{aligned}
(2.7)\qquad & \pi\int |x|^2e^{-\pi|x|^2}dx-\pi\int |x|^2|u(x)|^2e^{-\pi|x|^2}dx+\frac{1}{\pi}\int |\nabla u(x)|^2e^{-\pi|x|^2}dx\\
&\le \sqrt{2n}\Big(\frac{1}{\pi}\int |\nabla u|^2e^{-\pi|x|^2}dx-\int |u|^2\ln|u|^2e^{-\pi|x|^2}dx\Big)^{\frac12}\\
&+\Big(\frac{1}{\pi}\int |\nabla u|^2e^{-\pi|x|^2}dx-\int |u|^2\ln|u|^2e^{-\pi|x|^2}dx\Big).
\end{aligned}
$$

Note that supposing

$$\Big(\int |x|^2e^{-\pi|x|^2}dx-\int |x|^2|u_k(x)|^2e^{-\pi|x|^2}dx\Big)\to 0,$$

& $\delta_*(u_k)\to 0$, one then obtains thanks to (2.7) that

$$\int |\nabla u_k(x)|^2e^{-\pi|x|^2}dx\to 0$$

and this implies $H^1(e^{-\pi|x|^2}dx)$ convergence. Conversely, assuming $\sqrt{f}\in W^{1,2}(\mathbb{R}^n,d\gamma)$ where $d\gamma=(2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}dx$ denotes the Gaussian measure, the end of the proof of [23, Proposition C.1] implies

$$\frac12\int \frac{|\nabla f|^2}{f}d\gamma-\int f\ln f d\gamma\ge \frac{1}{4n}\Big(2\int f\ln f d\gamma+(m_2(\gamma)-m_2(fd\gamma))\Big)^2.$$

Thus set

$$f_k=|u_k|^2\big(\frac{x}{\sqrt{2\pi}}\big).$$

Then

$$\int f_ke^{-\frac{|x|^2}{2}}(2\pi)^{-\frac{n}{2}}dx=\int |u_k|^2e^{-\pi|x|^2}dx=1,$$

$$\frac12\int \frac{|\nabla f_k|^2}{f_k}d\gamma-\int f_k\ln f_k d\gamma=\frac{1}{\pi}\int |\nabla u_k|^2e^{-\pi|x|^2}dx-\int |u_k|^2\ln|u_k|^2e^{-\pi|x|^2}dx.$$

In particular

$$
\begin{aligned}
(2.8)\qquad & \frac{1}{4n}\Big(2\int |u_k|^2\ln|u_k|^2e^{-\pi|x|^2}dx+2\pi\int |x|^2e^{-\pi|x|^2}dx-2\pi\int |x|^2|u_k|^2e^{-\pi|x|^2}dx\Big)^2\\
&\le \frac{1}{\pi}\int |\nabla u_k|^2e^{-\pi|x|^2}dx-\int |u_k|^2\ln|u_k|^2e^{-\pi|x|^2}dx.
\end{aligned}
$$

Observe via $H^1(e^{-\pi|x|^2}dx)$ convergence and the LSI that

$$\int |u_k|^2\ln|u_k|^2e^{-\pi|x|^2}dx\to 0,$$

thus thanks to

$$\delta_*(u_k)=\Big(\frac{1}{\pi}\int |\nabla u_k|^2e^{-\pi|x|^2}dx-\int |u_k|^2\ln|u_k|^2e^{-\pi|x|^2}dx\Big)\to 0,$$

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(2.8) implies
$$\Big(\int |x|^2 e^{-\pi|x|^2}dx - \int |x|^2|u_k|^2 e^{-\pi|x|^2}dx\Big) \to 0.$$
2.
$$\int |x|^4|u(x)|^2 e^{-\pi|x|^2}dx \leq A$$
implies
$$\begin{aligned}
&\Big|\pi \int |x|^2 e^{-\pi|x|^2}dx - \pi\int |x|^2|u(x)|^2 e^{-\pi|x|^2}dx\Big| = \pi\Big|\int |x|^2[1-|u|][1+|u|]e^{-\pi|x|^2}dx\Big|\\
&\leq \pi\Big[\int |x|^4|1+|u||^2 e^{-\pi|x|^2}dx\Big]^{1/2}\Big[\int |1-|u||^2 e^{-\pi|x|^2}dx\Big]^{1/2}\\
&\leq \pi\Big[2(A+\int |x|^4 e^{-\pi|x|^2}dx)\Big]^{1/2}\Big[\int |1-|u||^2 e^{-\pi|x|^2}dx\Big]^{1/2}.
\end{aligned}$$

