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Discrete Analogue of Fishburn's Fractional-Order Stochastic Dominance

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Abstract: A stochastic dominance (SD) relation can be defined by two different perspectives: One from the view of distributions, and the other one from the view of expected utilities. In early days, Fishburn investigated SD from the view of distributions and we refer this perspective as Fishburn's SD. One of his many results was the development of fractional-order SD for continuous distributions. However, discrete fractional-order SD may not be generalized directly since some properties of fractional calculus do not have a discrete counterpart. In this paper, we develop a discrete analogue of fractional-order SD from the view of distributions. We generalize the order of SD by Lizama's fractional delta operator, show the preservation of SD hierarchy, and formulate the utility classes that are congruent with our SD relations. This work brings a message that some results of discrete SD cannot be generalized directly from continuous SD. We characterize the difference between discrete and continuous fractional-order SD, as well as the way to handle them.

Keywords: fractional-order stochastic dominance; discrete stochastic dominance; discrete utility; fractional sum

1. Introduction

Stochastic dominance (SD) is a tool to analyze risky decisions under uncertainty [1,2], analyze risk attitudes [3,4], formulate robust utility maximization [5,6], etc. It is a kind of stochastic ordering on two probability distributions that can be viewed in two different perspectives. The first perspective is solely based on the probability distributions that are usually associated with risky investments or portfolios. The second one is based on the expected utilities of the probability distributions in the sense of von Neumann-Morgenstern utility [7], i.e., the higher the expected utility the better the decision. Although many applications use first- and second-order SD only, higher-order risk attitudes, e.g., mixed risk aversion [8], have their economic implications and they can be analyzed by higher-order SD [9].

The two perspectives are equivalent for first- and second-order SD only, so extra constraints are imposed on either of the perspectives to formulate an equivalent definition [10]. The convention nowadays is to impose restrictions on the distributions so that higher-order SD works for all utility functions that are monotone (in alternative signs) up to certain order of derivatives. More specifically, it requires a consistent numerical order on the end-points of the higher-order cumulative distributions. One reason is that the exact formulation of the utility to be used may not be known in practice when analyzing risks, so we desire a definition that works for all utilities with a simple assumption. Beyond risk management, there are applications that only concern some specific utilities only, e.g., in the analysis of transmission schemes in batched network codes [11,12]. Therefore, it is also a practical problem to impose constraints on the utility classes instead of the distributions.

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In early days, researchers also investigated the other way that considers a specific class of utilities so that higher-order SD works without restrictions on the probability distributions. The class of utilities is usually a convex cone in function space in practice [13]. Fishburn is one of the scholars that investigated SD from this point of view, and gave many cornerstone results, e.g., the relation between SD and moments [14], that inspired many later works. In this paper, we call the definition of SD that imposes constraints on the utility classes Fishburn's SD.

One of Fishburn's important result is the development of the continuum of SD rules that extends the orders of SD from positive integers to real numbers no less than 1 [15,16]. Fractional-order SD is currently still an active research topic, e.g., [17–19], as integral-order SD is simply too coarse in some applications. For example, first-order SD models the non-satiable individuals with increasing utilities, and second-order SD models the non-satiable and risk-averse individuals with increasing concave utilities, but the cases between the substantial gap of the two orders are not captured. Just like what happens in fractional calculus, Fishburn's fractional-order SD, which is based on the Riemann-Liouville integral, is only one of the many ways to formulate fractional-order SD. The conventional restriction on the end-points of non-integral-order cumulative distributions is not well-defined, so we need to develop the relation between fractional-order SD and utilities.

On the other hand, most literature considered continuous distributions for SD. However, discrete distributions are often used in experimental tests due to the fact that the number of empirical data points are finite [20–22]. This also happens in pedagogical presentations, e.g., [23,24], because discretized small examples are easier to be understood. Most (integral-order) SD results on continuous distributions also work for discrete distributions with continuous utilities, as one can unify the cases by using Lebesgue integrals or Riemann-Stieltjes integrals [25]. For discrete utilities, these results still work but we need to cope with the discreteness directly since finite differences are not the same as derivatives for real functions [26].

When we come to discrete fractional-order SD, it is a different story. For example, Fishburn's (continuous) fractional-order SD (on bounded distributions) is based on the observation that the definition of higher-order SD matches with the form in the Cauchy formula for repeated integration. However, there is no such elegant analogue for summations. In other words, the form of the discrete analogue is not the same as the continuous case.

In this paper, we develop a discrete analogue of Fishburn's continuous fractional-order SD on bounded distributions, i.e., we follow Fishburn's perspective that we do not impose restrictions on the probability distributions. In Section 2, we give a brief introduction on SD, and also discuss the fundamental reason on the disagreement about the conventional and Fishburn's definitions. An interesting consequence is that, for any two bounded discrete distributions, one can dominate the other in some integral-order Fishburn's SD, which is not valid for conventional SD. Then, we present our main results in Section 3. First, we rewrite the definition of discrete SD into a single summation in terms of factorials. By doing so, we can generalize it into fractional-order SD that is in terms of Lizama's fractional delta operator [27] by replacing the factorials into gamma functions. Under this definition, we show that the hierarchy of SD is preserved, i.e., a smaller-order SD implies a larger-order SD. After that, we define a class of utility for each order so that it is congruent with our fraction-order SD definition. At last, we show that our utility classes for integral-orders are consistent with the traditional utilities that are monotone (in alternative signs) up to the corresponding order. Finally, we conclude this paper in Section 4.

2. Integral-Order Stochastic Dominance

Stochastic dominance (SD) is a stochastic ordering for two probability distributions. In this section, we give the definitions of SD in both perspectives from the distributions and the utilities. Then, we discuss the fundamental reason on the disagreement about the conventional and Fishburn's definitions.

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Throughout this paper, we adopt the following set notations. Let $\mathbb{Z}^+$ be the set of positive integers. Define $\mathbb{N}_c := \{0, 1, \ldots, c\}$ for any non-negative integer c. Denote by $\mathbb{R}$ and $\mathbb{R}^+$ the sets of real numbers and positive real numbers respectively.

