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Mutual-Inductance Computation between Coaxial Thin Conical Sheet Inductor and the Circular Loop

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Mutual-Inductance Computation between Coaxial Thin Conical Sheet Inductor and the Circular Loop

Slobodan Babic

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Featured Application: For the first time the semi-analytical formula for calculating the mutual inductance between the thin conical sheet inductor and the circular loop is given. The coils are coaxial. The potential applications of the presented work are in wireless power systems, and in ultra-broadband applications since the conical shapes limit effects of the stray capacitance creating the high impedance over a wide bandwidth.

Abstract: The paper describes a new formula for calculating the mutual inductance between a thin conical sheet inductor and a filamentary circular loop, which are coaxial. The presented formula is derived semi-analytically using the complete elliptic integrals of the first, second, and the third kind, along with the integral term which will be solved numerically. The results are validated using double and single integration methods, as well as the semi-analytical formula. The mutual inductance between a thin wall solenoid and a filamentary circular loop can be obtained using the new formula for the conical coil and circular loop. Presented formulas can be useful in various applications, such as broadband RF and wireless power transfer systems that utilize conical inductors. Overall, the paper presents a valuable contribution to the field of inductor design and can be useful in various applications involving conical inductors.

Keywords: mutual-inductance; thin conical sheet inductor; filamentary circular loop; complete elliptical integrals of the first; second and third kind; Heuman Lambda function

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1. Introduction

The paper discusses the challenge of computing the mutual inductance of conventional coils, which has been a topic of research since the time of Maxwell, [1]. While analytical solutions in the form of elementary functions exist for linear coils, more complex configurations, such as circular and elliptical coils require solutions in terms of elliptic integrals, Bessel and Struve functions, and hypergeometric convergent series, [2-8]. Numerical method and commercial software packages are also available, but there is interest in developing analytical and semi-analytical methods for more efficient computations. Reviewing the corresponding literature in the physics, electromagnetics as well as in the scientific paper in the engineering one cannot find too much the calculation of the self-inductance and the mutual inductance of the coils of the conical form. Recently in [9] the calculation of the self-inductance of thin sheet inductor is obtained in the semi-analytical form. Using the same reasoning, in this paper, a semi-analytical formula for calculating the mutual inductance between a thin conical sheet and a filamentary circular loop is given. Coils are coaxial. The new presented method is based on complete elliptical integrals of the first, second, and third kind, along with a term to be solved by numerical integration. As the special case of this new developed formula is the formula for calculating the mutual inductance between the wall solenoid and the filamentary circular loop. The calculation of the previously mentioned coils is useful for conical inductors which are of ideal form for ultra-broadband applications up to 40 GHz since the conical shapes limit effects of stray capacitances and effectively substitute a series of narrow-band inductors, creating the high impedance over a very wide bandwidth, and in wireless power transfer systems that utilize conical inductors [10-20]. All cases either the regular or the singular are explained with precise explications. The validation of the presented method is performed using the single and double integration as well as the semi- analytical formula. The Mathematica files with implemented formulas are available upon request.

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2. Basic formulas

Let us consider a thin conical sheet and a circular loop as showed in Figure 1. The thin conical sheet has the radii of basis R_1 and r_1 $(R_1 > r_1)$ and the axial positions z_1 and z_2, with the number of sheets turns N. The circular loop has the radii R and radial position z_Q.

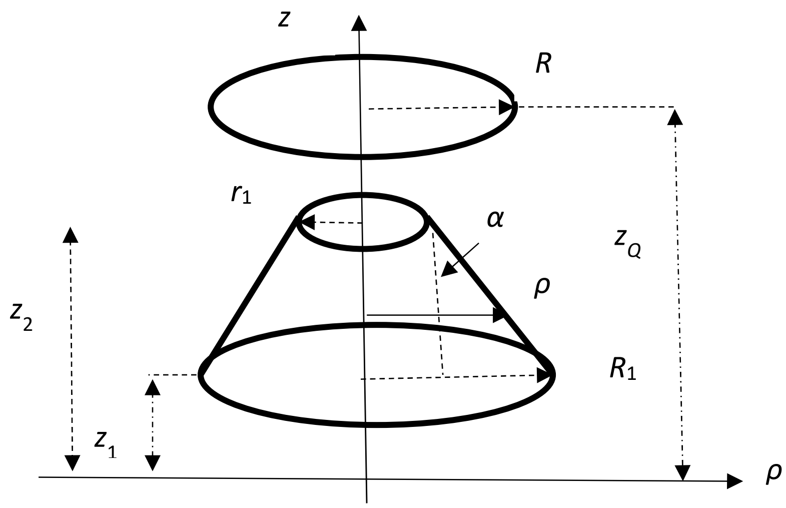

Figure 1. Thin conical sheet inductor and circular loop $(R_1 > r_1)$.

Thus, let us begin the complete analysis for the first case $(R_1 > r_1)$, Figure 1.

From Figure 1. one has,

$$\frac{R_1 - r_1}{z_2 - z_1} = tag(\alpha) = \eta\,, \qquad R_1 = r_1 + \eta(z_2 - z_1) \ \text{ or } \ r_1 = R_1 - \eta(z_2 - z_1) \tag{1}$$

$$\frac{z_2 - z}{\rho - r_1} = \frac{z_2 - z_1}{R_1 - r_1} = \frac{1}{\eta}\,, \qquad \rho = r_1 - \eta(z - z_2) = R_1 - \eta(z - z_1) \tag{2}$$

The mutual inductance between the thin conical sheet inductor and the circular loop can be calculated by,

$$= \frac{\mu_0 N}{z_2 - z_1}\int_0^{\pi}\int_{z_1}^{z_2}\frac{R\rho\cos(\theta)}{r_0}dzd\theta \tag{3}$$

with

$$r_0 = \sqrt{(R_1 - \eta z + \eta z_1)^2 - 2(R_1 - \eta z + \eta z_1)R\cos(\theta) + R^2 + (z - z_Q)^2} \tag{4}$$

Even though $R_1 > r_1$ $(\eta > 0)$ one must use $\rho = R_1 - \eta(z - z_1)$. By the simple verification for $R_1 = r_1$ $(\eta = 0)$ the thin conical sheet degenerates to the thin wall solenoid. For this case, the formula (3) begins the formula for calculating the mutual inductance between the thin wall solenoid and the circular loop, [50].

Introducing the substitution, $\theta = \pi - 2\beta$ in (3) one has,

$$M = -\frac{2\mu_0 NR}{z_2 - z_1}\int_0^{\pi/2}\int_{z_1}^{z_2}\frac{(R_1 - \eta z + \eta z_1)\cos(2\beta)}{r_0}dzd\beta \tag{5}$$

where,

$$r_0 = \sqrt{(R_1 - \eta z + \eta z_1)^2 + 2(R_1 - \eta z + \eta z_1)R\cos(2\beta) + R^2 + (z - z_Q)^2}$$

Let us calculate the first integral in (5),

$$I_1 = \int_{z_1}^{z_2} \frac{(R_1 - \eta z + \eta z_1)dz}{r_0} \tag{6}$$

with

$$r_0 = \frac{1}{\eta}\sqrt{c_1 t^2 + b_1 t + a_1}$$

$$c_1 = \eta^2 + 1$$

$$b_1 = 2\left[\eta^2 R\cos(2\beta) - R_1 + \eta(z_Q - z_1)\right]$$

$$a_1 = \left[R_1 - \eta(z_Q - z_1)\right]^2 + \eta^2 R^2$$

$$\Delta_1 = 4a_1c_1 - b_1^2 =$$

$$4\eta^2\left\{\eta^2 R^2 sin^2(2\beta) - 2R\left[R_1 + \eta\left(z_1 - z_Q\right)\right]\cos(2\beta) + \left[R_1 + \eta\left(z_1 - z_Q\right)\right]^2 + R^2\right\} = 4\eta^2 D_{10}$$

$$D_{10} = \left\{\eta^2 R^2 sin^2(2\beta) + 2R\left[R_1 - \eta\left(z_Q - z_1\right)\right]\cos(2\beta) + \left[R_1 - \eta\left(z_Q - z_1\right)\right]^2 + R^2\right\}$$

$$I_1 = \int_{r_1}^{R_1} \frac{tdt}{\sqrt{c_1 t^2 + b_1 t + a_1}} \tag{8}$$

Using [21],

$$I_1 = \left\{\frac{\sqrt{c_1R_1^2 + b_1R_1 + a_1}}{\eta^2 + 1} - \frac{\sqrt{c_1r_1^2 + b_1r_1 + a_1}}{\eta^2 + 1} - \frac{\eta^2 R cos(2\beta) - R_1 + \eta(z_1 - z_Q)}{(\eta^2+1)^{3/2}} \times \right.$$
$$\left.\left[Asinh\frac{\eta R cos(2\beta) + \eta R_1 + z_Q - z_1}{\sqrt{D_{10}}} - Asinh\frac{\eta R cos(2\beta) + \eta r_1 + z_Q - z_2}{\sqrt{D_{10}}}\right]\right\} \tag{9}$$

Thus, the solution of this integral is obtained in the close form.

