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Resolution of the $3n+1$ Problem Using Inequality Relation between Indices of 2 and 3

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Abstract: Collatz conjecture states that an integer n reduces to 1 when certain simple operations are applied to it. Mathematically, the Collatz function is written as $f^k(n) = \frac{3^k n+C}{2^z}$, where $z, k, C \geq 1$. Suppose the integer n violates Collatz conjecture by reappearing, then the equation modifies to $2^z n = 3^k n + C$. The article takes an elementary approach to this problem by stating that the inequality $2^z > 3^k$ or the inequality $2^z < 3^k$ holds when n reappears in the sequence. The first inequality leads to the solution set $(n, k, z) = (1, 1, 2)$. The second inequality leads to the solution set $n < 0$. Finally, it is shown that no integer chain exists that does not lead to 1.

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1. Introduction

Collatz conjecture, or the $3n+1$ problem, is a simple arithmetic function applied to positive integers. If the integer is odd, triple it and add one. It is called the odd step. If the integer is even, it is divided by two and is denoted as the even step. It is conjectured that every integer will eventually reach the number 1. Much work has been done to prove or disprove this conjecture [1–4].
The problem is easy to understand, and since it has attracted much attention from the general public and experts alike, the literature is endless. Still, the efforts made to tackle the $3n+1$ problem can generally be categorized under the following headings:

- Experimental or computational method: This method uses computational optimizations to verify Collatz conjecture by checking numbers for convergence [5–7]. Numbers as large as 10^{20} have shown no divergence from the conjecture.
- Arguments based on probability: It is suggested that, on average, the sequence of numbers tends to shrink in size so that divergence do not occur. On average, each odd number is 3/4 of the previous odd integer [8].
- Evaluation of stopping times: Many researchers seem to work on the $3n+1$ problem from this approach [9–13]. In essence, it is sought to prove that the Collatz conjecture yields a number smaller than the starting number.
- Mathematical induction: It is perhaps the most common method to "prove" the Collatz conjecture. The literature involving this particular method seems endless [14,15].

The issue is that the Collatz conjecture is a straightforward arithmetic operation, while the methods used are not. The mismatch is created because the problem has attracted the attention of brilliant people in mathematics who are used to dealing with complex issues with equally complex tools. Therefore, an elementary analysis of the problem might be lacking.

This article takes a rudimentary approach to the Collatz conjecture and treats it as a problem of inequality between indices of 2 and 3. The inequality relation will be turned into equality using variables. The values of these variables will be investigated, and it will be shown that the Collatz conjecture does not need complex analysis.

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2. Prerequisite

Consider that n is an odd integer, and the following Collatz function f is applied to it

$$f(n) = \begin{cases} 3n+1, & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$$

A sequence is formed by performing this operation repeatedly, taking the result at each step as the input for the next. Collatz conjecture states that, for all n, $f^k(n) = 1$ for some non-negative integer k, where the function is applied to n exactly k times. Let the sequence of integers obtained be:

$$1^{st} \textbf{ odd, } z_1 \textbf{ evens, } 2^{nd} \textbf{ odd, } z_2 \textbf{ evens, } \cdots \textbf{, } k^{th} \textbf{ odd, } z_a \text{ evens.}$$

The function $f^k(n)$ is computed to be

$$f^k(n) = \frac{3\left\{\frac{\left\{3\frac{\left\{3\frac{\left\{\frac{3n+1}{2^{z_1}}+1\right\}}{2^{z_2}}+1\right\}}{2^{z_3}}+1\right\}}{\vdots}\right\}+1}{2^{z_a}}$$

$$f^k(n) = \frac{3^k n + 3^{k-1} + 3^{k-2}2^{z_1} + \cdots + 3^0 2^{z_1+z_2+\cdots+z_{a-1}}}{2^{z_1+z_2+\cdots+z_{a-1}+z_a}}$$

$$f^k(n) = \frac{3^k n + C}{2^z} \tag{1}$$

where $z = z_1 + z_2 + \cdots + z_{a-1} + z_a$ and

$$C = 3^{k-1} + 3^{k-2}2^{z_1} + \cdots + 3^0 2^{z_1+z_2+\cdots+z_{a-1}} \tag{2}$$

It is noted that $(z, C) > 0$.