2.1. First- and Second-Order Stochastic Dominance

Let F and G be two random variables, and $f(x)$ and $g(x)$ be their probability density functions respectively. Define

$$F_{n+1}(x) = \int_{-\infty}^{x} F_n(t)dt, \qquad G_{n+1}(x) = \int_{-\infty}^{x} G_n(t)dt$$

for all $n \in \mathbb{Z}^+ \cup \{0\}$, where $F_0 = f(x)$ and $G_0 = g(x)$. For two discrete random variables F and G, we can use a similar definition by replacing the integrals into summations [25,26], i.e.,

$$F_{n+1}(x) = \sum_{i=-\infty}^{x} F_n(i), \qquad G_{n+1}(x) = \sum_{i=-\infty}^{x} G_n(i).$$

Notice that F_1 and G_1 are the cumulative distributions of F and G respectively. We can obtain F_2 by accumulating the values in F_1, which is the second-order cumulative distribution. In a similar way, we can obtain higher-order cumulative distributions of F and G.

Definition 1 (First-Order SD)**.** *F dominates G in the first order if and only if $F_1(x) \leq G_1(x)$ for all x. The dominance is strict if and only if the strict inequality holds for some x.*

From this definition, we can see that the cumulative distribution of F is always no larger than the cumulative distribution of G. Therefore, the expectation of F is no smaller than that of G. If we view x as the reward and $f(x), g(x)$ as the chances of getting the reward, we prefer a larger expected value on the reward. That is, we prefer F more than G, which is one of the basic explanations of F dominates G in the first order. As we will discuss below, first-order SD has a stronger meaning than the above explanation. However, this explanation is one of the most useful properties that can be applied in other fields, e.g., in modeling communication channels with packet loss [12].

In a more general sense, von Neumann-Morgenstern utility is an extension of the theory of consumer preferences. In this theory, the optimal decision is the one that maximizes the expected utility derived from the choice made, i.e., the distribution that maximizes the expected utility. The simplest case is a monotonically increasing utility, which means that we prefer a largest reward.

In this perspective, we have another definition of first-order SD. Denote by $\mathbb{E}$ the expectation operator. Let U_1 be the set of all monotonically increasing utilities.

Definition 2 (First-Order SD (Utility))**.** *F dominates G in the first order if and only if $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ for all $u \in U_1$. The dominance is strict if and only if the strict inequality holds for at least one $u \in U_1$.*

In fact, the two definitions are equivalent. We will see that in the discussion later. Notices that not all distributions may have a first-order SD relation.

We may further refine the relation to capture more information such as which one has a better mean while involving less risk. This is the tendency to prefer outcomes with low uncertainty than those with high uncertainty, known as risk aversion. This relation is the second-order SD. Similarly, we have two equivalent definitions. Let U_2 be the set of all monotonically increasing concave utilities.

Definition 3 (Second-Order SD)**.** *F dominates G in the second order if and only if $F_2(x) \leq G_2(x)$ for all x. The dominance is strict if and only if the strict inequality holds for some x.*

Definition 4 (Second-Order SD (Utility))**.** *F dominates G in the second order if and only if $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ for all $u \in U_2$. The dominance is strict if and only if the strict inequality holds for at least one $u \in U_2$.*

2.2. Higher-Order Stochastic Dominance

In a similar manner, we can define higher-order SD. First, we define the SD used by Fishburn in his works, which is a direct extension of the above definition.

Definition 5 (Fishburn's nth-Order SD)**.** *F dominates G in the nth order if and only if $F_n(x) \leq G_n(x)$ for all x. The dominance is strict if and only if the strict inequality holds for some x.*

For the utility-based definition, let

$$U_n = \left\{ u(x) \colon (-1)^{j+1} \frac{d^j u(x)}{dx^j} \geq 0, j \in \{1, 2, \ldots, n\} \right\}$$

be the set of all increasing utilities that odd derivatives are non-negative and even derivatives are non-positive, up to the nth derivative.

Definition 6 (nth-Order SD (Utility))**.** *F dominates G in the nth order if and only if $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ for all $u \in U_n$. The dominance is strict if and only if the strict inequality holds for at least one $u \in U_n$.*

However, the two definitions are not equivalent. To see the reason, we consider the following evaluation of bounded continuous distributions [25]. The evaluation for unbounded distributions can be found in [28,29]. Let f and g be two distributions whose supports are bounded by $[a, b]$. Note that $F_1(b) = G_1(b) = 1$ and $F_n(a) = G_n(a) = 0$ for all n.

$$\begin{aligned}
\mathbb{E}_F[u] - \mathbb{E}_G[u] &= \int_a^b u(x)(f(x) - g(x))dx \\
&= [(F_1(x) - G_1(x))u(x)]_a^b - \int_a^b (F_1(x) - G_1(x))u'(x)dx \\
&= \int_a^b (G_1(x) - F_1(x))u'(x)dx \qquad (1) \\
&= [(G_2(x) - F_2(x))u'(x)]_a^b - \int_a^b (G_2(x) - F_2(x))u''(x)dx \\
&= u'(b)(G_2(b) - F_2(b)) - \int_a^b (G_2(x) - F_2(x))u''(x)dx \qquad (2) \\
&= u'(b)(G_2(b) - F_2(b)) - u''(b)(G_3(b) - F_3(b)) + \int_a^b (G_3(x) - F_3(x))u'''(x)dx \qquad (3) \\
&= \cdots \\
&= \sum_{j=1}^{n-1} (-1)^{j+1} \frac{d^j u(x)}{dx^j}\bigg|_{x=b} (G_{j+1}(b) - F_{j+1}(b)) + (-1)^{n+1} \int_a^b (G_n(x) - F_n(x)) \frac{d^n u(x)}{dx^n} dx. \qquad (4)
\end{aligned}$$

For first-order SD, the evaluation stops at Equation (1). Because $G_1(x) \geq F_1(x)$ and $u'(x) \geq 0$ for all x, we know that $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$. For second-order SD, the evaluation stops at Equation (2). This time, we have $G_2(x) \geq F_2(x)$, $u'(x) \geq 0$ and $u''(x) \leq 0$ for all x, we know that $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$.