Now, the formula (5) can be given as follows,

$$M = -\frac{2\mu_0 NR}{(z_2 - z_1)(\eta^2+1)^{3/2}} \int_0^{\pi/2} V\, cos(2\beta)d\beta \tag{10}$$

where,

$$V = \sqrt{\eta^2+1}\left[\sqrt{c_1R_1^2 + b_1R_1 + a_1} - \sqrt{c_1r_1^2 + b_1r_1 + a_1}\right] -$$
$$\left[\eta^2 R cos(2\beta) - R_1 + \eta\left(z_Q - z_1\right)\right] \times \tag{11}$$
$$\left[Asinh\frac{\eta R cos(2\beta) + \eta R_1 + z_Q - z_1}{\sqrt{D_{10}}} - Asinh\frac{\eta R cos(2\beta) + \eta r_1 + z_Q - z_2}{\sqrt{D_{10}}}\right]$$

Let us solve the following integrals in (10) and (11).

$$I_2 = \int_0^{\pi/2} V\, cos(2\beta)d\beta = I_2^{(1)} + I_2^{(2)} + I_2^{(3)} + I_2^{(4)} \tag{12}$$

$$I_2^{(1)} = \sqrt{\eta^2+1}\int_0^{\pi/2} \sqrt{cR_1^2 + bR_1 + a\, cos(2\beta)d\beta} = \sqrt{\eta^2+1}\int_0^{\pi/2} \sqrt{T_1}\, cos(2\beta)d\beta \tag{13}$$

$$T_1 = c_1R_1^2 + b_1R_1 + a_1 = \eta^2\left[(R + R_1)^2 + (z_Q - z_1)^2 - 4RR_1\, sin^2(\beta)\right]$$

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$$I_2^{(1)} = \sqrt{\eta^2+1}\sqrt{\eta^2\left[(R+R_1)^2+(z_Q-z_1)^2\right]}\int_0^{\pi/2} \Delta_1(1-2sin^2(\beta))d\beta$$

$$\Delta_1 = \sqrt{1-k_1^2 sin^2(\beta)}\,,\;\; k_1^2 = \frac{4RR_1}{(R+R_1)^2+\left(z_Q-z_1\right)^2}$$

Using [21] one obtains,

$$I_2^{(1)} = 2|\eta|\sqrt{\eta^2+1}\frac{\sqrt{RR_1}}{k_1}\int_0^{\frac{\pi}{2}} \Delta_1(1-2sin^2(\beta))d\beta =$$

$$2|\eta|\sqrt{\eta^2+1}\frac{\sqrt{RR_1}}{3k_1^3}\{(2k_1^2-2)K(k_1)+(2-k_1^2)E(k_1)\}$$

or

$$I_2^{(1)} = 2|\eta|\sqrt{\eta^2+1}\frac{\sqrt{RR_1}}{3k_1^3}\{(2k_1^2-2)K(k_1)+(2-k_1^2)E(k_1)\} \quad (14)$$

This solution is obtained in the close form where $K(k_1)$ and $E(k_1)$ are the complete integrals of the first and second kind [21],[22].

Similarly,

$$I_2^{(2)} = -\sqrt{\eta^2+1}\int_0^{\frac{\pi}{2}} \sqrt{cr_1^2+br_1+a}\,cos(2\beta)d\beta \quad (15)$$

the solution of this integral is also obtained in the close form as follows,

$$I_2^{(2)} = -2|\eta|\sqrt{\eta^2+1}\frac{\sqrt{Rr_1}}{3k_2^3}\{(2k_2^2-2)K(k_2)+(2-k_2^2)E(k_2)\} \quad (16)$$

with

$$k_2^2 = \frac{4Rr_1}{(R+r_1)^2+\left(z_2-z_Q\right)^2}$$

The next integral is,

$$I_2^{(3)} = -\int_0^{\frac{\pi}{2}} \left[\eta^2 Rcos(2\beta)-R_1+\eta\left(z_Q-z_1\right)\right]\times$$
$$cos(2\beta)Asinh\frac{\eta Rcos(2\beta)+\eta R_1+z_Q-z_1}{\sqrt{D_{10}}}d\beta \quad (17)$$

or

$$I_2^{(3)} = -\frac{\eta^2 R}{2}\int_0^{\frac{\pi}{2}} Asinh\frac{\eta Rcos(2\beta)+\eta R_1+z_Q-z_1}{\sqrt{D_{10}}}d\beta -$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}}\{\eta^2 Rcos(4\beta)-2[R_1-\eta(z_Q-z_1)]cos(2\beta)\}Asinh\frac{\eta Rcos(2\beta)+\eta R_1+z_Q-z_1}{\sqrt{D_{10}}}d\beta = \tag{18}$$

$$-\frac{1}{2}\int_0^{\frac{\pi}{2}}\{\eta^2 Rcos(4\beta)-2[R_1-\eta(z_Q-z_1)]cos(2\beta)\}\,Asinh\frac{\eta Rcos(2\beta)+\eta R_1+z_Q-z_1}{\sqrt{D_{10}}}d\beta =$$

$$-\frac{\eta^2 R}{2}J_{10}$$

where,

$$J_{10} = \int_0^{\frac{\pi}{2}} Asinh\frac{\eta Rcos(2\beta)+\eta R_1+z_Q-z_1}{\sqrt{D_{10}}}d\beta \tag{19}$$

This integral does not have the analytical solution so that it must be solved numerically. The kernel function of this integral is continuous function on the interval of the integration. Let us solve the second integral by the partial integration.

$$v = \int_0^{\frac{\pi}{2}}\{\eta^2 Rcos(4\beta)-2[R_1-\eta(z_Q-z_1)]cos(2\beta)\}d\beta$$

or

$$v = \frac{sin(2\beta)}{2}\{\eta^2 Rcos(2\beta)-2[R_1-\eta(z_Q-z_1)]\}$$

$$u = Asinh\frac{\eta Rcos(2\beta)+\eta R_1+z_Q-z_1}{\sqrt{D_{10}}}$$

$$du = \frac{R_1\sqrt{\eta^2+1}\,k_1 sin(2\beta)[cos(2\beta)+a_{22}]d\beta}{\eta\sqrt{RR_1}\Delta[cos^2(2\beta)-a_{11}cos(2\beta)-a_{33}]}$$

with

$$a_{11} = \frac{2[R_1-\eta(z_Q-z_1)]}{\eta^2 R}$$

$$a_{22} = \frac{\eta R^2+\eta(z_Q-z_1)^2-R_1(z_Q-z_1)}{\eta R R_1}$$

$$a_{33} = \frac{[R_1-\eta(z_Q-z_1)]^2+(\eta^2+1)R^2}{\eta^2 R^2}$$

Now,

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$$I_2^{(3)} = -\frac{\eta^2 R}{2} J_{10} - \frac{1}{2}\left\{ uv \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} v du \right\} = -\frac{\eta^2 R}{2} J_{10} +$$

$$\int_0^{\frac{\pi}{2}} \frac{\eta^2 R R_1 \sqrt{\eta^2+1}\, k_1 sin(2\beta)[cos(2\beta)+a_{22}] sin(2\beta)[cos(2\beta)-a_{11}] d\beta}{4\eta\sqrt{RR_1}\Delta[cos^2(2\beta) - a_{11}cos(2\beta) - a_{33}]} =$$

$$\frac{\eta^2 R}{2} J_{10} - \frac{\eta\sqrt{\eta^2+1}\, k_1 \sqrt{RR_1}}{4} \int_0^{\frac{\pi}{2}} \frac{[cos^2(2\beta)-1][cos(2\beta)-a_{11})][cos(2\beta)+a_{22}] d\beta}{[cos^2(2\beta) - a_{11}cos(2\beta) - a_{33}]\Delta_1} = \tag{20}$$

$$\frac{\eta^2 R}{2} J_{10} - \frac{k_1 \eta\sqrt{\eta^2+1}\sqrt{RR_1}}{4} \int_0^{\frac{\pi}{2}} \{cos^2(2\beta) + a_{22}cos(2\beta) + a_{33} - 1\} \frac{d\beta}{\Delta_1} -$$

$$\frac{k_1 \eta\sqrt{\eta^2+1}\sqrt{RR_1}}{4} \int_0^{\frac{\pi}{2}} \frac{(Acos(2\beta)+B) d\beta}{[cos^2(2\beta) - a_{11}cos(2\beta) - a_{33}]\Delta_1} d\beta$$

with

$$A = (a_{22} + a_{11})a_{33} - a_2, \quad B = a_{33}^2 - a_{33} + a_{11}a_{22}$$