3. Closed chain solutions of $3n + 1$

Suppose the integer n re-appears in the $3n + 1$ sequence and forms a closed chain, i.e., $f^k(n) = n$. The integer n repeats after k odd steps and z even steps, therefore

$$2^z n = 3^k n + C \tag{3}$$

Modify Eq. (3) to the following forms:

$$n = \frac{C}{2^z - 3^k} \tag{4}$$

$$n = \frac{3^k}{2^z} n + \frac{C}{2^z} \tag{5}$$

$$C = n(2^z - 3^k) \tag{6}$$

When the integer n rea-appears, one of the following relation between z and k will be valid:

$$2^z > 3^k$$
$$2^z < 3^k$$
$$2^z = 3^k$$

Equation $2^z = 3^k$ gives trivial solution $(k, z) = (0, 0)$ and hence will be ignored. The remaining two relations are analyzed in the following sections.

3.1. Closed chain solutions when $2^z > 3^k$

Obtain the relation between z and k in the following manner:

$$2^z > 3^k \tag{7}$$
$$(2^3)^{\frac{z}{3}} > (3^2)^{\frac{k}{2}}$$
$$8^{\frac{z}{3}} > (8+1)^{\frac{k}{2}}$$
$$8^{\frac{z}{3}} > 8^{\frac{k}{2}}$$
$$z > \frac{3k}{2} \tag{8}$$

The minimum integer solution to Eq. (8) is $(k, z) = (1, 2)$.
For Eq. (4) to be valid and noting that the difference of integers is also an integer,

$$2^z - 3^k \geq 1$$
$$1 - \frac{3^k}{2^z} \geq \frac{1}{2^z}$$
$$\frac{3^k}{2^z} \leq \frac{3}{4} \tag{9}$$
$$\frac{3^k}{2^z} = \frac{3}{4} - p \tag{10}$$

Where the minimum value of z from $(k, z) = (1, 2)$ is used to find the lowest bound on $3^k/2^z$ and $p \in [0, 3/4)$.
Bounds on $2^z - 3^k$ is found using Eq. (9)

$$\frac{3^k}{2^z} \leq \frac{3}{4}$$
$$2^z - 3^k \geq \frac{2^z}{4}$$
$$2^z - 3^k = \frac{2^z}{4} + q \tag{11}$$

Where $q \geq 0$.
The value of C calculated according to Eq. (11) and (6) is

$$C = n\left\{\frac{2^z}{4} + q\right\} \tag{12}$$

Equations (10) and (12) are substituted in Eq. (5).

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$$n = \left\{\frac{3}{4} - p\right\} n + \left\{\frac{2^z}{4} + q\right\} \frac{n}{2^z}$$
$$p = \frac{q}{2^z} \tag{13}$$

Following two cases arise from Eq. (13).

3.1.1. Case 1: $p = q = 0$

For $p = 0$, Equation (10) gives $(k, z) = (1, 2)$.
Similarly, for $q = 0$, Eq. (13) gives $p = 0$, which in turn gives $(k, z) = (1, 2)$ from Eq. (10).

3.1.2. Case 2: $p = \frac{q}{2^z}$

Substitute the value of q in Eq. (12) to obtain the value of C

$$C = n\left\{\frac{2^z}{4} + 2^z p\right\}$$
$$C = 2^z n\left\{\frac{1}{4} + p\right\} \tag{14}$$