In particular [23, Theorem 1.17], Lemma 1.3 ($|u|\geq 0$, $|\nabla|u|| = |\nabla u|$ a.e.), and [12, Theorem 2] imply

$$\begin{aligned}
&\frac{1}{\pi}\int |\nabla u(x)|^2 e^{-\pi|x|^2}dx\\
&\leq \Big|\pi \int |x|^2 e^{-\pi|x|^2}dx - \pi\int |x|^2|u(x)|^2 e^{-\pi|x|^2}dx\Big| + \sqrt{2n}\Big(\frac{1}{\pi}\int |\nabla u|^2 e^{-\pi|x|^2}dx - \int |u|^2\ln|u|^2 e^{-\pi|x|^2}dx\Big)^{\frac12}\\
&+\Big(\frac{1}{\pi}\int |\nabla u|^2 e^{-\pi|x|^2}dx - \int |u|^2\ln|u|^2 e^{-\pi|x|^2}dx\Big)\\
&\leq \pi\Big[2(A+\int |x|^4 e^{-\pi|x|^2}dx)\Big]^{1/2}\Big[\int |1-|u||^2 e^{-\pi|x|^2}dx\Big]^{1/2}+\\
&\sqrt{2n}\Big(\frac{1}{\pi}\int |\nabla u|^2 e^{-\pi|x|^2}dx - \int |u|^2\ln|u|^2 e^{-\pi|x|^2}dx\Big)^{\frac12} + \Big(\frac{1}{\pi}\int |\nabla u|^2 e^{-\pi|x|^2}dx - \int |u|^2\ln|u|^2 e^{-\pi|x|^2}dx\Big)\\
&\leq \pi\Big[2(A+\int |x|^4 e^{-\pi|x|^2}dx)\Big]^{1/2}\sqrt{\frac{\pi}{\kappa}}\Big[\frac{1}{\pi}\int |\nabla u|^2 e^{-\pi|x|^2}dx - \int |u|^2\ln|u|^2 e^{-\pi|x|^2}dx\Big]^{1/2}+\\
&\sqrt{2n}\Big(\frac{1}{\pi}\int |\nabla u|^2 e^{-\pi|x|^2}dx - \int |u|^2\ln|u|^2 e^{-\pi|x|^2}dx\Big)^{\frac12} + \Big(\frac{1}{\pi}\int |\nabla u|^2 e^{-\pi|x|^2}dx - \int |u|^2\ln|u|^2 e^{-\pi|x|^2}dx\Big).
\end{aligned}$$

3. Suppose $fd\gamma$ is a probability measure & let $T=\nabla\Phi$ be the Brenier map between $fd\gamma$ and $d\gamma$. Then it follows from the proof of LSI via optimal transport [10] that
$$\int |T(x)-x+\nabla \ln f|^2 f d\gamma \leq 2\delta(f).$$
The argument is as follows: define $\theta := \Phi - \frac12|x|^2$. Note that
$$f(x)e^{-|x|^2/2} = \det(I+D^2\theta(x))e^{-|x+\nabla\theta(x)|^2/2}.$$
Next, taking the logarithm and then integrating:

$$\begin{aligned}\int f\ln f d\gamma &\leq \int f\Big[\Delta\theta - x\cdot\nabla\theta\Big]d\gamma - \frac{1}{2}\int |\nabla\theta|^2 f d\gamma\\ &= -\int \nabla\theta\cdot\nabla f d\gamma - \frac{1}{2}\int |\nabla\theta|^2 f d\gamma\\ &= -\frac{1}{2}\int \Big|\nabla\theta + \frac{\nabla f}{f}\Big|^2 f d\gamma + \frac{1}{2}\int \frac{|\nabla f|^2}{f}d\gamma.\end{aligned}$$

Therefore

$$\frac{1}{2}\int \frac{|\nabla f|^2}{f}d\gamma - \int f\ln f d\gamma \geq \frac{1}{2}\int \Big|T - x + \nabla \ln f\Big|^2 f d\gamma.$$

Thus, Jensen's inequality yields

$$2\delta(f) \geq \int |T(x) - x + \nabla \ln f|^2 f d\gamma \geq \Big(\int |T(x) - x + \nabla \ln f| f d\gamma\Big)^2;$$

in particular

$$\int |T(x) - x + \nabla \ln f| f d\gamma \leq \sqrt{2\delta(f)}.$$

Observe $T$ is the Brenier map, nevertheless supposing one considers the Monge-cost, it yields an upper bound on $W_1$ (in the one-dimensional case, the inequality is an equality):

$$W_1(fd\gamma,\gamma) \leq \int |T(x) - x| f d\gamma.$$

This then implies

$$\begin{aligned}W_1(fd\gamma,\gamma) &\leq \sqrt{2\delta(f)} + \int |\nabla f| d\gamma\\ &\leq \sqrt{2\delta(f)} + \Big(\int \frac{|\nabla f|^2}{f}d\gamma\Big)^{\frac{1}{2}}.\end{aligned}$$

Therefore let $\{f_k d\gamma\}$ be a sequence of probability measures with

$$\liminf_k W_1(f_k d\gamma,\gamma) \to \infty$$

$$\delta(f_k) \to 0,$$

[25]; thus

$$\liminf_k \int \frac{|\nabla f_k|^2}{f_k}d\gamma \to \infty.$$

Set

$$f_k = |u_k|^2(\frac{x}{\sqrt{2\pi}}).$$

One then has

$$\int f_k e^{-\frac{|x|^2}{2}}(2\pi)^{-\frac{n}{2}}dx = \int |u_k|^2 e^{-\pi|x|^2}dx = 1$$

$$\int \frac{|\nabla f_k|^2}{f_k} d\gamma = \frac{2}{\pi} \int |\nabla u_k|^2 e^{-\pi|x|^2} dx.$$