The issue arises when we consider third-order SD [10]. The evaluation stops at Equation (3). We have $G_3(x) \geq F_3(x)$, $u'(x) \geq 0$, $u''(x) \leq 0$, and $u'''(x) \geq 0$ for all x. However, we cannot conclude whether $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ or not due to the term $u'(b)(G_2(b) - F_2(b))$. Similarly, for nth-order SD, we have the problematic terms $u'(b)(G_2(b) - F_2(b))$, $-u''(b)(G_3(b) - F_3(b)), \ldots, (-1)^n \frac{d^{n-1}u(x)}{dx^{n-1}}\Big|_{x=b}(G_n(b) - F_n(b))$.

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In risk management, we concern about all possible utilities in U_n. To ensure that $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$, the convention is to assume $G_j(b) - F_j(b) \geq 0$ for all $j \in \{1, 2, \ldots, n-1\}$, so that all problematic terms become non-negative. This leads to the conventional definition of nth-order SD below, which is equivalent to the utility-based definition.

Definition 7 (Conventional nth-Order SD)**.** *F dominates G in the nth order if and only if*

1. $F_n(x) \leq G_n(x)$ *for all* x*; and*
2. $F_j(b) \leq G_j(b)$ *for all* $j \in \{1, 2, \ldots, n-1\}$.

The dominance is strict if and only if there is at least one strict inequality.

Note that this is a compromise to impose the restrictions on the distributions to fulfill the utility definition. If we start from Fishburn's SD, then we need to restrict the choices of utilities to a specific utility class in the corresponding equivalent definition. The choice of definition depends on the application. For example, if we only concern whether the expectation of F is no less than that of G, then we do not need to use the stronger definition that works for utilities other than the identity function, thus Fishburn's definition is sufficient.

From the evaluation, we can see that first-order SD implies second-order SD, and so on and so forth. This SD hierarchy, together with the equivalent of utility-based definition and the conventional definition, are valid for discrete distributions, no matter if the utility is continuous or discrete. For discrete utilities on $\{x_1, x_2, \ldots, x_m\}$ in ascending order [26], the characterization becomes

$$U_n = \left\{ u(x) \colon (-1)^{j+1} u^{(j)}(i) \geq 0, i \in \{1, 2, \ldots, m-j\}, j \in \{1, 2, \ldots, n\} \right\}.$$

where

$$u^{(k)}(i) = \begin{cases} \frac{u^{(k-1)}(x_{i+1}) - u^{(k-1)}(x_i)}{x_{i+1} - x_i} & \text{if } k = 1, 2, \ldots, m-1, \text{ and } i = 1, 2, \ldots, m-k, \\ u(x_i) & \text{if } k = 0 \text{ and } i = 1, 2, \ldots, m. \end{cases}$$

When the domain is a sequence of consecutive integers, we can simplify $u^{(k)}(i)$ as $\Delta^n u(i)$, where Δ^n is the nth-order forward difference operator defined as

$$\Delta^n u(i) = \begin{cases} \Delta^{n-1} u(i+1) - \Delta^{n-1} u(i) & \text{if } n > 1 \\ u(i+1) - u(i) & \text{if } n = 1. \end{cases}$$

From the definition, we can see that when the order is too high, the nth-order forward difference is undefined if the utility has a bounded domain. There is no such issue when the domain is not right-bounded.

Note that it is not guaranteed for the existence of n such that either F dominates G or G dominates F in some integral order in the conventional definition, even when the definition is extended to infinite-order [30]. The necessary and sufficient condition for such existence in conventional SD was shown in [30] that applies negative moments and the Bernstein's theorem on the totally monotone utilities. Surprisingly, such n exists for bounded discrete random variables in the sense of Fishburn's SD. For simplicity, we write $F \succeq_n G$ to denote F dominates G in the nth order using the definition of Fishburn's SD. When the dominance is strict, we write $F \succ G$.

Theorem 1. *For any two bounded discrete random variables* F, G*, there exists an* $n \in \mathbb{Z}^+$ *such that either* $F \succeq_n G$ *or* $G \succeq_n F$*. Furthermore, if the two distributions are distinct, then there exists an* $n \in \mathbb{Z}^+$ *such that* $F \succ_n G$ *if* $f(\lambda) < g(\lambda)$*, or* $G \succ_n F$ *if* $g(\lambda) < f(\lambda)$*, where* λ *is the smallest value such that* $f(\lambda) \neq g(\lambda)$.

Proof. See Section A. □

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The idea of the proof is that, if $F_j(i) > G_j(i)$ for some $i > \lambda$ and some $j > 1$, we can keep accumulating the sum by increasing the order of summations. Eventually, we will reach $F_{j'}(i) < G_{j'}(i)$ for some $j' > j$ since $f(\lambda) < g(\lambda)$.

3. Discrete Fishburn's Fractional-Order Stochastic Dominance

We assume F and G are $\{0, 1, \ldots, b\}$-valued random variables in this section.

3.1. Definition

To begin with, we first revisit how Fishburn defined his fractional-order SD in [15] for continuous distributions on $[0, b]$. By expanding the recursive relation, we have the nth repeated integral

$$F_n(x) = \int_0^x F_{n-1}(t)dt = \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_n)dt_n \cdots dt_2 dt_1.$$

This form matches with the Cauchy formula for repeated integration, which gives

$$F_n(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)dt, \tag{5}$$

where the order n is now a value in the formula. By writing the factorial in gamma function, i.e., $(n-1)! = \Gamma(n)$, we have the Riemann-Liouville integral

$$F_n(x) = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t)dt,$$

which extends the order of SD into a continuum. To distinguish fractional-order SD from integral-order, we use $\alpha \in [1, \infty)$ as the notation for the order, i.e., αth-order SD.

Now, we come to our problem on discrete distributions. In a similar manner, we have

$$F_n(x) = \sum_{i_n=0}^{x} \sum_{i_{n-1}=0}^{i_n} \cdots \sum_{i_1=0}^{i_2} f(i_1).$$

However, there is no discrete analogue of the Cauchy formula for repeated integration that gives a similar form as Equation (5) in a single summation.