The expression (20) can be written as follows,

$$I_2^{(3)} = -\frac{\eta^2 R}{2} J_{10} - \frac{k_1 \eta\sqrt{\eta^2+1}\sqrt{RR_1}}{4} \int_0^{\frac{\pi}{2}} \left\{cos^2(2\beta) + a_2 cos(2\beta) - 1 + a_3\right\} \frac{d\beta}{\Delta} -$$

$$\frac{k_1 \eta\sqrt{\eta^2+1}\sqrt{RR_1}}{4} \int_0^{\frac{\pi}{2}} \frac{(Acos(2\beta)+B) d\beta}{[(cos(2\beta)-p_1)(cos(2\beta)-p_2)]\Delta_1} \tag{21}$$

Let us find the solutions of the following equation,

$$cos^2(2\beta) - a_{11}cos(2\beta) - a_{33} = (cos(2\beta)-p_1)(cos(2\beta)-p_2) = 0$$

$$p_{1,2} = \frac{a_{11} \pm \sqrt{a_{11}^2 + 4a_{33}}}{2} = \frac{a_{11} \pm \sqrt{D_1}}{2} = \frac{[R_1 - \eta(z_Q - z_1)]}{\eta^2 R} + \frac{\sqrt{(\eta^2+1)}}{\eta^2 R}\sqrt{T_{10}}$$

$$D_1 = a_{11}^2 + 4a_{33} = 4\frac{[R_1 - \eta(z_Q - z_1)]^2}{\eta^4 R^2} + 4\frac{[R_1 - \eta(z_Q - z_1)]^2 + (\eta^2+1)R^2}{\eta^2 R^2}$$

$$D_1 = \frac{4(\eta^2+1)}{\eta^4 R^2}\left\{[R_1 - \eta(z_Q - z_1)]^2 + \eta^2 R^2\right\} = \frac{4(\eta^2+1)}{\eta^4 R^2} T_{10} > 0$$

$$T_{10} = \left\{[R_1 - \eta(z_Q - z_1)]^2 + \eta^2 R^2\right\}$$

The following fraction can be obtained in the following form,

$$\frac{(Acos(2\beta)+B)}{[cos^2(2\beta) - a_{11}cos(2\beta) - a_{33}]\Delta} = \frac{Acos(2\beta)+B}{(cos(2\beta)-p_1)(cos(2\beta)-p_2)} =$$

$$\frac{V_1}{cos(2\beta)-p_1} + \frac{V_2}{cos(2\beta)-p_2} \tag{22}$$

where,

$$V_1 = -\frac{Ap_1 + B}{p_2 - p_1}, \quad V_2 = -\frac{Ap_2 + B}{p_2 - p_1}$$

The simplified form is,

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$$I_2^{(3)} = -\frac{\eta^2 R}{2} J_{10} - \frac{k_1 \eta \sqrt{\eta^2+1}\sqrt{RR_1}}{4} \int_0^{\frac{\pi}{2}} \left\{ cos^2(2\beta) + a_{22} cos(2\beta) - 1 + a_{33} \right\} \frac{d\beta}{\Delta_1} -$$

$$\frac{k_1 \eta \sqrt{\eta^2+1}\sqrt{RR_1}}{4} \int_0^{\frac{\pi}{2}} \frac{Acos(2\beta) + B}{(cos(2\beta) - p_1)(cos(2\beta) - p_2)\Delta} = -\frac{\eta^2 R}{2} J_{10} -$$

$$\frac{k_1 \eta \sqrt{\eta^2+1}\sqrt{RR_1}}{4} \int_0^{\frac{\pi}{2}} \left\{ 4sin^4(\beta) - (2a_2+4)sin^2(\beta) + a_2 + a_3 \right\} \frac{d\beta}{\Delta_1} - \quad (23)$$

$$\frac{k_1 \eta \sqrt{\eta^2+1}\sqrt{RR_1}}{4} \left\{ T_1 \int_0^{\frac{\pi}{2}} \frac{d\beta}{(cos(2\beta) - p_1)\Delta_1} + T_2 \int_0^{\frac{\pi}{2}} \frac{d\beta}{(cos(2\beta) - p_2)\Delta_1} \right\} = -\frac{\eta^2 R}{2} J_{10} -$$

$$\frac{k_1 \eta \sqrt{\eta^2+1}\sqrt{RR_1}}{4} \left\{ 4\left[\frac{2+k_1^2}{3k_1^4} K(k_1) - \frac{2+2k_1^2}{3k_1^4} E(k_1) \right] - \frac{(2a_2+4)}{k_1^2} [K(k_1) - E(k_1)] - \right.$$

$$\left. (a_{22} + a_{33}) K(k_1) \right\} - \frac{k_1 \eta \sqrt{\eta^2+1}\sqrt{RR_1}}{4} \left[\frac{V_1}{1-p_1} \Pi(h_1, k_1) + \frac{V_2}{1-p_2} \Pi(h_2, k_1) \right]$$

with,

$$h_1 = \frac{2}{1-p_1}, \qquad h_2 = \frac{2}{1-p_2}$$

The final expression is obtained in the close form over the complete elliptic integrals of the first, second and third kind, $K(k_1)$, $E(k_1)$, $\Pi(h_1, k_1)$ and $\Pi(h_2, k_1)$, [21], [22].

Similarly,

$$I_2^{(4)} = \int_0^{\frac{\pi}{2}} \left[\eta^2 Rcos(2\beta) - R_1 + \eta(z_Q - z_1) \right] cos(2\beta) Asinh \frac{\eta Rcos(2\beta) + \eta R_1 + z_Q - z_2}{\sqrt{D_{10}}} d\beta \quad (24)$$

can be obtained as follows,

$$I_2^{(4)} = \frac{\eta^2 R}{2} J_{20} + \frac{\eta \sqrt{\eta^2+1}\sqrt{Rr_1}}{12k_2^3} \{ [3(a_{222} + a_{333})k_2^4 - (6a_{222} + 8)k_2^2 + 8] K(k_2) +$$

$$[(6a_{22}2 + 4)k_2^2 - 8] E(k_2) \} + \frac{k_2 \eta \sqrt{\eta^2+1}\sqrt{Rr_1}}{4} \left[\frac{V_3}{1-p_3} \Pi(h_3, k_3) + \frac{V_4}{1-p_4} \Pi(h_4, k_4) \right] \quad (25)$$