Therefore

$$\liminf_k \int |\nabla u_k|^2 e^{-\pi|x|^2} dx \to \infty$$

and this directly implies

$$||u_k||_{H^1(e^{-\pi|x|^2}dx)} \to \infty,$$
$$\delta(u_k) \to 0.$$

□

**Remark 2.1.** Observe that $H^1$ convergence is equivalent to $W_2$ convergence (when $\delta(u_k) \to 0$). The first $W_2$ bound was obtained in [24].

*Proof of Theorem 1.2.* If $a > 0$, define

$$u_a(x) = (2a+1)^{\frac{n}{4}} e^{-a\pi|x|^2}.$$

Now

$$||u_a||_{L^2(e^{-\pi|x|^2}dx)} = 1,$$
$$\int x|u_a|^2 e^{-\pi|x|^2} dx = 0,$$

$$\frac{\left(\int |\nabla u|^2 e^{-\pi|x|^2} dx - \pi \int |u|^2 \ln |u|^2 e^{-\pi|x|^2} dx\right)}{\int |u-1|^2 e^{-\pi|x|^2} dx}$$
$$= \pi \left( \frac{\frac{2na^2}{2a+1} - \frac{n}{2}\ln(2a+1) + \frac{na}{2a+1}}{2 - 2\frac{(2a+1)^{\frac{n}{4}}}{(a+1)^{\frac{n}{2}}}} \right)$$
$$= \frac{\pi}{2} \frac{2na^2 - \frac{n}{2}(2a+1)\ln(2a+1) + na}{2a+1 - \frac{(2a+1)^{\frac{n+4}{4}}}{(a+1)^{\frac{n}{2}}}},$$

$$\lim_{a \to 0^+} \frac{\left(\int |\nabla u|^2 e^{-\pi|x|^2} dx - \pi \int |u|^2 \ln |u|^2 e^{-\pi|x|^2} dx\right)}{\int |u-1|^2 e^{-\pi|x|^2} dx}$$
$$\lim_{a \to 0^+} \frac{\pi}{2} \frac{4na - n\ln(2a+1)}{2 - \left(\frac{n+4}{2}\frac{(2a+1)^{\frac{n}{4}}}{(a+1)^{\frac{n}{2}}} - \frac{n}{2}\frac{(2a+1)^{\frac{n+4}{4}}}{(a+1)^{\frac{n}{2}+1}}\right)}$$
$$\lim_{a \to 0^+} -\pi \frac{2n - n\frac{1}{2a+1}}{\left(\left(\frac{(a+1)^{\frac{n}{2}}\frac{n}{2}\frac{n+4}{2}(2a+1)^{\frac{n-4}{4}} - (2a+1)^{\frac{n}{4}}\frac{n+4}{2}\frac{n}{2}(a+1)^{\frac{n-2}{2}}}{(a+1)^n}\right) - \left(\frac{(a+1)^{\frac{n}{2}+1}\frac{n}{2}\frac{n+4}{2}(2a+1)^{\frac{n}{4}} - (2a+1)^{\frac{n+4}{4}}\frac{n}{2}(a+1)^{\frac{n}{2}}(\frac{n}{2}+1)}{(a+1)^{n+2}}\right)\right)}$$
$$= 2\pi.$$

In particular, assume via contradiction

$$\omega\Big(\int |\nabla u|^2 e^{-\pi|x|^2}dx - \pi\int |u|^2\ln|u|^2 e^{-\pi|x|^2}dx\Big) \geq \kappa\int |u-1|^2 e^{-\pi|x|^2}dx$$

when $u$ is normalized and centered, with

$$\omega(a) = o(a).$$

Hence

$$\begin{aligned}\kappa &\leq \frac{\omega\Big(\int |\nabla u_a|^2 e^{-\pi|x|^2}dx - \pi\int |u_a|^2\ln|u_a|^2 e^{-\pi|x|^2}dx\Big)}{\int |u_a-1|^2 e^{-\pi|x|^2}dx}\\ &= \frac{\omega\Big(\int |\nabla u_a|^2 e^{-\pi|x|^2}dx - \pi\int |u_a|^2\ln|u_a|^2 e^{-\pi|x|^2}dx\Big)}{\int |\nabla u_a|^2 e^{-\pi|x|^2}dx - \pi\int |u_a|^2\ln|u_a|^2 e^{-\pi|x|^2}dx}\frac{\int |\nabla u_a|^2 e^{-\pi|x|^2}dx - \pi\int |u_a|^2\ln|u_a|^2 e^{-\pi|x|^2}dx}{\int |u_a-1|^2 e^{-\pi|x|^2}dx}\\ &\to 0\end{aligned}$$

as $a \to 0^+$; this therefore contradicts $\kappa > 0$. □

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