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*Article*

# A Duality Principle and a Concerning Convex Dual Formulation Suitable for Non-Convex Variational Optimization

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**Abstract:** This article develops a duality principle and a related convex dual formulation suitable for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a model in non-linear elasticity.

**Keywords:** Convex dual variational formulation, duality principle for non-convex optimization, model in non-linear elasticity

**MSC:** 49N15

## 1. Introduction

In this article we establish a duality principle and a related convex dual formulation for a large class of models in non-convex optimization.

More specifically, the main duality principle is applied to a model in non-linear elasticity.

Such results are based on the works of J.J. Telega and W.R. Bielski [2,3,10,11] and on a D.C. optimization approach developed in Toland [12].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [4–7,9]. Finally, the model in non-linear elasticity here presented may be found in [8].

**Remark.** *In this text we adopt the standard Einstein convention of summing up repeated indices unless otherwise indicated.*

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = \frac{1}{2}\int_\Omega H_{ijkl}e_{ij}(u)e_{kl}(u)\,dx - \langle u_i, f_i\rangle_{L^2}. \tag{1}$$

Here $\{H_{ijkl}\}$ is a fourth order symmetric positive definite tensor and

$$\{e_{ij}(u)\} = \left\{\frac{1}{2}(u_{i,j}+u_{j,i}) + \frac{1}{2}(u_{m,i}u_{m,j})\right\},$$

where

$$u = (u_1, u_2, u_3) \in V = W_0^{1,4}(\Omega;\mathbb{R}^3)$$

denotes the field of displacements resulting from the action of the external forces $f = (f_1, f_2, f_2) \in L^2(\Omega;\mathbb{R}^3)$ on the elastic solid comprised by $\Omega \subset \mathbb{R}^3$.

Moreover, denoting $Y = Y^* = L^2(\Omega;\mathbb{R}^{3\times 3})$, the stress tensor $\sigma \in Y^*$ is defined by

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$$\{\sigma_{ij}(u)\} = \{H_{ijkl}e_{kl}(u)\}.$$

At this point we define the functionals $F_1 : V \times Y \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $G : V \to \mathbb{R}$ by

$$\begin{aligned} F_1(u,\sigma) &= -\sum_{i=1}^{3}\frac{K}{2}\int_\Omega (u_{i,i})^2\,dx \\ &+\sum_{i=1}^{3}\frac{K_1}{2}\int_\Omega (\sigma_{ij,j}+(\sigma_{im}u_{m,j})_{,j}+f_i)^2\,dx \\ &+\frac{K_2}{2}\int_\Omega u_{i,j}u_{i,j}\,dx - \langle u_i, f_i\rangle_{L^2}, \end{aligned} \tag{2}$$

for appropriate positive real constants, $K, K_1, K_2$ to be specified,

$$F_2(u) = \frac{K_2}{2}\int_\Omega u_{i,j}u_{i,j}\,dx,$$

and

$$G(u) = \frac{1}{2}\int_\Omega H_{ijkl}e_{ij}(u)e_{kl}(u)\,dx + \sum_{i=1}^{3}\frac{K}{2}\int_\Omega (u_{i,i})^2\,dx.$$

Here, it is worth highlighting that

$$F_1(u,\sigma) - F_2(u) + G(u) = J(u) + \sum_{i=1}^{3}\frac{K_1}{2}\int_\Omega (\sigma_{ij,j}+(\sigma_{im}u_{m,j})_{,j}+f_i)^2\,dx, \forall u \in V,\ \sigma \in Y^*.$$

