Short Note

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Modular Dvoretzky, Type-Cotype, Khinchin-Kahane and Grothendieck Inequality Problems

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Article

Modular Dvoretzky, Type-Cotype, Khinchin-Kahane and Grothendieck Inequality Problems

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Abstract: We recently formulated Modular Dvoretzky, Type-Cotype, Khinchin-Kahane and Grothendieck Inequality problems in the Appendix of *[arXiv:2302.03718v1]*. For the sake of wide accessibility we give a separate treatment of them.

Keywords: Dvoretzky Theorem; Type-Cotype; Khinchin-Kahane Inequality; Grothendieck Inequality; C*-algebra; Hilbert C*-module

MSC: 47B10; 47L20; 46L08; 46L05; 42C15

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1. Modular Problems

Our first kind of problems come from the Dvoretzky theorem [36–50]. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite dimensional Banach spaces such that $\dim(\mathcal{X}) = \dim(\mathcal{Y})$. Remember that the Banach-Mazur distance between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X}, \mathcal{Y}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{X} \to \mathcal{Y} \text{ is invertible linear operator }\}.$$

For $n \in \mathbb{N}$, let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the standard Euclidean Hilbert space.

Theorem 1. *[28,51] **(John Theorem)** If $\mathcal{X}$ is any n-dimensional real Banach space, then*

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^n, \langle \cdot, \cdot \rangle)) \leq \sqrt{n}.$$

Theorem 2. *[28,52] **(Dvoretzky Theorem)** There is a universal constant $C > 0$ satisfying the following property: If $\mathcal{X}$ is any n-dimensional real Banach space and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq C \log n \frac{\varepsilon^2}{|\log \varepsilon|},$$

there exists a k-dimensional Banach subspace $\mathcal{Y}$ of $\mathcal{X}$ such that

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

Let $\mathcal{A}$ be a unital C*-algebra with invariant basis number property (see [53] for a study on such C*-algebras) and $\mathcal{E}, \mathcal{F}$ be finite rank Banach modules over $\mathcal{A}$ such that $\operatorname{rank}(\mathcal{E}) = \operatorname{rank}(\mathcal{F})$. Modular Banach-Mazur distance between $\mathcal{E}$ and $\mathcal{F}$ is defined as

$$d_{MBM}(\mathcal{E}, \mathcal{F}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{E} \to \mathcal{F} \text{ is invertible module homomorphism}\}.$$

Given a unital C*-algebra $\mathcal{A}$ and $n \in \mathbb{N}$, by $\mathcal{A}^n$ we mean the standard (left) module over $\mathcal{A}$. We equip $\mathcal{A}^n$ with the C*-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}$ defined by

$$\langle (a_j)_{j=1}^n, (b_j)_{j=1}^n \rangle := \sum_{j=1}^n a_j b_j^*, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n.$$

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Hence norm on $\mathcal{A}^n$ is given by

$$\|(a_j)_{j=1}^n\| := \left\|\sum_{j=1}^n a_j a_j^*\right\|^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n \in \mathcal{A}^n.$$

Then it is well-known that $\mathcal{A}^n$ is a Hilbert C*-module. We denote this Hilbert C*-module by $(\mathcal{A}^n, \langle\cdot,\cdot\rangle)$.

Problem 3. *(Modular Dvoretzky Problem) Let $\mathbb{A}$ be the set of all unital C*-algebras with invariant basis number property. What is the best function $\Psi : \mathbb{A} \times \left(0, \frac{1}{3}\right) \times \mathbb{N} \to (0, \infty)$ satisfying the following property: If $\mathcal{E}$ is any n-rank Banach module over a unital C*-algebra $\mathcal{A}$ with IBN property and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq \Psi(\mathcal{A}, \varepsilon, n),$$

there exists a k-rank Banach submodule $\mathcal{F}$ of $\mathcal{E}$ such that

$$d_{MBM}(\mathcal{F}, (\mathcal{A}^k, \langle\cdot,\cdot\rangle)) < 1 + \varepsilon.$$

A particular case of Problem 3 is the following conjecture.