where,

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$$k_2^2 = \frac{4Rr_1}{(R+r_1)^2 + \left(z_Q - z_2\right)^2}$$
$$a_{111} = \frac{2\left[r_1 - \eta\left(z_Q - 2\right)\right]}{\eta^2 R}$$
$$a_{222} = \frac{\eta R^2 + \eta\left(z_Q - z_2\right)^2 - r_1\left(z_Q - z_2\right)}{\eta R r_1}$$
$$a_{333} = \frac{\left[r_1 - \eta\left(z_Q - z_2\right)\right]^2 + (\eta^2+1)R^2}{\eta^2 R^2} \tag{26}$$
$$\frac{(Ccos(2\beta) + D)}{[cos^2(2\beta) - a_{11}cos(2\beta) - a_{33}]} = \frac{Ccos(2\beta) + D}{(cos(2\beta) - p_3)(cos(2\beta) - p_4)} =$$
$$\frac{V_3}{cos(2\beta) - p_3} + \frac{V_4}{cos(2\beta) - p_4}$$
$$C = (a_{22} + a_{11})a_{33} - a_{22}\ , \qquad D = a_{33}^2 - a_{33} + a_{11}a_{22},$$
$$p_{3,4} = \frac{a_{111} \pm \sqrt{a_{111}^2 + 4a_{333}}}{2} = \frac{a_{11} \pm \sqrt{D_2}}{2} = \frac{\left[r_1 - \eta\left(z_Q - z_2\right)\right]}{\eta^2 R} + \frac{\sqrt{(\eta^2+1)}}{\eta^2 R}\sqrt{D_2}$$
$$D_2 = a_{111}^2 + 4a_{333} = 4\frac{\left[r_1 - \eta\left(z_Q - z_2\right)\right]^2}{\eta^4 R^2} + 4\frac{\left[r_1 - \eta\left(z_Q - z_2\right)\right]^2 + (\eta^2+1)R^2}{\eta^2 R^2}$$
$$D_2 = \frac{4(\eta^2+1)}{\eta^4 R^2}\left\{\left[r_1 - \eta\left(z_Q - z_2\right)\right]^2 + \eta^2 R^2\right\} = \frac{4(\eta^2+1)}{\eta^4 R^2}T_{20} > 0$$
$$T_{20} = \left\{\left[r_1 - \eta\left(z_Q - z_2\right)\right]^2 + \eta^2 R^2\right\} > 0$$
$$V_3 = -\frac{Cp_3 + D}{p_4 - p_3}\ ,\ V_4 = -\frac{Cp_3 + D}{p_4 - p_3}$$
$$J_{20} = \int_0^{\frac{\pi}{2}} Asinh\frac{\eta Rcos(2\beta) + \eta R_1 + z_Q - z_2}{\sqrt{D_{10}}}d\beta$$

Finally, from (14), (16), (21) and (25) the mutual inductance between the thin conical sheet and the circular loop can be calculated in the semi-analytical form as follows,

$$M = -\frac{2\mu_0 NR}{(z_2 - z_1)(\eta^2+1)^{\frac{3}{2}}}V_0 \tag{27}$$

where,

$$
\begin{aligned}
V_0 = &-\frac{\eta^2 R}{2} J_0 - \frac{\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{12k_1^3}\{[3(a_2+a_3)k_1^4-(6a_2+8)k_1^2+8]K(k_1)+ \\
&[(6a_2+4)k_1^2-8]E(k_1)\} + |\eta|\sqrt{\eta^2+1}\frac{\sqrt{RR_1}}{12k_1^3}\{(16k_1^2-16)K(k_1)+ \\
&(16-8k_1^2)E(k_1)\} + \frac{\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{12k_2^3}\{[3(a_{22}+a_{33})k_2^4-(6a_{22}+8)k_2^2+8]K(k_2)+ \\
&[(6a_{22}+4)k_2^2-8]E(k_2)\} - |\eta|\sqrt{\eta^2+1}\frac{\sqrt{Rr_1}}{12k_2^3}\times \\
&\{(16k_2^2-16)K(k_2)+(16-8k_2^2)E(k_2)\} - \frac{k_1\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{4}\times \\
&\left[\frac{T_1}{1-p_1}\Pi(h_1,k_1)+\frac{T_2}{1-p_2}\Pi(h_2,k_1)\right] + \frac{k_2\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{4} + \\
&\left[\frac{T_3}{1-p_3}\Pi(h_3,k_2)+\frac{T_4}{1-p_4}\Pi(h_4,k_2)\right]
\end{aligned}
\tag{28}
$$

where,

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$$k_1^2 = \frac{4RR_1}{(R+R_1)^2 + \left(z_Q - z_1\right)^2}, \qquad k_2^2 = \frac{4Rr_1}{(R+r_1)^2 + \left(z_Q - z_2\right)^2}$$

$$J_0 = \int_0^{\frac{\pi}{2}} Asinh \frac{\eta R cos(2\beta) + \eta R_1 + z_Q - z_1}{\sqrt{D_0}} d\beta - \int_0^{\frac{\pi}{2}} Asinh \frac{\eta R cos(2\beta) + \eta r_1 + z_Q - z_2}{\sqrt{D_0}} d\beta$$

$$a_1 = \frac{2\left[R_1 - \eta\left(z_Q - z_1\right)\right]}{\eta^2 R}$$

$$a_2 = \frac{\eta R^2 + \eta(z_Q - z_1)^2 - R_1(z_Q - z_1)}{\eta R R_1}$$

$$a_3 = \frac{\left[R_1 - \eta\left(z_Q - z_1\right)\right]^2 + (\eta^2+1)R^2}{\eta^2 R^2}$$

$$A = (a_2 + a_1)a_3 - a_2 \, , B = a_3^2 - a_3 + a_1 a_2$$

$$p_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 + 4a_3}}{2}, \quad h_1 = \frac{2}{1-p_1}, \qquad h_2 = \frac{2}{1-p_2} \tag{}$$

$$V_1 = -\frac{Ap_1 + B}{p_2 - p_1}, \quad V_2 = -\frac{Ap_2 + B}{p_2 - p_1}$$

$$a_{11} = \frac{2\left[r_1 - \eta\left(z_Q - z_2\right)\right]}{\eta^2 R}$$

$$a_{22} = \frac{\eta R^2 + \eta(z_Q - z_2)^2 - r_1(z_Q - z_2)}{\eta R r_1}$$

$$a_{33} = \frac{\left[r_1 - \eta\left(z_Q - z_2\right)\right]^2 + (\eta^2+1)R^2}{\eta^2 R^2}$$

$$C = (a_{22} + a_{11})a_{33} - a_{22} \, , D = a_{33}^2 - a_{33} + a_{11} a_{22}$$

$$p_{3,4} = \frac{a_{11} \pm \sqrt{a_{11}^2 + 4a_{33}}}{2}, \quad h_3 = \frac{2}{1-p_3}, \qquad h_4 = \frac{2}{1-p_4}$$

$$V_3 = -\frac{Cp_3 + D}{p_4 - p_3}, \quad V_4 = -\frac{Cp_3 + D}{p_4 - p_3}$$

$$\eta = \frac{R_1 - r_1}{z_2 - z_1} > 0 \text{ because } R_1 > r_1$$

Thus, the general solution of equation (26) with (27) is expressed by the complete elliptic integrals $K(k)$, $E(k)$ and $\Pi(h, k)$ as well as one term J_0 which must be solved numerically.

2.2. *Singular cases*

The equation (27) with (28) can be directly applied for the general cases. It is possible to have in (28) four singular cases so that one must do some corrections to overcome these problems.

2.2.1. $p_1 = p_3 = -1$

From (26) one can see that case that for $p_1 = -1$ the singular case appears and $A = B$ so that,

$$\frac{(Acos(2\beta)+B)}{(cos(2\beta)-p_1)(cos(2\beta)-p_2)}=\frac{A(cos(2\beta)+1)}{(cos(2\beta)+1)(cos(2\beta)-p_2)}=\frac{A}{cos(2\beta)-p_2}=\frac{A}{1-p_2}\frac{1}{1-h_2sin^2(\beta)} \quad (29)$$

Also, in (26) the singularity appears for $p_3=-1$ and $C=D$ so that,

$$\frac{(Ccos(2\beta)+D)}{(cos(2\beta)-p_3)(cos(2\beta)-p_4)}=\frac{C(cos(2\beta)+1)}{(cos(2\beta)+1)(cos(2\beta)-p_4)}=\frac{C}{cos(2\beta)-p_4}=\frac{C}{1-p_4}\frac{1}{1-h_4sin^2(\beta)} \quad (30)$$

From (29) and (30), it is obvious that the complete elliptic integrals of the third kind $\Pi(h_1,k_1)$ and $\Pi(h_3,k_3)$ will vanish.