Furthermore, we define the functionals $F_1^* : [Y^*]^3 \to \mathbb{R}$, $F_2^* : Y^* \to \mathbb{R}$ and $G^* : [Y^*]^2 \to \mathbb{R}$ by

$$F_1^*(\sigma, Q, \tilde{Q}) = \sup_{u\in V}\{-\langle u_{i,j}, \sigma_{ij}\rangle_{L^2} - \langle u_{i,j}, Q_{ij}\rangle_{L^2} + \langle u_{i,j}, \tilde{Q}_{ij}\rangle_{L^2} - F_1(u,\sigma)\},$$

$$\begin{aligned} F_2^*(\tilde{Q}) &= \sup_{v_2\in Y}\left\{\langle (v_2)_{ij}, \tilde{Q}_{ij}\rangle_{L^2} - \frac{K_2}{2}\int_\Omega (v_2)_{ij}(v_2)_{ij}\,dx\right\} \\ &= \frac{1}{2K_2}\int_\Omega \tilde{Q}_{ij}\tilde{Q}_{ij}\,dx, \end{aligned} \tag{3}$$

and

$$G^*(\sigma, Q) = \sup_{(v_1,v_2)\in Y\times Y}\{\langle (v_1)_{ij}, \sigma_{ij}\rangle_{L^2} + \langle (v_2)_{ij}, Q_{ij}\rangle_{L^2} - \hat{G}(v_1,v_2)\},$$

where

$$\begin{aligned} \hat{G}(v_1,v_2) &= \frac{1}{2}\int_\Omega H_{ijkl}\left((v_1)_{ij} + \frac{1}{2}(v_2)_{mi}(v_2)_{mj}\right)\left((v_1)_{kl} + \frac{1}{2}(v_2)_{mk}(v_2)_{ml}\right)\,dx \\ &+\sum_{i=1}^{3}\frac{K}{2}\int_\Omega ((v_2)_{ii})^2\,dx, \end{aligned} \tag{4}$$

so that

$$G^*(\sigma, Q) = \frac{1}{2}\int_\Omega (\overline{\sigma_{ij}^K})Q_{mi}Q_{mj}\,dx + \frac{1}{2}\int_\Omega \overline{H}_{ijkl}\sigma_{ij}\sigma_{kl}\,dx,$$

if $\sigma \in B^*$ where

$$B^* = \{\sigma \in Y^* \ :\ \|\sigma_{ij}\|_\infty \le K/8,\ \forall i,j \in \{1,2,3\} \text{ and } \{\sigma_{ij}\} < -\varepsilon I_d\},$$

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for some small parameter $\varepsilon > 0$ and where $I_d$ denotes the $3 \times 3$ identity matrix. Observe that such a definition for $B^*$ corresponds to the case of negative definite stress tensors, which refers to compression in a solid mechanics context.

Here

$$\{\sigma_{ij}^K\} = \begin{bmatrix} \sigma_{11} + K & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} + K & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} + K \end{bmatrix} \tag{5}$$

$$\overline{\sigma_{ij}^K} = \{\sigma_{ij}^K\}^{-1},$$

and

$$\{\overline{H}_{ijkl}\} = \{H_{ijkl}\}^{-1}$$

in an appropriate tensor sense.

At this point we define

$$J^*(\sigma, Q, \tilde{Q}) = -F_1^*(\sigma, Q, \tilde{Q}) + F_2^*(\tilde{Q}) - G^*(\sigma, Q).$$

Specifically for

$$K_2 \gg K_1 \gg K \gg \max\{1/\varepsilon^2, 1, K_3, \|H_{ijkl}\|\},$$

we define

$$D^* = \{Q \in Y^* \; : \; \|Q_{ij}\|_\infty \leq K_3, \; \forall i, j \in \{1, 2, 3\}\}.$$

By direct computation, we may obtain

$$\left\{ \frac{\partial^2 J^*(\sigma, Q, \tilde{Q})}{\partial \sigma \partial Q} \right\} < \mathbf{0},$$

and

$$\left\{ \frac{\partial^2 J^*(\sigma, Q, \tilde{Q})}{\partial \tilde{Q}^2} \right\} > \mathbf{0},$$

on $B^* \times D^* \times Y^*$, so that $J^*$ is concave in $(\sigma, Q)$ and convex in $\tilde{Q}$ on $B^* \times D^* \times Y^*$.