Theorem 2. *$F_n(x) = \sum_{i=0}^{x} \binom{n-1+x-i}{n-1} f(i)$ for $x = 0, 1, \ldots, b$ and $n \in \mathbb{Z}^+$.*

Proof. We expand the recursive form of $F_n(x)$ and group the terms as follows:

$$F_n(x) = \sum_{i_n=0}^{x} \sum_{i_{n-1}=0}^{i_n} \cdots \sum_{i_1=0}^{i_2} f(i_1) = \sum_{i=0}^{x} N(n, x, i) f(i),$$

where

$$N(n, x, i) = |\{(i_n, i_{n-1}, \ldots, i_2, i_1) \in \mathbb{Z}^+ : x \geq i_n \geq i_{n-1} \geq \ldots \geq i_2 \geq i_1 = i\}|.$$

Note that in the definition of $N(n, x, i)$, i_1 is fixed as $i_1 = i$, so $N(n, x, i)$ is equivalent to a stars and bars counting problem with $x - i$ stars and $n - 1$ bars. That is, we have $N(n, x, i) = \binom{n-1+x-i}{n-1}$. □

We can see that the form for the discrete case is totally different from the Riemann-Liouville integral. To extend the form into a continuum, consider

$$\binom{n-1+x-i}{n-1} = \frac{(n-1+x-i)!}{(n-1)!(x-i)!} = \frac{\Gamma(n+x-i)}{\Gamma(n)\Gamma(1+x-i)}.$$

Define
$$k^{\alpha}(j) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)}.$$
Then, the αth-order sum becomes
$$F_{\alpha}(x) = \sum_{j=0}^{x} k^{\alpha}(x-j)f(j), \tag{6}$$
which has the same form as Lizama's fractional delta operator [27]. This discrete fractional sum notation can be obtained via translation from other notations such as [31–33].

After obtaining the definition in Equation (6), we need to show its consistency with the properties of integral-order SD. That is, we need to show

- The preservation of SD hierarchy: For any $\nu > 0$ and $\alpha \in [1,\infty)$, $F \succeq_{\alpha} G$ implies $F \succeq_{\alpha+\nu} G$;
- Equivalent definition by utilities: Find the utility classes that are congruent with the αth-order SD; and
- Monotonicity of utility classes: For integral nth-order SD, show that every u in the utility class we found satisfies $(-1)^{p+1}\Delta^{p}u(i) \geq 0$ for all $p = 1,2,\ldots,n$.

3.2. Stochastic Dominance Hierarchy

The proof for showing the preservation of SD hierarchy applies a few mathematical tools. The first one is Abel's lemma, which is the discrete version of "integration by parts". Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two real sequences. Abel's lemma states that
$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n-1}\left(\sum_{j=1}^{i} a_j\right)(b_i - b_{i+1}) + \left(\sum_{j=1}^{n} a_j\right)b_n.$$

Next, we also use the beta function $B(x,y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, which has the property that
$$B(x+1,y) = \frac{xB(x,y)}{x+y}.$$
Also, when $x, y > 0$, we have $\Gamma(x) > 0$ and $\Gamma(y) > 0$, thus $B(x,y) > 0$.

Theorem 3. *For any $\nu > 0$ and $\alpha \in [1,\infty)$,*

- *$F \succeq_{\alpha} G$ implies $F \succeq_{\alpha+\nu} G$; and*
- *$F \succ_{\alpha} G$ implies $F \succ_{\alpha+\nu} G$.*

Proof. First, $F \succeq_{\alpha} G$ implies that $G_{\alpha}(n) - F_{\alpha}(n) \geq 0$ for all $n \in \mathbb{N}_b$. Consider $n > 0$. For any $j \in \mathbb{N}_{n-1} \subset \mathbb{N}_b$, we have $n > n-1 \geq j$, so we know that
$$\frac{\nu}{\alpha+\nu+n-j-1}\left(\prod_{k=1}^{j}\frac{\alpha+n-k}{\alpha+\nu+n-k}\right) > 0.$$
Therefore, we have
$$\sum_{j=0}^{n-1}(G_{\alpha}(j) - F_{\alpha}(j))\frac{\nu}{\alpha+\nu+n-j-1}\left(\prod_{k=1}^{j}\frac{\alpha+n-k}{\alpha+\nu+n-k}\right) \geq 0$$
$$\sum_{j=0}^{n-1}(G_{\alpha}(j) - F_{\alpha}(j))\left(\prod_{k=1}^{j}\frac{\alpha+n-k}{\alpha+\nu+n-k} - \prod_{k=1}^{j+1}\frac{\alpha+n-k}{\alpha+\nu+n-k}\right) \geq 0. \tag{7}$$

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On the other hand, we know that

$$(G_\alpha(n) - F_\alpha(n)) \prod_{k=1}^{n} \frac{\alpha+n-k}{\alpha+\nu+n-k} \geq 0. \tag{8}$$

By summing up Section 3.2 and Equation (8), and expressing

$$G_\alpha(j) - F_\alpha(j) = \sum_{i=0}^{j} k^\alpha(n-i)(g(i) - f(i))$$

for all $j \in \mathbb{N}_n$, we obtain

$$\sum_{j=0}^{n-1} \left(\sum_{i=0}^{j} k^\alpha(n-i)(g(i) - f(i)) \right) \left(\prod_{k=1}^{j} \frac{\alpha+n-k}{\alpha+\nu+n-k} - \prod_{k=1}^{j+1} \frac{\alpha+n-k}{\alpha+\nu+n-k} \right) + \left(\sum_{i=0}^{n} k^\alpha(n-i)(g(i) - f(i)) \right) \prod_{k=1}^{n} \frac{\alpha+n-k}{\alpha+\nu+n-k} \geq 0. \tag{9}$$

By Abel's lemma, we have

$$\sum_{j=0}^{n} k^\alpha(n-j)(g(j) - f(j)) \prod_{k=1}^{j} \frac{\alpha+n-k}{\alpha+\nu+n-k} \geq 0.$$

Next, note that $\frac{B(\alpha,\nu)}{B(\alpha+n,\nu)} > 0$, so we have

$$\frac{B(\alpha,\nu)}{B(\alpha+n,\nu)} \sum_{j=0}^{n} k^\alpha(n-j)(g(j) - f(j)) \prod_{k=1}^{j} \frac{\alpha+n-k}{\alpha+\nu+n-k} \geq 0.$$