Thus, (28) begins,

$$V_0=\frac{\eta^2R}{2}J_0-\frac{\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{12k_1^3}\{[3(a_2+a_3)k_1^4-(6a_2+8)k_1^2+8]K(k_1)+[(6a_2+4)k_1^2-8]E(k_1)\}+|\eta|\sqrt{\eta^2+1}\frac{\sqrt{RR_1}}{12k_1^3}\{(16k_1^2-16)K(k_1)+(16-8k_1^2)E(k_1)\}+\frac{\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{12k_2^3}\{[3(a_{22}+a_{33})k_2^4-(6a_{22}+8)k_2^2+8]K(k_2)+[(6a_{22}+4)k_2^2-8]E(k_2)\}-|\eta|\sqrt{\eta^2+1}\frac{\sqrt{Rr_1}}{12k_2^3}\{(16k_2^2-16)K(k_2)+(16-8k_2^2)E(k_2)\}-\frac{k_1\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{4}\frac{A}{1-p_2}\Pi(h_2,k_1)+\frac{k_2\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{4}\frac{C}{1-p_4}\Pi(h_4,k_2) \quad (31)$$

2.2.2. $p_1=p_3=1$

From (22) one can see that case that for $p_1=1$ the singular case appears and $A=-B$ so that,

$$\frac{(Acos(2\beta)+B)}{(cos(2\beta)-p_1)(cos(2\beta)-p_2)}=\frac{A(cos(2\beta)-1)}{(cos(2\beta)-1)(cos(2\beta)-p_2)}=\frac{A}{cos(2\beta)-p_2}=\frac{A}{1-p_2}\frac{1}{1-h_2sin^2(\beta)} \quad (32)$$

Also, in (26) the singularity appears for $p_3=1$ and $C=-D$ so that,

$$\frac{(Ccos(2\beta)+D)}{(cos(2\beta)-p_3)(cos(2\beta)-p_4)}=\frac{C(cos(2\beta)-1)}{(cos(2\beta)-1)(cos(2\beta)-p_4)}=\frac{C}{cos(2\beta)-p_4}=\frac{C}{1-p_4}\frac{1}{1-h_4sin^2(\beta)} \quad (33)$$

From (32) and (33), it is obvious that the complete elliptic integrals of the third kind $\Pi(h_1,k_1)$ and $\Pi(h_3,k_3)$ will vanish.

Thus, for this case we use (31).

2.2.3. $p_2=p_4=-1$

From (26) one can see that case that for $p_2=-1$ the singular case appears and $A=B$ so that,

$$\frac{(Acos(2\beta)+B)}{(cos(2\beta)-p_1)(cos(2\beta)-p_2)}=\frac{A(cos(2\beta)+1)}{(cos(2\beta)+1)(cos(2\beta)-p_1)}=\frac{A}{cos(2\beta)-p_1}=\frac{A}{1-p_1}\frac{1}{1-h_1sin^2(\beta)} \quad (34)$$

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Also, in (26) the singularity appears for $p_4 = -1$ and $C = D$ so that,

$$\frac{(Ccos(2\beta)+D)}{(cos(2\beta)-p_3)(cos(2\beta)-p_4)} = \frac{C(cos(2\beta)+1)}{(cos(2\beta)+1)(cos(2\beta)-p_3)} = \frac{C}{cos(2\beta)-p_3} = \frac{C}{1-p_3}\frac{1}{1-h_3 sin^2(\beta)} \tag{35}$$

From (34) and (35), it is obvious that the complete elliptic integrals of the third kind $\Pi(h_2,k_1)$ and $\Pi(h_4,k_3)$ will vanish.

Thus, (28) begins,

$$V_0 = \frac{\eta^2 R}{2}J_0 - \frac{\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{12k_1^3}\{[3(a_2+a_3)k_1^4-(6a_2+8)k_1^2+8]K(k_1)+ +[(6a_2+4)k_1^2-8]E(k_1)\} + +|\eta|\sqrt{\eta^2+1}\frac{\sqrt{RR_1}}{12k_1^3}\{(16k_1^2-16)K(k_1)+(16-8k_1^2)E(k_1)\} + +\frac{\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{12k_2^3}\{[3(a_{22}+a_{33})k_2^4-(6a_{22}+8)k_2^2+8]K(k_2)+[(6a_{22}+4)k_2^2-8]E(k_2)\} - -|\eta|\sqrt{\eta^2+1}\frac{\sqrt{Rr_1}}{12k_2^3}\{(16k_2^2-16)K(k_2)+(16-8k_2^2)E(k_2)\} - -\frac{k_1\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{4}\frac{A}{1-p_1}\Pi(h_1,k_1)+\frac{k_2\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{4}\frac{C}{1-p_3}\Pi(h_3,k_2) \tag{36}$$

2.2.4. $p_2 = p_4 = 1$

From (26) one can see that case that for $p_2 = 1$ the singular case appears and $A = -B$ so that,

$$\frac{(Acos(2\beta)+B)}{(cos(2\beta)-p_1)(cos(2\beta)-p_2)} = \frac{A(cos(2\beta)+1)}{(cos(2\beta)+1)(cos(2\beta)-p_1)} = \frac{A}{cos(2\beta)-p_1} = \frac{A}{1-p_1}\frac{1}{1-h_1 sin^2(\beta)} \tag{37}$$

Also, in (26) the singularity appears for $p_4 = -1$ and $C = -D$ so that,

$$\frac{(Ccos(2\beta)+D)}{(cos(2\beta)-p_3)(cos(2\beta)-p_4)} = \frac{C(cos(2\beta)+1)}{(cos(2\beta)+1)(cos(2\beta)-p_3)} = \frac{C}{cos(2\beta)-p_3} = \frac{C}{1-p_3}\frac{1}{1-h_3 sin^2(\beta)} \tag{38}$$

From (36) and (37), it is obvious that the complete elliptic integrals of the third kind $\Pi(h_2,k_1)$ and $\Pi(h_4,k_3)$ will vanish.

Thus, for this case we use (36).

From this detailed analysis detailed all partial singular cases can be found in the previous discussed cases.

2.3. Case $r_1 > R_1$

Now, let us consider a thin conical sheet and a circular loop as showed in Figure 2. The thin conical sheet has the radii of basis R_1 and r_1 ($r_1 > R_1$) and the axial positions z_1 and z_2, with the number of sheets turns N. The circular loop has the radii R and radial position z_Q.

In this case $r_1 > R_1$ one can use the same reasoning as in the previous cases. This is why the final expressions for each calculation.

From Figure 2, we have,

$$\frac{r_1 - R_1}{z_2 - z_1} = tag(\alpha) = \eta\,, \quad r_1 = R_1 + \eta(z_2 - z_1) \tag{39}$$

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$$\frac{z-z_1}{\rho-R_1}=\frac{z_2-z_1}{r_1-R_1}=\frac{1}{\eta}, \qquad \rho=R_1+\eta(z-z_1) \tag{40}$$

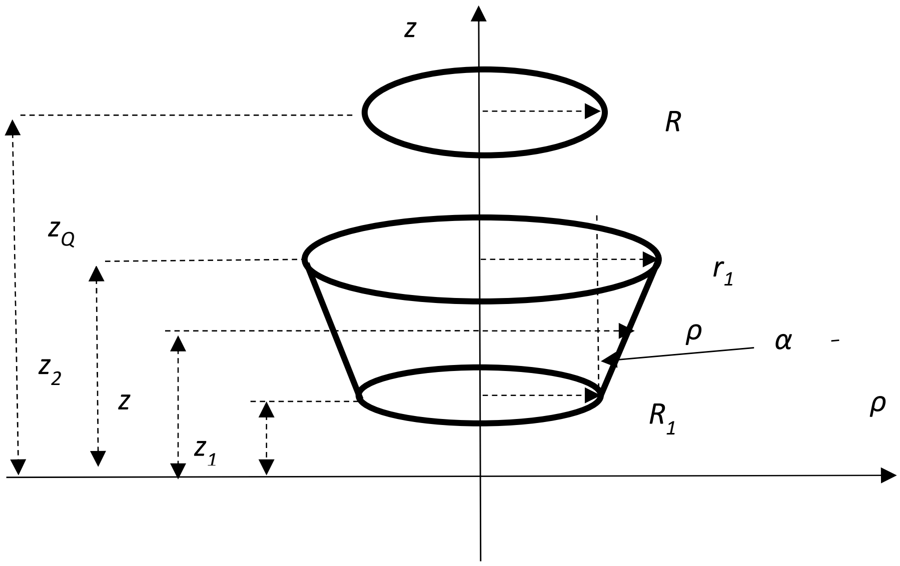

Figure 2. Circular loop-thin conical sheet inductor ($r_1 > R_1$)