Compare Eq. (2) and (14) by rearranging as follow

$$2^z n\left\{\frac{1}{4} + p\right\} = 3^{k-1} + 3^{k-2}2^{z_1} + \cdots + 3^0 2^{z_1+z_2+\cdots+z_{a-1}}$$
$$2^z n\left\{\frac{1}{4} + p\right\} = 2^z\left\{\frac{3^{k-1}}{2^z} + \frac{3^{k-2}2^{z_1}}{2^z} + \cdots + \frac{3^0 2^{z_1+z_2+\cdots+z_{a-1}}}{2^z}\right\}$$
$$2^z n\left\{\frac{1}{4} + p\right\} = 2^z\left\{\frac{2^{z_1+z_2+\cdots+z_{a-1}}}{2^z} + \frac{3^{k-1}}{2^z} + \frac{3^{k-2}2^{z_1}}{2^z} + \cdots + \frac{3^1 2^{z_1+\cdots z_{a-2}}}{2^z}\right\}$$
$$2^z n\left\{\frac{1}{4} + p\right\} = 2^z\left\{\frac{1}{2^{z_a}} + \textbf{some fraction}\right\} \tag{15}$$

Compare coefficients to obtain the solution set $(n, z_a) = (1, 2)$. Value of p and z can not be determined from Eq. (15), but since $n = 1$, the only valid solution set is $(n, k, z, p, q) = (1, 1, 2, 0, 0)$. Thus, the only possible closed chain when $3n + 1$ series converges is $1, 4, 2, 1$.

3.2. Closed chain solutions when $2^z < 3^k$

Equations (2) and (6) are rewritten along with the relation between z and k

$$3^k > 2^z$$
$$C = \begin{cases} n(2^z - 3^k) \\ 3^{k-1} + 3^{k-2}2^{z_1} + \cdots + 3^0 2^{z_1+z_2+\cdots+z_{a-1}} \end{cases}$$

The first value of C becomes negative when $3^k > 2^z$. The value of C is always positive as it is a sum of positive integers. This is contradictory and not possible. The only solution to the problem of C is obtained when $n < 0$.

Using a negative integer in the $3n + 1$ series is akin to using a positive integer in the $3n - 1$ series. It is already known that $3n - 1$ series has several closed chain solutions in addition to closed

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chain $1, 2, 1$. Thus, the appearance of several closed chains of negative integers in the $3n + 1$ series is consistent with the findings in this section.

4. Do all positive integers reach 1?

Let there exist a number chain that does not converge to 1. Since the only closed chain in the $3n + 1$ series is $1, 4, 2, 1$, this n-chain is an open chain. The n-chain converges to n from infinity and then diverges to infinity.

The n-chain contains all terms of the form $2^i n$ where $i \geq 0$. Further, terms arising from the arithmetic function $3n + 1$ are also part of this chain. Every even integer x in the n-chain is connected to a precursor even number and a precursor odd number (iff $3m + 1 = x$ is possible for some m). Similarly, every odd integer is connected to a precursor even number. The branches that arise out of the n-chain are infinite. In short, the n-chain contains every integer greater than n up to infinity.

However, there should exist no linkage between the n-chain and the 1-chain. Because, if there were some linkage, all the integers in the n-chain would converge to 1 using the said linkage.

It is absurd, as shown in Figure 1, as this means the 1-chain ends abruptly below n. It implies that there exists no integer $2x$ in the n-chain such that $x < n$. Conversely, there exists no x in the 1-chain such that $3x + 1 > n$.

It is concluded that a n-chain that does not converge to 1 is impossible.

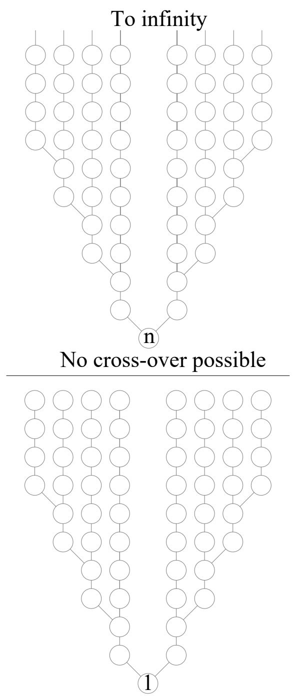

Figure 1. n-chain and 1-chain shown for representation purpose only; may not be factually correct.

5. Conclusions

This article demonstrates the following three properties of the $3n + 1$ series:

- The only closed chain of positive integers is $1, 4, 2, 1$.
- There exists several closed chains of negative integers when the $3n + 1$ series is diverging
- There exist no chain of integers that is not connected to 1.

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