By the property of beta function, we know that

$$B(\alpha+n,\nu) = B(\alpha+n-1,\nu)\frac{\alpha+n-1}{\alpha+\nu+n-1} = \ldots = B(\alpha+n-j,\nu) \prod_{k=1}^{j} \frac{\alpha+n-k}{\alpha+\nu+n-k}.$$

Then, we have

$$B(\alpha,\nu) \sum_{j=0}^{n} \frac{k^\alpha(n-j)(g(j) - f(j))}{B(\alpha+n-j,\nu)} \geq 0$$

$$\sum_{j=0}^{n} \frac{\Gamma(\alpha)\Gamma(\nu)}{\Gamma(\alpha+\nu)} \frac{\Gamma(\alpha+n-j)}{\Gamma(\alpha)\Gamma(n-j+1)} \frac{g(j) - f(j)}{B(\alpha+n-j,\nu)} \geq 0.$$

Finally, note that

$$B(\alpha+n-j,\nu) = \frac{\Gamma(\alpha+n-j)\Gamma(\nu)}{\Gamma(\alpha+\nu+n-j)},$$

thus we have

$$\sum_{j=0}^{n} \frac{\Gamma(\alpha+\nu+n-j)}{\Gamma(\alpha+\nu)\Gamma(n-j+1)}(g(j) - f(j)) \geq 0$$

$$\sum_{j=0}^{n} k^{\alpha+\nu}(g(j) - f(j)) \geq 0.$$

That is, we have $G_{\alpha+\nu}(n) - F_{\alpha+\nu}(n) \geq 0$ for all $n \in \mathbb{N}_b$, which implies that $F \succeq_{\alpha+\nu} G$.

For $F \succ_\alpha G$, we have $G_\alpha(n) - F_\alpha(n) \geq 0$ for all $n \in \mathbb{N}_b$ and there is some $n' \in \mathbb{N}_b$ such that $G_\alpha(n') - F_\alpha(n') > 0$. The proof for this case is almost the same as that for

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$F \succeq_\alpha G$. First, at least one of Section 3.2 and eq. (8) has a strict inequality. So, the inequality in Equation (9) is strict, and so are the remaining inequalities. As a result, we have $\sum_{j=0}^{n} k^{\alpha+v}(g(j) - f(j)) > 0$, which means that $G_{\alpha+v}(n) - F_{\alpha+v}(n) \geq 0$ for all $n \in \mathbb{N}_b$, and there is some $n' \in \mathbb{N}_b$ such that $G_{\alpha+v}(n') - F_{\alpha+v}(n') > 0$. In other words, $F \succ_{\alpha+v} G$. □

3.3. Utility Classes

We first define the utility class for each $\alpha \in [1, \infty)$, then show that this utility class is congruent with the αth-order SD. Define

$$U_\alpha := \left\{ u\colon u(i) = -\sum_{x=i}^{b} t(x) k^\alpha(x-i) + c; c \in \mathbb{R}; t\colon \mathbb{N}_b \to \mathbb{R}^+ \cup \{0\} \right\},$$
$$U'_\alpha := \left\{ u\colon u(i) = -\sum_{x=i}^{b} t(x) k^\alpha(x-i) + c; c \in \mathbb{R}; t\colon \mathbb{N}_b \to \mathbb{R}^+ \right\}.$$

Fix an α. Now, we define the corresponding notations for utilities:

- $F \succeq_u G$ means that $\sum_{x=0}^{b} u(x) f(x) \geq \sum_{x=0}^{b} u(x) g(x)$ for all $u \in U_\alpha$; and
- $F \succ_u G$ means that $\sum_{x=0}^{b} u(x) f(x) > \sum_{x=0}^{b} u(x) g(x)$ for all $u \in U'_\alpha$.

Note that our definition of $F \succ_u G$ is a very strong one: We require all but not only for some utilities to achieve a strict inequality.

Theorem 4. *$\succeq_\alpha \cong \succeq_u$ and $\succ_\alpha \cong \succ_u$ for each $\alpha \in [1, \infty)$.*

Proof. Fix an α. We first prove the relation $\succ_\alpha \cong \succ_u$. For any F, G, it is sufficient to prove that $F \succ_\alpha G$ implies $F \succ_u G$ for all $u \in U'_\alpha$, and also prove that if $F \succ_\alpha G$ is false, then there exists a $u \in U'_\alpha$ such that $\sum_{i=0}^{b} u(i) g(i) \geq \sum_{i=0}^{b} u(i) f(i)$.

We start with the first part. Given that $F \succ_\alpha G$, we have $F_\alpha(x) \leq G_\alpha(x)$ for all $x \in \mathbb{N}_b$, and $F_\alpha(x) < G_\alpha(x)$ for some $x \in \mathbb{N}_b$. As $t(x) > 0$ for all $x \in \mathbb{N}_b$, we have

$$\sum_{x=0}^{b} -t(x) F_\alpha(x) > \sum_{x=0}^{b} -t(x) G_\alpha(x) \tag{10}$$
$$\sum_{x=0}^{b} -t(x) \sum_{i=0}^{x} k^\alpha(x-i) f(i) > \sum_{x=0}^{b} -t(x) \sum_{i=0}^{x} k^\alpha(x-i) g(i)$$
$$\sum_{i=0}^{b} \left(-\sum_{x=i}^{b} t(x) k^\alpha(x-i) \right) f(i) > \sum_{i=0}^{b} \left(-\sum_{x=i}^{b} t(x) k^\alpha(x-i) \right) g(i).$$

For any $c \in \mathbb{R}$, we have $\sum_{i=0}^{b} c \cdot f(i) = \sum_{i=0}^{b} c \cdot g(i) = c$, so the above inequality becomes

$$\sum_{i=0}^{b} u(i) f(i) > \sum_{i=0}^{b} u(i) g(i)$$

for any $u \in U'_\alpha$.

We now prove the second part. When $F \succ_\alpha G$ is false, we have two cases. The first case is $F = G$, which gives $\sum_{i=0}^{b} u(i) f(i) = \sum_{i=0}^{b} u(i) g(i)$ for all $u \in U'_\alpha$. The second case is that there exists some $i \in \mathbb{N}_b$ such that $F_\alpha(i) > G_\alpha(i)$.