The mutual inductance can be calculated as follows,

$$M=\frac{\mu_0 N}{z_2-z_1}\int_0^{\pi}\int_{z_1}^{z_2}\frac{R\rho\cos(\theta)}{r_0}dzd\theta=\frac{\mu_0 N}{z_2-z_1}\int_0^{\pi}\int_{z_1}^{z_2}\frac{R(R_1+\eta z-\eta z_1)\cos(\theta)}{r_0}dzd\theta \tag{41}$$

with

$$r_0=\sqrt{(R_1+\eta z-\eta z_1)^2-2(R_1+\eta z-\eta z_1)R\cos(\theta)+R^2+(z-z_Q)^2} \tag{42}$$

Introducing the substitution, $\theta=\pi-2\beta$ in (41) one has,

$$M=-\frac{2\mu_0 NR}{z_2-z_1}\int_0^{\pi/2}\int_{z_1}^{z_2}\frac{(R_1+\eta z-\eta z_1)\cos(2\beta)}{r_0}dzd\beta \tag{43}$$

With

$$r_0=\sqrt{(R_1+\eta z-\eta z_1)^2+2(R_1+\eta z-\eta z_1)R\cos(2\beta)+R^2+(z-z_Q)^2}$$

The mutual inductance can be calculated by the double integration (43). Following the procedures as in **2.1**, after the first integration over the variable z, one has,

$$M=\frac{2\mu_0 NR}{(z_2-z_1)(\eta^2+1)^{3/2}}\int_0^{\pi/2} V\,cos(2\beta)d\beta \tag{44}$$

where,

$$V=\sqrt{\eta^2+1}\left[\sqrt{c_0R_1^2+b_0R_1+a_0}-\sqrt{c_0r_1^2+b_0r_1+a_0}\right]-\left[\eta^2Rcos(2\beta)-R_1-\eta\left(z_Q-z_1\right)\right]$$
$$\times\left[Asinh\frac{\eta Rcos(2\beta)+\eta R_1-z_Q+z_1}{\sqrt{D_{20}}}-Asinh\frac{\eta Rcos(2\beta)+\eta r_1-z_Q+z_2}{\sqrt{D_{20}}}\right]$$
$$c_0=\eta^2+1$$
$$b_0=2\left[\eta^2R\cos(2\beta)-R_1-\eta(z_Q-z_1)\right] \quad (45)$$
$$a_0=\left[R_1+\eta(z_Q-z_1)\right]^2+\eta^2R^2$$
$$\Delta_{\mathbf{0}}=4\eta^2D_{00}$$
$$D_{10}=\left\{\eta^2R^2sin^2(2\beta)+2R\left[R_1+\eta\left(z_Q-z_1\right)\right]\cos(2\beta)+\left[R_1+\eta\left(z_Q-z_1\right)\right]^2+R^2\right\}$$

Finally, using the same procedures as in **2.1** after the second integration the mutual inductance in the semi-analytical form is given as follows,

$$M=\frac{2\mu_0NR}{(z_2-z_1)(\eta^2+1)^{3/2}}\int_0^{\pi/2}V_{00}\,d\beta \quad (46)$$

The solution of the four integrals in (46),

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$$V_{00} = V_{00}^{(1)} + V_{00}^{(2)} + V_{00}^{(3)} + V_{44}^{(4)} =$$

$$-\frac{\eta^2 R}{2} J_{00} - \frac{\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{12k_1^3}\{[3(d_2+d_3)k_1^4 - (6d_2+8)k_1^2 + 8]K(k_1) +$$

$$[(6d_2+4)k_1^2 - 8]E(k_1)\} + |\eta|\sqrt{\eta^2+1}\frac{\sqrt{RR_1}}{12k_1^3}\{(16k_1^2-16)K(k_1) +$$

$$(16-8k_1^2)E(k_1)\} + \frac{\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{12k_2^3}\{[3(d_{22}+d_{33})k_2^4 - (6d_{22}+8)k_2^2 + 8]K(k_2) +$$

$$[(6d_{22}+4)k_2^2 - 8]E(k_2)\} - |\eta|\sqrt{\eta^2+1}\frac{\sqrt{Rr_1}}{12k_2^3}\times$$

$$\{(16k_2^2-16)K(k_2) + (16-8k_2^2)E(k_2)\} - \frac{k_1\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{4}\times$$

$$\left[\frac{T_{11}}{1-p_{11}}\Pi(h_{11},k_1) + \frac{T_{22}}{1-p_{22}}\Pi(h_{22},k_1)\right] + \frac{k_2\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{4}$$

$$\left[\frac{T_{33}}{1-p_{33}}\Pi(h_{33},k_2) + \frac{T_{44}}{1-p_{44}}\Pi(h_{44},k_2)\right]$$

$$k_1^2 = \frac{4RR_1}{(R+R_1)^2 + (z_Q - z_1)^2},\ k_2^2 = \frac{4Rr_1}{(R+r_1)^2 + (z_2 - z_Q)^2}$$

$$J_{00} = \int_0^{\frac{\pi}{2}} Asinh\frac{\eta R\cos(2\beta) + \eta R_1 + z_1 - z_Q}{\sqrt{D_{00}}}d\beta -$$

$$\int_0^{\frac{\pi}{2}} Asinh\frac{\eta R\cos(2\beta) + \eta r_1 + z_2 - z_Q}{\sqrt{D_{00}}}d\beta \quad (47)$$

$$d_1 = \frac{2[R_1 + \eta(z_Q - z_1)]}{\eta^2 R}$$

$$d_2 = \frac{\eta R^2 + \eta(z_Q - z_1)^2 + R_1(z_Q - z_1)}{\eta R R_1}$$

$$d_3 = \frac{[R_1 + \eta(z_Q - z_1)]^2 + (\eta^2+1)R^2}{\eta^2 R^2}$$

$$A_1 = (d_2 + d_1)d_3 - d_2,\ B_1 = d_3^2 - d_3 + d_1 d_2$$

$$p_{11,22} = \frac{d_1 \pm \sqrt{d_1^2 + 4d_3}}{2},\quad h_{11} = \frac{2}{1-p_{11}},\quad h_{22} = \frac{2}{1-p_{22}}$$

$$V_{11} = -\frac{A_1 p_{11} + B_1}{p_{22} - p_{11}},\ V_{22} = -\frac{A_1 p_2 + B_1}{p_{22} - p_{11}}$$

$$d_{11} = \frac{2[r_1 + \eta(z_Q - z_2)]}{\eta^2 R}$$

$$d_{22} = \frac{\eta R^2 + \eta(z_Q - z_2)^2 + r_1(z_Q - z_2)}{\eta R r_1}$$

$$d_{33} = \frac{[r_1 + \eta(z_Q - z_2)]^2 + (\eta^2+1)R^2}{\eta^2 R^2}$$

$$C_1 = (d_{22} + d_{11})d_{33} - d_{22},\ D_1 = d_{33}^2 - d_{33} + d_{11} d_{22}$$

$$p_{33,44} = \frac{d_{11} \pm \sqrt{d_{11}^2 + 4d_{33}}}{2},\quad h_{33} = \frac{2}{1-p_{33}},\quad h_{44} = \frac{2}{1-p_{44}}$$

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$$V_{33} = -\frac{C_1 p_{33} + D_1}{p_{44} - p_{33}}\ ,\ V_{44} = -\frac{C_1 p_{33} + D_1}{p_{44} - p_{34}}$$

Thus, the general solution of equation (46) with (47) is expressed by the complete elliptic integrals $K(k)$, $E(k)$ and $\Pi(h, k)$ as well as one term J_{00} which must be solved numerically.

2.4. Singular cases

It is possible to have in (47) four singular cases so that one must do some corrections to overcome these problems.

