Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle

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Abstract: Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^m, \{\omega_k\}_{k=1}^m)$ be p-Schauder frames for a finite dimensional Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X} \setminus \{0\}$, we show that

$$(1)\qquad \|\theta_f x\|_0^{\frac{1}{p}} \|\theta_g x\|_0^{\frac{1}{q}} \geq \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |f_j(\omega_k)|} \quad \text{and} \quad \|\theta_g x\|_0^{\frac{1}{p}} \|\theta_f x\|_0^{\frac{1}{q}} \geq \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|}.$$

where

$$\theta_f : \mathcal{X} \ni x \mapsto (f_j(x))_{j=1}^n \in \ell^p([n]); \quad \theta_g : \mathcal{X} \ni x \mapsto (g_k(x))_{k=1}^m \in \ell^p([m])$$

and q is the conjugate index of p. We call Inequality (1) as **Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle**. Inequality (1) improves Ricaud-Torrésani uncertainty principle *[IEEE Trans. Inform. Theory, 2013]*. In particular, it improves Elad-Bruckstein uncertainty principle *[IEEE Trans. Inform. Theory, 2002]* and Donoho-Stark uncertainty principle *[SIAM J. Appl. Math., 1989]*.

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1. Introduction

While Heisenberg's uncertainty principle [3] (English translation of original 1927 paper) is one of the greatest inequalities in the first half of the 20 century, Donoho-stark uncertainty principle [1] is one of the greatest inequalities in the second half of the 20 century. For $h \in \mathbb{C}^d$, let $\|h\|_0$ be the number of nonzero entries in h. Let $\hat{}: \mathbb{C}^d \to \mathbb{C}^d$ be the Fourier transform.

Theorem 1.1. *(**Donoho-Stark Uncertainty Principle**) [1] For every $d \in \mathbb{N}$,*

$$(2)\qquad \left(\frac{\|h\|_0 + \|\widehat{h}\|_0}{2}\right)^2 \geq \|h\|_0 \|\widehat{h}\|_0 \geq d, \quad \forall h \in \mathbb{C}^d \setminus \{0\}.$$

In 2002, Elad and Bruckstein generalized Inequality (2) to pairs of orthonormal bases [2]. To state the result we need some notations. Given a collection $\{\tau_j\}_{j=1}^n$ in a finite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we define

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h := (\langle h, \tau_j \rangle)_{j=1}^n \in \mathbb{K}^n.$$

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Theorem 1.2. *(**Elad-Bruckstein Uncertainty Principle**) [2] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$\left(\frac{\|\theta_\tau h\|_0+\|\theta_\omega h\|_0}{2}\right)^2 \geq \|\theta_\tau h\|_0\|\theta_\omega h\|_0 \geq \frac{1}{\max\limits_{1\leq j,k\leq n}|\langle \tau_j,\omega_k\rangle|^2}, \quad \forall h \in \mathcal{H}\setminus\{0\}.$$

Note that Theorem 1.1 follows from Theorem 1.2. In 2013, Ricaud and Torrésani showed that orthonormal bases in Theorem 1.2 can be improved to Parseval frames [6].

Theorem 1.3. *(**Ricaud-Torrésani Uncertainty Principle**) [6] Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$\left(\frac{\|\theta_\tau h\|_0+\|\theta_\omega h\|_0}{2}\right)^2 \geq \|\theta_\tau h\|_0\|\theta_\omega h\|_0 \geq \frac{1}{\max\limits_{1\leq j,k\leq n}|\langle \tau_j,\omega_k\rangle|^2}, \quad \forall h \in \mathcal{H}\setminus\{0\}.$$

In this paper, we derive uncertainty principle for finite dimensional Banach spaces which contains Theorem 1.3 as a particular case.

2. Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle

In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{X}$ denotes a finite dimensional Banach space over $\mathbb{K}$. Identity operator on $\mathcal{X}$ is denoted by $I_\mathcal{X}$. Dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$. Whenever $1<p<\infty$, q denotes conjugate index of p. For $d\in\mathbb{N}$, standard finite dimensional Banach space $\mathbb{K}^d$ over $\mathbb{K}$ equipped with standard $\|\cdot\|_p$ norm is denoted by $\ell^p([d])$. Canonical basis for $\mathbb{K}^d$ is denoted by $\{\delta_j\}_{j=1}^d$ and $\{\zeta_j\}_{j=1}^d$ be the coordinate functionals associated with $\{\delta_j\}_{j=1}^d$. We need the following variant of p-approximate Schauder frames defined by Krishna and Johnson in [5].