For the second case, let $S := \{x\colon F_\alpha(x) > G_\alpha(x)\}$. Pick an $x' \in S$. Let $F_\alpha(x') - G_\alpha(x') > \lambda$ for some $\lambda > 0$. As we only need to find a $u \in U'_\alpha$ that gives $\sum_{i=0}^{b} u(i) g(i) \geq \sum_{i=0}^{b} u(i) f(i)$, we consider a $u \in U'_\alpha$ where $c = 0$ and

$$t(x) = \begin{cases} 1 & \text{if } x = x', \\ \delta & \text{otherwise} \end{cases}$$

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for some $\delta > 0$. For this u, we have

$$\begin{aligned}\sum_{i=0}^{b} u(i)f(i) &= \sum_{i=0}^{b}\left(-\sum_{x=i}^{b} t(x)k^{\alpha}(x-i)\right)f(i)\\ &= -\sum_{x=0}^{b} t(x)\sum_{i=0}^{x} k^{\alpha}(x-i)f(i)\\ &= -\sum_{x=0}^{b} t(x)F_{\alpha}(x)\\ &= -F_{\alpha}(x') - \delta\sum_{\substack{x=0\\ x\neq x'}}^{b} F_{\alpha}(x).\end{aligned}$$

Similarly, we have

$$\sum_{i=0}^{b} u(i)g(i) = -G_{\alpha}(x') - \delta\sum_{\substack{x=0\\ x\neq x'}}^{b} G_{\alpha}(x).$$

Now, consider

$$\begin{aligned}\sum_{i=0}^{b} u(i)(g(i)-f(i)) &= F_{\alpha}(x') - G_{\alpha}(x') + \delta\sum_{\substack{x=0\\ x\neq x'}}^{b}(F_{\alpha}(x) - G_{\alpha}(x))\\ &> \lambda + b\delta \min_{x\in\mathbb{N}_b\setminus\{x'\}}(F_{\alpha}(x) - G_{\alpha}(x)).\end{aligned}$$

For a small enough $\delta > 0$, the last equation is positive. Therefore, there exists a $u \in U'_{\alpha}$ such that $\sum_{i=0}^{b} u(i)g(i) \geq \sum_{i=0}^{b} u(i)f(i)$, where the equality is from the first case that $F = G$. This finishes the proof for $\succ_{\alpha}\cong\succ_{u}$.

Now, we prove the relation $\succeq_{\alpha}\cong\succeq_{u}$. The proof is similar as that for $\succ_{\alpha}\cong\succ_{u}$. For any F, G, it is sufficient to prove that $F \succeq_{\alpha} G$ implies $F \succeq_{u} G$ for all $u \in U_{\alpha}$, and also prove that if $F \succeq_{\alpha} G$ is false, then there exists a $u \in U_{\alpha}$ such that $\sum_{i=0}^{b} u(i)g(i) > \sum_{i=0}^{b} u(i)f(i)$.

For the first part, this time we have $F_{\alpha}(x) \leq G_{\alpha}(x)$ and $t(x) \geq 0$ for all $x \in \mathbb{N}_b$. So, we have a $\geq$ instead of a $>$ in Section 3.3 and the inequalities below it. That is, we achieve

$$\sum_{i=0}^{b} u(i)f(i) \geq \sum_{i=0}^{b} u(i)g(i)$$

for any $u \in U_{\alpha}$.

For the second part, when $F \succeq_{\alpha} G$ is false, then the only case is that there exists some $i \in \mathbb{N}_b$ such that $F_{\alpha}(i) > G_{\alpha}(i)$. We can construct the same u as used in the proof for $\succ_{\alpha}\cong\succ_{u}$, but this time this u is in U_{α}. Using the same argument, we can conclude that there exists a $u \in U_{\alpha}$ such that $\sum_{i=0}^{b} u(i)g(i) > \sum_{i=0}^{b} u(i)f(i)$. This finishes the proof for $\succeq_{\alpha}\cong\succeq_{u}$. □

Next, we aim to show the monotonicity of the utility classes, i.e., for any $n \in \mathbb{Z}^+$, every $u \in U_n \cup U'_n$ satisfies $(-1)^{p+1}\Delta^p u(i) \geq 0$ for all $p = 1, 2, \ldots, n$. A direct way is to find the exact formula of $\Delta^n u(i)$. As a convention in combinatorics, define $\binom{-1}{0} = 1$ and $\binom{j-1}{j} = 0$ for all $j \in \mathbb{Z}^+$.

Theorem 5. *For any $n \in \mathbb{Z}^+$ and $u \in U_{\alpha} \cup U'_{\alpha}$,*

$$\Delta^n u(i) = (-1)^{n+1}\left(\sum_{j=0}^{n-1}\binom{\alpha-n+j-1}{j}t(i+j) + \prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n}^{b}\frac{t(x)k^{\alpha}(x-i-n)}{\prod_{\ell=1}^{n}(x-i+1-\ell)}\right).$$

Proof. See Section B. □

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We can now show the monotonicity result.

Corollary 1. *For any $n \in \mathbb{Z}^+$, every $u \in U_n \cup U'_n$ satisfies $(-1)^{p+1}\Delta^p u(i) \geq 0$ for all $p = 1, 2, \ldots, n$.*

Proof. Substituting $\alpha = n$ into Theorem 5. Our goal is to show that

$$(-1)^{p+1}\Delta^p u(i) = \sum_{j=0}^{p-1} \binom{n-p+j-1}{j} t(i+j) + \prod_{j=1}^{p}(n-j) \sum_{x=i+p}^{b} \frac{t(x)k^n(x-i-p)}{\prod_{\ell=1}^{p}(x-i+1-\ell)} \geq 0$$

for all $p = 1, 2, \ldots, n$.