Conjecture 4. *(Modular Dvoretzky Conjecture) Let $\mathcal{A}$ be a unital C*-algebra with IBN property. There is a universal constant $C > 0$ (which may depend upon $\mathcal{A}$) satisfying the following property: If $\mathcal{E}$ is any n-rank Banach module and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq C \log n \frac{\varepsilon^2}{|\log \varepsilon|},$$

there exists a k-rank Banach submodule $\mathcal{F}$ of $\mathcal{E}$ such that

$$d_{MBM}(\mathcal{F}, (\mathcal{A}^k, \langle\cdot,\cdot\rangle)) < 1 + \varepsilon.$$

Our second kind of problems come from the type-cotype theory of Banach spaces [8,12,13,18,19,28,54,55]. Let $\mathcal{H}$ be a Hilbert space, $n \in \mathbb{N}$. Recall that for any n points $x_1, \ldots, x_n \in \mathcal{H}$, we have

$$\frac{1}{2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}} \left\|\sum_{j=1}^n \varepsilon_j x_j\right\|^2 = \sum_{j=1}^n \|x_j\|^2. \tag{1}$$

It is Equality (1) which motivated the definition of type and cotype for Banach spaces.

Definition 4. *[28] Let $1 \leq p \leq 2$. A Banach space $\mathcal{X}$ is said to be of* **(Rademacher) type** *p if there exists $T_p(\mathcal{X}) > 0$ such that*

$$\left(\frac{1}{2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}} \left\|\sum_{j=1}^n \varepsilon_j x_j\right\|^p\right)^{\frac{1}{p}} \leq T_p(\mathcal{X}) \left(\sum_{j=1}^n \|x_j\|^p\right)^{\frac{1}{p}}, \quad \forall x_1, \ldots, x_n \in \mathcal{X}, \ \forall n \in \mathbb{N}.$$

Definition 4. *[28] Let $2 \leq q < \infty$. A Banach space $\mathcal{X}$ is said to be of* **(Rademacher) cotype** *q if there exists $C_q(\mathcal{X}) > 0$ such that*

$$\left(\sum_{j=1}^n \|x_j\|^q\right)^{\frac{1}{q}} \leq C_q(\mathcal{X}) \left(\frac{1}{2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}} \left\|\sum_{j=1}^n \varepsilon_j x_j\right\|^q\right)^{\frac{1}{q}}, \quad \forall x_1, \ldots, x_n \in \mathcal{X}, \ \forall n \in \mathbb{N}.$$

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Let $\mathcal{E}$ be a (left) Hilbert C*-module over a unital C*-algebra $\mathcal{A}$, $n \in \mathbb{N}$. We see that for any n points $x_1, \dots, x_n \in \mathcal{E}$, we have

$$\frac{1}{2^n} \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}} \left\langle \sum_{j=1}^{n} \varepsilon_j x_j, \sum_{k=1}^{n} \varepsilon_k x_k \right\rangle = \sum_{j=1}^{n} \langle x_j, x_j \rangle. \tag{2}$$

Problem 5. *(Modular Type-Cotype Problems) Whether there is a way to define type (we call modular-type) and cotype (we call modular-cotype) for Banach modules over C*-algebras which reduces to Equality (2) for Hilbert C*-modules?*

Problem 6. *Whether there is a notion of type and cotype for Banach modules over C*-algebras such that Kwapien theorems holds?, In other words, whether following statements hold?*

(i) A Banach module $\mathcal{M}$ over a unital C-algebra $\mathcal{A}$ has modular-type 2 and modular-cotype 2 if and only if $\mathcal{M}$ is isomorphic to a Hilbert C*-module over $\mathcal{A}$.*
(ii) If $\mathcal{M}$ and $\mathcal{N}$ are Banach modules over a unital C-algebra $\mathcal{A}$ of modular-type 2 and modular-cotype 2, respectively, then a bounded module morphism $T : \mathcal{M} \to \mathcal{N}$ factors through a Hilbert C*-module over $\mathcal{A}$.*

Problem 7. *(Modular Khinchin-Kahane Inequalities Problems) Whether there is a Khinchin-Kahane inequalities for Banach modules over C*-algebras which reduce to Equality (2) for Hilbert C*-modules?*

Our third kind of problems come from Grothendieck inequality [1,2,28,56–63].