Definition 2.1. *Let $\mathcal{X}$ be a finite dimensional Banach space over $\mathbb{K}$. Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathcal{X}$ and $\{f_j\}_{j=1}^n$ be a sequence in $\mathcal{X}^*$. The pair $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is said to be a **p-Schauder frame** $(1<p<\infty)$ for $\mathcal{X}$ if the following conditions hold.*

(i) *For every $x\in\mathcal{X}$,*

$$\|x\|^p=\sum_{j=1}^n |f_j(x)|^p.$$

(ii) *For every $x\in\mathcal{X}$,*

$$x=\sum_{j=1}^n f_j(x)\tau_j.$$

We easily see that condition (i) in Definition 2.1 says that the map

$$\theta_f:\mathcal{X}\ni x\mapsto (f_j(x))_{j=1}^n\in \ell^p([n])$$

is a linear isometry. Like Holub's characterization of frames for Hilbert spaces [4], following theorem characterizes p-Schauder frames.

Theorem 2.2. *A pair $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is a p-Schauder frame for $\mathcal{X}$ if and only if*

$$f_j=\zeta_j U, \quad \tau_j=V\delta_j, \quad \forall j=1,\dots,n,$$

where $U:\mathcal{X}\to\ell^p([n])$, $V:\ell^p([n])\to\mathcal{X}$ are linear operators such that $VU=I_\mathcal{X}$ and U is an isometry.

Following is the crucial result of this paper.

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Theorem 2.3. *(Functional Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle) Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^m, \{\omega_k\}_{k=1}^m)$ be p-Schauder frames for a Banach space $\mathcal{X}$. Then for every $x \in \mathcal{X}\setminus\{0\}$, we have*

$$\|\theta_f x\|_0^{\frac{1}{p}}\|\theta_g x\|_0^{\frac{1}{q}} \geq \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|} \quad \text{and} \quad \|\theta_g x\|_0^{\frac{1}{p}}\|\theta_f x\|_0^{\frac{1}{q}} \geq \frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m}|g_k(\tau_j)|}. \tag{3}$$

Proof. Let $x \in \mathcal{X}\setminus\{0\}$ and q be the conjugate index of p. First using θ_f is an isometry and later using θ_g is an isometry, we get

$$\begin{aligned}
\|x\|^p = \|\theta_f x\|^p &= \sum_{j=1}^n |f_j(x)|^p = \sum_{j\in \operatorname{supp}(\theta_f x)} |f_j(x)|^p \\
&= \sum_{j\in \operatorname{supp}(\theta_f x)} \left|f_j\left(\sum_{k=1}^m g_k(x)\omega_k\right)\right|^p = \sum_{j\in \operatorname{supp}(\theta_f x)} \left|\sum_{k=1}^m g_k(x)f_j(\omega_k)\right|^p \\
&= \sum_{j\in \operatorname{supp}(\theta_f x)} \left|\sum_{k\in \operatorname{supp}(\theta_g x)} g_k(x)f_j(\omega_k)\right|^p \leq \sum_{j\in \operatorname{supp}(\theta_f x)} \left(\sum_{k\in \operatorname{supp}(\theta_g x)} |g_k(x)f_j(\omega_k)|\right)^p \\
&\leq \left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)^p \sum_{j\in \operatorname{supp}(\theta_f x)} \left(\sum_{k\in \operatorname{supp}(\theta_g x)} |g_k(x)|\right)^p \\
&= \left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)^p \|\theta_f x\|_0 \left(\sum_{k\in \operatorname{supp}(\theta_g x)} |g_k(x)|\right)^p \\
&\leq \left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)^p \|\theta_f x\|_0 \left(\sum_{k\in \operatorname{supp}(\theta_g x)} |g_k(x)|^p\right)^{\frac{p}{p}} \left(\sum_{k\in \operatorname{supp}(\theta_g x)} 1^q\right)^{\frac{p}{q}} \\
&= \left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)^p \|\theta_f x\|_0 \|\theta_g x\|^p \|\theta_g x\|_0^{\frac{p}{q}} \\
&= \left(\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|\right)^p \|\theta_f x\|_0 \|x\|^p \|\theta_g x\|_0^{\frac{p}{q}}.
\end{aligned}$$

Therefore

$$\frac{1}{\displaystyle\max_{1\leq j\leq n, 1\leq k\leq m}|f_j(\omega_k)|} \leq \|\theta_f x\|_0^{\frac{1}{p}}\|\theta_g x\|_0^{\frac{1}{q}}.$$