Note that empty summation is defined to be 0. First, for every p, we have

$$\prod_{j=1}^{p}(n-j) \sum_{x=i+p}^{b} \frac{t(x)k^n(x-i-p)}{\prod_{\ell=1}^{p}(x-i+1-\ell)} \geq 0$$

as every term on the left side is non-negative. Next, for $1 \leq p < n$, we have $n - p \geq 1$, thus

$$\sum_{j=0}^{p-1} \binom{n-p+j-1}{j} t(i+j) \geq 0.$$

The inequality is strict if $u \in U'_n$ (and $n > 1$) because $t > 0$. The last piece of the puzzle is the case $1 \leq p = n$. We have

$$\sum_{j=0}^{p-1} \binom{n-p+j-1}{j} t(i+j) = \binom{-1}{0} t(i) + \sum_{j=1}^{p-1} \binom{j-1}{j} t(i+j) = t(i) \geq 0.$$

The inequality is strict if $u \in U'_n$ because $t > 0$. Combining the above discussion, the proof is done. □

4. Conclusions

In this paper, we discussed the diverse perspectives in higher-order stochastic dominance (SD) from the view of distributions and the view of utilities. The conventional SD makes a compromise to restrict the choices of distributions so that the utility-based definition must work for all utilities in the class. This is very useful in risk analysis as the exact form of the utility may not be known. Other than risk management, some applications only concern some specific utilities. This way, we prefer imposing restrictions on the utility classes instead. We called this type of SD Fishburn's SD in this paper.

Our motivation is due to practical needs of developing SD between integral orders and applying discrete distributions. The existing SD development mostly focus on continuous distributions as the results may be generalized to discrete distributions with ease. However, when we investigate fractional-order SD, there may not have a straightforward analogue between fractional calculus and fractional sum. This leads to the investigation of this paper: If we consider the natural extension of discrete SD using fractional sum, can it be a candidate of fractional-order SD that satisfying some requirements? We proved that our fractional-order SD definition is a candidate: The SD hierarchy is preserved, and it is congruent to certain utility classes that are consistent with integral-order SD.

There are some future directions on this research. First, we crafted some specific utility classes to suit our needs. However, this construction only considered discrete utilities. A question would be, are there any utility classes of continuous utilities that congruent with the discrete fractional-order SD? Second, we considered bounded distributions in this paper as preliminary results. How to extend these results to unbounded distributions? We suspect that some properties may not be valid in the unbounded case. Third, if we consider conventional SD but discrete utilities, how to formulate the corresponding fractional-order

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SD? At last, although the formulations and the proofs are different for continuous and discrete fractional-order SD, is there any way to unify them in a general form? These directions may help us to understand more about discrete SD.

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Abbreviations

The following abbreviation is used in this manuscript:

SD	Stochastic dominance

Appendix A. Proof of Theorem 1

If F and G are identical, then we have $F \succeq_1 G$ by definition. Now consider distinct F and G. We simplify the proof by mapping the union of the supports of F and G to $\{0, 1, \ldots, b\}$. Without loss of generality, assume that $f(\lambda) < g(\lambda)$. The case $g(\lambda) < f(\lambda)$ can be proved by similar arguments.

Define $F \succ_K^m G$ if and only if $F_K(x) \leq G_K(x)$ for all $0 \leq x \leq m$ and there is at least one strict inequality. Note that $F \succ_K^b G$ is equivalent to $F \succ_K G$ as the distributions of F and G are bounded on $\{0, 1, \ldots, b\}$. When we have $F \succ_K^m G$, we also have

$$F_{K+1}(i) = \sum_{x=0}^{i} F_K(x) \leq \sum_{x=0}^{i} G_K(x) = G_{K+1}(i)$$

for all $0 \leq i \leq m$, and the equality between the summations does not hold when $i = m$. So, $F \succ_K^m G$ implies $F \succ_{K+1}^m G$.

We use induction to prove the proposition: For any $\lambda \leq m \leq b$, there exists an integer K_m such that $F \succ_{K_m}^m G$.

We start with $m = \lambda$. By the definition of λ, we have $f(\lambda) < g(\lambda)$ and $f(i) = g(i)$ for all $0 \leq i < \lambda$. So, we have

$$G_1(i) - F_1(i) = \sum_{x=0}^{i} (g(x) - f(x)) \geq 0$$

for all $0 \leq i \leq \lambda$, where the equality does not hold when $i = \lambda$. That is, $F \succ_1^\lambda G$.

If $\lambda = b$, then we do not need the induction step below. For $\lambda \neq b$, we assume that there exists an integer K such that $F \succ_K^{k-1} G$, where $\lambda \leq k - 1 < b$. Consider $m = k$. If $F_K(k) \leq G_K(k)$, then it is clear that $F \succ_K^k G$.

Suppose $F_K(k) > G_K(k)$. Define

$$\omega_K^k = \sum_{i=0}^{k-1} (G_K(i) - F_K(i)) > 0.$$

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Note that

$$\begin{aligned}\omega_{K+1}^k &= \sum_{i=0}^{k-1}\sum_{i'=0}^{i}(G_K(i')-F_K(i'))\\ &= \omega_K^k + \sum_{i=0}^{k-2}\sum_{i'=0}^{i}(G_K(i')-F_K(i')) > \omega_K^k.\end{aligned}$$

Similarly, we have $\omega_{K+j}^k > \omega_K^k > 0$ for all $j \in \mathbb{Z}^+$.

Define $\varsigma_K^k = F_K(k) - G_K(k)$ and $N = \lfloor \varsigma_K^k/\omega_K^k \rfloor + 1$, which give $\varsigma_K^k < N\omega_K^k$. Then, we have

$$F_{K+N}(k) - G_{K+N}(k) = \varsigma_K^k - \sum_{j=K}^{K+N-1}\omega_j^k \le \varsigma_K^k - N\omega_K^k < 0.$$

As $F \succ_K^{k-1} G$ implies $F \succ_{K+N}^{k-1} G$, we have $F_{K+N}(i) \le G_{K+N}(i)$ for all $0 \le i \le k$ with at least one strict inequality, i.e., $F \succ_{K+N}^{k} G$.

By induction, we have $F \succ_{K_b}^{b} G$ for some integer K_b. The proof is done.