2.4.1. $p_{11} = p_{33} = -1$

From (47) one can see that case that for $p_{11} = -1$ the singular case appears and $A_1 = B_1$ so that,

$$\frac{(A_1 cos(2\beta) + B_1)}{(cos(2\beta) - p_{11})(cos(2\beta) - p_{22})} = \frac{A_1(cos(2\beta) + 1)}{(cos(2\beta) + 1)(cos(2\beta) - p_{22})} = \frac{A_1}{cos(2\beta) - p_{22}} = \frac{A_1}{1 - p_{22}}\frac{1}{1 - h_{22} sin^2(\beta)} \tag{48}$$

For $p_{33} = -1$ in (47) $C_1 = D_1$ so that

$$\frac{(C_1 cos(2\beta) + D_1)}{(cos(2\beta) - p_{33})(cos(2\beta) - p_{44})} = \frac{C_1(cos(2\beta) + 1)}{(cos(2\beta) + 1)(cos(2\beta) - p_{44})} = \frac{C_1}{cos(2\beta) - p_{44}} = \frac{C_1}{1 - p_{44}}\frac{1}{1 - h_{44} sin^2(\beta)} \tag{49}$$

From (48) and (49), it is obvious that the complete elliptic integrals of the third kind $\Pi(h_{22}, k_1)$ and $\Pi(h_{44}, k_3)$ will vanish.

Thus, (47) begins,

$$\begin{aligned} V_{00} = {} & \frac{\eta^2 R}{2} J_{00} - \frac{\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{12k_1^3}\{[3(d_2 + d_3)k_1^4 - (6d_2 + 8)k_1^2 + 8]K(k_1) + \\ & [(6d_2 + 4)k_1^2 - 8]E(k_1)\} + |\eta|\sqrt{\eta^2+1}\frac{\sqrt{RR_1}}{12k_1^3}\{(16k_1^2 - 16)K(k_1) + (16 - 8k_1^2)E(k_1)\} + \\ & \frac{\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{12k_2^3}\{[3(d_{22} + d_{33})k_2^4 - (6d_{22} + 8)k_2^2 + 8]K(k_2) + [(6d_{22} + 4)k_2^2 - 8]E(k_2)\} - \\ & |\eta|\sqrt{\eta^2+1}\frac{\sqrt{Rr_1}}{12k_2^3}\{(16k_2^2 - 16)K(k_2) + (16 - 8k_2^2)E(k_2)\} - \\ & \frac{k_1\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{4}\frac{A_1}{1 - p_{22}}\Pi(h_{22}, k_1) + \frac{k_2\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{4}\frac{C_1}{1 - p_{44}}\Pi(h_{44}, k_2) \end{aligned} \tag{50}$$

2.4.2. $p_{11} = p_{33} = 1$

From (47) one can see that case that for $p_{11} = 1$ the singular case appears and $A_1 = -B$ so that,

$$\frac{(A_1 cos(2\beta) + B_1)}{(cos(2\beta) - p_{11})(cos(2\beta) - p_{22})} = \frac{A_1(cos(2\beta) - 1)}{(cos(2\beta) - 1)(cos(2\beta) - p_{22})} = \frac{A_1}{cos(2\beta) - p_{22}} = \frac{A_1}{1 - p_{22}}\frac{1}{1 - h_{22} sin^2(\beta)} \tag{51}$$

Also, in (47) the singularity appears for $p_{33} = 1$ and $C_1 = -D_1$ so that,

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$$\frac{(C_1 cos(2\beta)+D_1)}{(cos(2\beta)-p_{33})(cos(2\beta)-p_{44})}=\frac{C_1(cos(2\beta)-1)}{(cos(2\beta)-1)(cos(2\beta)-p_{44})}=\frac{C_1}{cos(2\beta)-p_{44}}=\frac{C_1}{1-p_{44}}\frac{1}{1-h_{44}sin^2(\beta)} \tag{52}$$

Thus, for this case we use (50).

2.4.3. $p_{22}=p_{44}=-1$

From (47) one can see that case that for $p_{22}=-1$ the singular case appears and $A_1=B_1$ so that,

$$\frac{(A_1 cos(2\beta)+B_1)}{(cos(2\beta)-p_{11})(cos(2\beta)-p_{22})}=\frac{A_1(cos(2\beta)+1)}{(cos(2\beta)+1)(cos(2\beta)-p_{11})}=\frac{A_1}{cos(2\beta)-p_{11}}=\frac{A_1}{1-p_{11}}\frac{1}{1-h_{11}sin^2(\beta)} \tag{53}$$

Also, in (47) the singularity appears for $p_{44}=-1$ and $C_1=D_1$ so that,

$$\frac{(C_1 cos(2\beta)+D_1)}{(cos(2\beta)-p_{33})(cos(2\beta)-p_{44})}=\frac{C_1(cos(2\beta)+1)}{(cos(2\beta)+1)(cos(2\beta)-p_{33})}=\frac{C_1}{cos(2\beta)-p_{33}}=\frac{C_1}{1-p_3}\frac{1}{1-h_3 sin^2(\beta)} \tag{54}$$

From (53) and (54), it is obvious that the complete elliptic integrals of the third kind $\Pi(h_{22},k_1)$ and $\Pi(h_{44},k_3)$ will vanish.

Thus, (47) begins,

$$\begin{aligned}V_{00}=&\frac{\eta^2 R}{2}J_{00}-\frac{\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{12k_1^3}\{[3(d_2+d_3)k_1^4-(6d_2+8)k_1^2+8]K(k_1)+\\&+[(6d_2+4)k_1^2-8]E(k_1)\}++|\eta|\sqrt{\eta^2+1}\frac{\sqrt{RR_1}}{12k_1^3}\{(16k_1^2-16)K(k_1)+(16-8k_1^2)E(k_1)\}+\\&+\frac{\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{12k_2^3}\{[3(d_{22}+d_{33})k_2^4-(6d_{22}+8)k_2^2+8]K(k_2)+[(6d_{22}+4)k_2^2-8]E(k_2)\}-\\&-|\eta|\sqrt{\eta^2+1}\frac{\sqrt{Rr_1}}{12k_2^3}\{(16k_2^2-16)K(k_2)+(16-8k_2^2)E(k_2)\}-\\&-\frac{k_1\eta\sqrt{\eta^2+1}\sqrt{RR_1}}{4}\frac{A_1}{1-p_{11}}\Pi(h_{11},k_1)+\frac{k_2\eta\sqrt{\eta^2+1}\sqrt{Rr_1}}{4}\frac{C_1}{1-p_{33}}\Pi(h_{33},k_2)\end{aligned} \tag{55}$$

2.4.4. $p_2=p_4=1$

From (47) one can see that case that for $p_{22}=1$ the singular case appears and $A_1=B_1$ so that,

$$\frac{(A_1 cos(2\beta)+B_1)}{(cos(2\beta)-p_{11})(cos(2\beta)-p_{22})}=\frac{A_1(cos(2\beta)-1)}{(cos(2\beta)+1)(cos(2\beta)-p_{11})}=\frac{A_1}{cos(2\beta)-p_{11}}=\frac{A_1}{1-p_{11}}\frac{1}{1-h_{11}sin^2(\beta)} \tag{56}$$

Also, in (47) the singularity appears for $p_{44}=1$ and $C_1=D_1$ so that,

$$\frac{(C_1 cos(2\beta)+D_1)}{(cos(2\beta)-p_{33})(cos(2\beta)-p_{44})}=\frac{C_1(cos(2\beta)+1)}{(cos(2\beta)+1)(cos(2\beta)-p_{33})}=\frac{C_1}{cos(2\beta)-p_{33}}=\frac{C_1}{1-p_3}\frac{1}{1-h_3 sin^2(\beta)} \tag{57}$$

From (56) and (57), it is obvious that the complete elliptic integrals of the third kind $\Pi(h_{22},k_1)$ and $\Pi(h_{44},k_3)$ will vanish.

Thus, for this case we use (55).

Finally, all partial singular cases can be found in the previous discussed cases.

2.5. $\eta = 0 \ (r_1 = R_1)$

This is the special case when the thin conical sheet degenerates to the thin wall solenoid ($r_1 = R_1$). The mutual inductance between the thin wall solenoid and the circular loop is given in [7],

$$M = \frac{\mu_0 N}{(z_2 - z_1)} \sum_{n=1}^{n=2} (-1)^{n-1} \Psi_n \tag{58}$$

where,

$$\Psi_n = -\frac{t_n \sqrt{R_1 R}}{k_n} [K(k_n) - E(k_n)] + \frac{\pi}{4} sgn(t_n) |R_1^2 - R^2| [1 - \Lambda_0(\varepsilon_n, k_n)] \tag{59}$$

$$k_n^2 = \frac{4RR_1}{(R+R_1)^2 + t_n^2}, \qquad t_n = z_n - z_Q, \qquad h = \frac{4RR_1}{(R+R_1)^2 + t_n^2}, \quad \varepsilon_n = arcsin \sqrt{\frac{1-h}{1-k_n^2}}$$

$\Lambda_0(\varepsilon_n, k_n)$ is Heuman's Lambda function, [22].