Theorem 8. *[1,2,28,56–58] (Grothendieck Inequality) There is a universal constant K_G satisfying the following: For any Hilbert space $\mathcal{H}$ and any $m, n \in \mathbb{N}$, if a scalar matrix $[a_{j,k}]_{1\leq j\leq m, 1\leq k\leq n}$ satisfy*

$$\left| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k} s_j t_k \right| \leq 1, \quad \forall s_j, t_k \in \mathbb{K}, |s_j| \leq 1, |t_k| \leq 1,$$

then

$$\left| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k} \langle u_j, v_k \rangle \right| \leq K_G, \quad \forall u_j, v_k \in \mathcal{H}, \|u_j\| \leq 1, \|v_k\| \leq 1.$$

Problem 8. *(Modular Grothendieck Inequality Problem - 1) Let $\mathcal{A}$ be the set of all unital C*-algebras. Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. Let $\mathcal{A}^+$ be the set of all positive elemnts in $\mathcal{A}$. What is the best function $\Psi : \mathcal{A} \times \mathbb{N} \times \mathbb{N} \to \mathcal{A}^+$ satisfying the following property: If $[a_{j,k}]_{1\leq j\leq m, 1\leq k\leq n} \in \mathbb{M}_{m\times n}(\mathcal{A})$ satisfy*

$$\left\langle \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k} s_j t_k, \sum_{p=1}^{m} \sum_{q=1}^{n} a_{p,q} s_p t_q \right\rangle \leq 1, \quad \forall s_j, t_k \in \mathcal{A},$$
$$s_j s_j^* = s_j^* s_j = 1, \forall 1 \leq j \leq m, t_k t_k^* = t_k^* t_k = 1, \forall 1 \leq k \leq n,$$

then

$$\left\langle \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k} \langle u_j, v_k \rangle, \sum_{p=1}^{m} \sum_{q=1}^{n} a_{p,q} \langle u_p, v_q \rangle \right\rangle \leq \Psi(\mathcal{A}, m, n), \quad \forall u_j, v_k \in \mathcal{E},$$
$$\langle u_j, u_j \rangle = 1, \forall 1 \leq j \leq m, \langle v_k, v_k \rangle = 1, \forall 1 \leq k \leq n.$$

In particular, whether Ψ depends on m and n?

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Problem 8. *(Modular Grothendieck Inequality Problem - 2) Let $\mathscr{A}$ be the set of all unital C*-algebras. Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. Let $\mathcal{A}^+$ be the set of all positive elemnts in $\mathcal{A}$. What is the best function $\Psi : \mathscr{A} \times \mathbb{N} \times \mathbb{N} \to \mathcal{A}^+$ satisfying the following property: If* $[a_{j,k}]_{1\leq j\leq m, 1\leq k\leq n} \in \mathbb{M}_{m\times n}(\mathcal{A})$ *satisfy*

$$\left\langle \sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}s_jt_k, \sum_{p=1}^{m}\sum_{q=1}^{n} a_{p,q}s_pt_q \right\rangle \leq 1, \quad \forall s_j, t_k \in \mathcal{A}, \|s_j\| \leq 1, \forall 1 \leq j \leq m, \|t_k\| \leq 1, \forall 1 \leq k \leq n,$$

then

$$\left\langle \sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}\langle u_j, v_k\rangle, \sum_{p=1}^{m}\sum_{q=1}^{n} a_{p,q}\langle u_p, v_q\rangle \right\rangle \leq \Psi(\mathcal{A}, m, n), \quad \forall u_j, v_k \in \mathcal{E},$$
$$\|u_j\| \leq 1, \forall 1 \leq j \leq m, \|v_k\| \leq 1, \forall 1 \leq k \leq n.$$

In particular, whether Ψ depends on m and n?

We believe strongly that Ψ depends on $\mathcal{A}$.

Remark 8. ***Modular Bourgain-Tzafriri restricted invertibility conjecture*** *and* ***Modular Johnson-Lindenstrauss flattening conjecture*** *have been stated in [64,65].*

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