Appendix B. Proof of Theorem 5

In this proof, we apply the extended Pascal's rule. Let $n \in \mathbb{R}$ and $k \in \mathbb{N}$. Then, the rule is that

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{\Gamma(n)}{\Gamma(k+1)\Gamma(n-k)} + \frac{\Gamma(n)}{\Gamma(k)\Gamma(n-k+1)}\\ &= \Gamma(n)\left(\frac{n-k}{\Gamma(k+1)\Gamma(n-k+1)} + \frac{k}{\Gamma(k+1)\Gamma(n-k+1)}\right)\\ &= \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}\\ &= \binom{n}{k}.\end{aligned}$$

Now, we start the proof. Fix a $u \in U_\alpha$ or $u \in U'_\alpha$. Write this u as

$$u(i) = -\sum_{x=i}^{b} t(x)k^\alpha(x-i) + c$$

for some t and c. We prove this theorem by induction. For $n = 1$, we have

$$\begin{aligned}\Delta^1 u(i) &= u(i+1) - u(i)\\ &= -\sum_{x=i+1}^{b} t(x)k^\alpha(x-(i+1)) + \sum_{x=i}^{b} t(x)k^\alpha(x-i)\\ &= t(i)k^\alpha(0) + \sum_{x=i+1}^{b} t(x)(k^\alpha(x-i) - k^\alpha(x-i-1))\\ &= t(i) + (\alpha-1)\sum_{x=i+1}^{b}\frac{t(x)k^\alpha(x-i-1)}{x-i}\\ &= (-1)^{(1)+1}\left(\sum_{j=0}^{(1)-1}\binom{\alpha-(1)+j-1}{j}t(i+j) + \prod_{j=1}^{(1)}(\alpha-j)\sum_{x=i+(1)}^{b}\frac{t(x)k^\alpha(x-i-(1))}{\prod_{\ell=1}^{(1)}(x-i+1-\ell)}\right).\end{aligned}$$

Assume that the statement we want to prove holds for some $n \in \mathbb{Z}^+$. As an induction step, we consider $\Delta^{n+1}u(i)$.

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$$
\begin{aligned}
\Delta^{n+1}u(i) &= \Delta^n u(i+1) - \Delta^n u(i) \\
&= (-1)^{n+1}\left(\sum_{j=0}^{n-1}\binom{\alpha-n+j-1}{j}(t(i+1+j)-t(i+j)) - \prod_{j=1}^{n}(\alpha-j)\frac{t(i+n)k^{\alpha}(0)}{\prod_{\ell=1}^{n}(n+1-\ell)}\right. \\
&\quad \left. +\prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n+1}^{b}t(x)\left(\frac{k^{\alpha}(x-i-1-n)}{\prod_{\ell=1}^{n}(x-i-\ell)} - \frac{k^{\alpha}(x-i-n)}{\prod_{\ell=1}^{n}(x-i+1-\ell)}\right)\right).
\end{aligned}
$$

We separate the formulation into two parts. First, we consider

$$
\begin{aligned}
&\sum_{j=0}^{n-1}\binom{\alpha-n+j-1}{j}(t(i+1+j)-t(i+j)) - \prod_{j=1}^{n}(\alpha-j)\frac{t(i+n)k^{\alpha}(0)}{\prod_{\ell=1}^{n}(n+1-\ell)} \\
&= \sum_{j=1}^{n-1}\left(\binom{\alpha-n+(j-1)-1}{j-1} - \binom{\alpha-n+j-1}{j}\right)t(i+j) + \binom{\alpha-n+(n-1)-1}{n-1}t(i+n) \\
&\quad -\binom{\alpha-n-1}{0}t(i) - \binom{\alpha-(n+1)+n}{n}t(i+n) \\
&= -\sum_{j=1}^{n-1}\binom{\alpha-n+(j-1)-1}{j}t(i+j) + \binom{\alpha-(n+1)+n-1}{n-1}t(i+n) - t(i) - \binom{\alpha-(n+1)+n}{n}t(i+n) \\
&= -\left(\sum_{j=1}^{n-1}\binom{\alpha-(n+1)+j-1}{j}t(i+j) + \binom{\alpha-(n+1)+n-1}{n}t(i+n) + t(i)\right) \\
&= -\sum_{j=0}^{(n+1)-1}\binom{\alpha-(n+1)+j-1}{j}t(i+j).
\end{aligned}
$$

Next, we consider

$$
\begin{aligned}
&\prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n+1}^{b}t(x)\left(\frac{k^{\alpha}(x-i-1-n)}{\prod_{\ell=1}^{n}(x-i-\ell)} - \frac{k^{\alpha}(x-i-n)}{\prod_{\ell=1}^{n}(x-i+1-\ell)}\right) \\
&= \prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n+1}^{b}t(x)k^{\alpha}(x-i-1-n)\left(\frac{1}{\prod_{\ell=1}^{n}(x-i-\ell)} - \frac{\alpha+x-i-1-n}{(x-i-n)\prod_{\ell=1}^{n}(x-i+1-\ell)}\right) \\
&= \prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n+1}^{b}t(x)k^{\alpha}(x-i-1-n)\left(\frac{1}{\prod_{\ell=2}^{n+1}(x-i+1-\ell)} - \frac{\alpha+x-i-1-n}{\prod_{\ell=1}^{n+1}(x-i+1-\ell)}\right) \\
&= \prod_{j=1}^{n}(\alpha-j)\sum_{x=i+n+1}^{b}t(x)k^{\alpha}(x-i-1-n)\frac{-(\alpha-(n+1))}{\prod_{\ell=1}^{n+1}(x-i+1-\ell)} \\
&= -\prod_{j=1}^{n+1}(\alpha-j)\sum_{x=i+n+1}^{b}\frac{t(x)k^{\alpha}(x-i-(n+1))}{\prod_{\ell=1}^{n+1}(x-i+1-\ell)}.
\end{aligned}
$$

Combining the two parts, we have

$$
\Delta^{n+1}u(i) = (-1)^{(n+1)+1}\left(\sum_{j=0}^{(n+1)-1}\binom{\alpha-(n+1)+j-1}{j}t(i+j) + \prod_{j=1}^{(n+1)}(\alpha-j)\sum_{x=i+(n+1)}^{b}\frac{t(x)k^{\alpha}(x-i-(n+1))}{\prod_{\ell=1}^{n+1}(x-i+1-\ell)}\right).
$$

The proof is done by induction.

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