R_1 is the radii of the thin wall solenoid,

z_1 and z_2 are the axial positions of the thin wall solenoid,

R is the radii of the circular loop,

z_Q is the axial position of the circular loop,

N is the number of turns of the thin wall solenoid.

3. Numerical validation

To verify the validity of the new presented formula for calculating the following set of examples is presented.

Example 1. *Calculate the mutual inductance between the thin conical sheet inductor and the circular loop for which $R_1 = 10$ m, $r_1 = 2$ m, $z_1 = -1$ m, $z_2 = 2$ m, N=1000 and $R = 5$ m, $z_Q = 0$ m.*

This is the case $R_1 > r_1$ $(Figure\ 1)$. *Let us begin with the basic formula (5) where the mutual inductance is given by the double integration.*

The mutual inductance is,

$$M(Double) = 7.401731104798464 \ mH$$

Using the formula (10) with (11) the mutual inductance is obtained by the single integration.

$$M(Single) = 7.401731104798464 \ mH$$

Let us use the semi-analytical formula (27) with (28),

$$M(Semi - analytical) = 7.401731104798464 \ mH$$

Example 2. *Calculate the mutual inductance between the thin conical sheet inductor and the circular loop for which $R_1 = 2$ m, $r_1 = 10$ m, $z_1 = -1$ m, $z_2 = 2$ m, N=1000 and $R = 5$ m, $z_Q = 0$ m.*

This is the case $R_1 < r_1$ $(Figure\ 2)$.

Using the basic formula (43) for the double integration the mutual inductance is,

$$M(Double) = 8.607861541512988 \ mH$$

Using the formula (44) with (45) the mutual inductance is obtained by the single integration,

$$M(Single) = 8.607861541512988 \ mH$$

Finally, let us use the semi-analytical formula (46) with (47) the mutual inductance is,

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$$M(Semi - analytical) = 8.607861541512988 \ \ mH$$

All results are in an excellent agreement.

Example 3. *Calculate the mutual inductance between the thin conical sheet inductor and the circular loop for which R_1 = 3 m, r_1 = 2 m, z_1 = 0 m, z_2 = 0.2 m, N=1000, R = 1 m, and z_Q = 0.2 m.*

This is the case $R_1 > r_1$ $(Figure\ 1)$. The basic formula for the mutual inductance is given by the double integration (5).

The mutual inductance is,

$$M(Double) = \mathbf{0.85591409191908}95\ mH$$

Using the formula (10) with (11) the mutual inductance is obtained by the single integration.

$$M(Single) = \mathbf{0.85591409191908}92\ mH$$

Finally, let us use the semi-analytical formula (27) with (28) for the mutual inductance which gives,

$$M(Semi - analytical) = \mathbf{0.85591409191908}74\ mH$$

All results are in an excellent agreement. The figures that agree are bolded.

Example 4. *Calculate the mutual inductance between the thin conical sheet inductor and the circular loop for which R_1 = 2 m, r_1 = 3 m, z_1 = 0 m, z_2 = 0.2 m, N=1000 and R = 1 m, z_Q = 0.2 m.*

This is the case $r_1 > R_1$ $(Figure\ 2)$. The basic formula for the mutual inductance is given by the double integration (43) which gives,

$$M(Double) = \mathbf{0.8529571443235432}\ mH$$

Using the formulas (44) with (45) the self-inductance is obtained by the single integration.

$$M(Single) = \mathbf{0.8529571443235433}\ mH$$

Let us use the semi-analytical formula (46) with (47),

$$M(Semi - analytical) = \mathbf{0.8529571443235445} mH$$

All results are in an excellent agreement. The figures that agree are bolded.

Example 5. *In this example one calculates the mutual inductance between the thin wall solenoid and the circular loop for which R_1 = r_1 = 2 m, z_1 = 0 m, z_2 = 0.2 m, N=1000 and R = 1 m, z_Q = 0.2 m.*

In this example the thin conical sheet inductor degenerates to the thin wall solenoid ($\eta = 0$).

The exact formula for calculating the mutual inductance between the thin wall solenoid and the circular loop is given by (58) with (59) as follows, [7],

$$M(Analytical) = \mathbf{1.08887170213681}\ \text{mH}$$

Using the formulas for the double and the single integration either for the case $R_1 > r_1$ or $R_1 < r_1$ the mutual inductance is respectively,

$$M(Double) = \mathbf{1.088871}5096342527\ mH$$

$$M(Single) = \mathbf{1.08887170213}67874 mH$$

Equations (5) and (43) for the double integration and (10) with (11) as well as (44) with (45) for the single integration are not singular for $\eta = 0$. It is not case for the semi-analytical solutions (27) with (28) or (46) with (77) when they are singular or indeterminate.

However, one can take, for example, R_1 = 2.0000001 m, and r_1 = 2.0000001 m equations (28) and (47) respectively so that they give,

$$M(Semi - analytical) = \mathbf{1.088871}6126957454 mH$$

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All figures that agree with the exact result, are bolded. Even though the results are in the particularly good agreement it is recommended to use the formula (58) with (59) for calculating the mutual inductance between the thin wall solenoid and the circular loop. This formula can be carefully obtained from (28) and (47) in the limit case when $R_1 \rightarrow r_1$ *and vice versa.*

Example 6. *Calculate the mutual inductance between the thin conical sheet inductor and the circular loop for which* $R_1 = 3\ m$, $r_1 = 2\ m$, $z_1 = 0\ m$, $z_2 = 0.2\ m$, *N=1000 and* $R = 1\ m$, $z_Q = 0.4\ m$.

In this example one finds that $p_1 = p_3 = -1$, *so that it is the singular case* **2.2.1.**

Applying (31) the mutual inductance is,

$$M(Semi-analytical) = \mathbf{0.83546472534096}38\ mH$$

Using the basic formula (5) for the double integration the mutual inductance is,

$$M(Double) = \mathbf{0.83546472534096}52 mH$$

Using the formula (10) with (11) the mutual inductance is obtained by the single integration.

$$M(Single) = \mathbf{0.83546472534096}52\ mH$$

All figures, that agree, are bolded. All results are in an excellent agreement.

Example 7. *Calculate the mutual inductance between the thin conical sheet and the circular loop for which* $R_1 = 3\ m$, $r_1 = 2\ m$, $z_1 = 0\ m$, $z_2 = 0.2\ m$, *N=1000 and* $R = 1\ m$, $z_Q = 0.8\ m$.

In this example one finds that $p_1 = p_3 = 1$, *so that it is the singular case* **2.2.2.**

Applying (31) the mutual inductance is,

$$M(Semi-analytical) = \mathbf{0.7397457129358}785 mH$$

Using the basic formula (5) for the double integration the mutual inductance is,

$$M(Double) = \mathbf{0.739745712935874}\ mH$$

Using the formula (11) the mutual inductance is obtained by the single integration.

$$M(Single) = \mathbf{0.7397457129358}738\ mH$$

All results are in an excellent agreement. The figures that agree are bolded.

All presented results have been obtained by the Mathematica programing. One can see that they are in an excellent agreement so that the potential users can use on of the presented formulas by their preference.

4. Discussion

For the first time in the literature the new formula for calculating the mutual inductance between the thin conical sheet inductor and the filamentary loop is given. Coils are coaxial. Conical coils used in RF /Microwave and mm Wave systems have extremely ultra-wideband electrical responses and are commonly attached to transmission lines to bias microwave devices. These coils have traditionally been designed experimentally without the aid of modern 3D electromagnetic simulators due to the difficult model creation process. The calculation of the presented mutual induction is obtained in the semi-analytical form over the complete elliptical integrals of the first, second and third kind as well as one term that does not have the analytical solution and it must be solved numerically. The kernel function of this integral is continuous function on the interval of integration. All procedures of the calculation are given promptly so that the potential users can easily use them choosing the appropriate formula. The representative numerical examples are given to validate the presented method. The presented method can serve as the base to calculate the mutual inductance between more complex configurations such as the coaxial thin sheet inductor the thin wall solenoid inductor and the two coaxial thin conical inductors. It will be the future work which is the continuation of the new presented method.

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