On the other way, first using θ_g is an isometry and θ_f is an isometry, we get

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$$
\begin{aligned}
\|x\|^p &= \|\theta_g x\|^p = \sum_{k=1}^{m} |g_k(x)|^p = \sum_{k\in \operatorname{supp}(\theta_g x)} |g_k(x)|^p \\
&= \sum_{k\in \operatorname{supp}(\theta_g x)} \left| g_k\left(\sum_{j=1}^{n} f_j(x)\tau_j\right)\right|^p = \sum_{k\in \operatorname{supp}(\theta_g x)} \left|\sum_{j=1}^{n} f_j(x) g_k(\tau_j)\right|^p \\
&= \sum_{k\in \operatorname{supp}(\theta_g x)} \left| \sum_{j\in \operatorname{supp}(\theta_f x)} f_j(x) g_k(\tau_j)\right|^p \leq \sum_{k\in \operatorname{supp}(\theta_g x)} \left(\sum_{j\in \operatorname{supp}(\theta_f x)} |f_j(x) g_k(\tau_j)|\right)^p \\
&\leq \left(\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|\right)^p \sum_{k\in \operatorname{supp}(\theta_g x)} \left(\sum_{j\in \operatorname{supp}(\theta_f x)} |f_j(x)|\right)^p \\
&= \left(\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|\right)^p \|\theta_g x\|_0 \left(\sum_{j\in \operatorname{supp}(\theta_f x)} |f_j(x)|\right)^p \\
&\leq \left(\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|\right)^p \|\theta_g x\|_0 \left(\sum_{j\in \operatorname{supp}(\theta_f x)} |f_j(x)|^p\right)^{\frac{p}{p}} \left(\sum_{j\in \operatorname{supp}(\theta_f x)} 1^q\right)^{\frac{p}{q}} \\
&= \left(\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|\right)^p \|\theta_g x\|_0 \|\theta_f x\|^p \|\theta_f x\|_0^{\frac{p}{q}} \\
&= \left(\max_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|\right)^p \|\theta_g x\|_0 \|x\|^p \|\theta_f x\|_0^{\frac{p}{q}}
\end{aligned}
$$

Therefore

$$
\frac{1}{\max\limits_{1\leq j\leq n, 1\leq k\leq m} |g_k(\tau_j)|} \leq \|\theta_g x\|_0^{\frac{1}{p}} \|\theta_f x\|_0^{\frac{1}{q}}.
$$

□

Corollary 2.4. *Theorem 1.3 follows from Theorem 2.3.*

Proof. Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Define

$$
f_j : \mathcal{H} \ni h \mapsto \langle h, \tau_j\rangle \in \mathbb{K}; \quad g_j : \mathcal{H} \ni h \mapsto \langle h, \omega_j\rangle \in \mathbb{K}, \quad \forall 1 \leq j \leq n.
$$

Then $p = q = 2$, $\theta_\tau = \theta_f$, $\theta_\omega = \theta_g$ and

$$
|f_j(\omega_k)| = |\langle \omega_k, \tau_j\rangle|, \quad \forall 1 \leq j, k \leq n.
$$

□

Theorem 2.3 brings the following question.

Question 2.5. *Given p, m, n and a Banach space $\mathcal{X}$, for which pairs of p-Schauder frames $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^m, \{\omega_k\}_{k=1}^m)$, we have equality in Inequality (3)?*

References

[1] David L. Donoho and Philip B. Stark. Uncertainty principles and signal recovery. *SIAM J. Appl. Math.*, 49(3):906–931, 1989.

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[2] Michael Elad and Alfred M. Bruckstein. A generalized uncertainty principle and sparse representation in pairs of bases. *IEEE Trans. Inform. Theory*, 48(9):2558–2567, 2002.

[3] W. Heisenberg. The physical content of quantum kinematics and mechanics. In John Archibald Wheeler and Wojciech Hubert Zurek, editors, *Quantum Theory and Measurement*, Princeton Series in Physics, pages 62–84. Princeton University Press, Princeton, NJ, 1983.

[4] James R. Holub. Pre-frame operators, Besselian frames, and near-Riesz bases in Hilbert spaces. *Proc. Amer. Math. Soc.*, 122(3):779–785, 1994.

[5] K. Mahesh Krishna and P. Sam Johnson. Towards characterizations of approximate Schauder frame and its duals for Banach spaces. *J. Pseudo-Differ. Oper. Appl.*, 12(1):Paper No. 9, 13, 2021.

[6] Benjamin Ricaud and Bruno Torrésani. Refined support and entropic uncertainty inequalities. *IEEE Trans. Inform. Theory*, 59(7):4272–4279, 2013.