## 2. The main duality principle and a related convex dual variational formulation

Our main duality principle is summarized by the following theorem.

**Theorem 1.** *Considering the statements and definitions of the previous section, suppose* $(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) \in B^* \times D^* \times Y^*$ *is such that*

$$\delta J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) = \mathbf{0}.$$

*Let* $u_0 \in V$ *be such that*

$$(u_0)_{i,j} = \frac{\partial F_2^*(\tilde{Q})}{\partial \tilde{Q}_{ij}}.$$

*Under such hypotheses, we have*

$$\begin{aligned} J(u_0) &= \min_{u \in V} \left\{ J(u) + \sum_{i=1}^{3} \frac{K_1}{2} \int_\Omega (\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im} u_{m,j})_{,j} + f_i)^2 \, dx \right\} \\ &= \inf_{\tilde{Q} \in Y^*} \left\{ \sup_{(\sigma, Q) \in B^* \times D^*} J^*(\sigma, Q, \tilde{Q}) \right\} \\ &= J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}). \end{aligned} \tag{6}$$

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**Proof.** Observe that there exists $\hat{u} \in V$ such that, defining

$$H(u,\sigma,Q,\tilde{Q}) = -\langle u_{i,j},\sigma_{ij}\rangle_{L^2} - \langle u_{i,j},Q_{ij}\rangle_{L^2} + \langle u_{i,j},\tilde{Q}_{ij}\rangle_{L^2} - F_1(u,\sigma),$$

we have

$$\frac{\partial H(\hat{u},\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial u} = \mathbf{0}$$

and

$$F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}}) = H(\hat{u},\hat{\sigma},\hat{Q},\hat{\tilde{Q}}).$$

Moreover, from the variation in of $J^*$ in $\tilde{Q}$, we obtain

$$-\frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} + \frac{\partial F_2^*(\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} = \mathbf{0},$$

where

$$\begin{aligned} & \frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} \\ = & \frac{\partial H(\hat{u},\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} \\ & + \frac{\partial H(\hat{u},\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial u}\frac{\partial \hat{u}}{\partial \tilde{Q}_{ij}} \\ = & \hat{u}_{i,j}. \end{aligned} \tag{7}$$

From such last two equations we get

$$(u_0)_{i,j} = \frac{\partial F_2^*(\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} = \frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} = \hat{u}_{i,j},$$

so that from the concerning boundary conditions,

$$u_0 = \hat{u}.$$

On the other hand, from the variation of $J^*$ in $Q$ we have

$$-\frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial Q_{ij}} - \frac{\partial G^*(\hat{\sigma},\hat{Q})}{\partial Q_{ij}} = \mathbf{0},$$

so that

$$(u_0)_{i,j} - \overline{\hat{\sigma}^K_{im}}\hat{Q}_{mj} = 0,$$

and therefore

$$\hat{Q}_{ij} = \hat{\sigma}_{im}(u_0)_{m,j} + K\delta_{ij}(u_0)_{i,j}.$$

Finally, from the variation of $J^*$ in $\sigma$ we obtain

$$-\frac{\partial F_1^*(\hat{\sigma},\hat{Q},\hat{\tilde{Q}})}{\partial \sigma_{ij}} - \frac{\partial G^*(\hat{\sigma},\hat{Q})}{\partial \sigma_{ij}} = \mathbf{0},$$

so that

$$(u_0)_{i,j} + \frac{1}{2}(u_0)_{m,i}(u_0)_{m,j} - \overline{H}_{ijkl}\hat{\sigma}_{kl} = 0.$$

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Thus, since $\{H_{ijkl}\}$ is symmetric, we get

$$\hat{\sigma}_{ij} = H_{ijkl}e_{kl}(u_0).$$

From these last results and from

$$\frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{\hat{Q}})}{\partial u} = \mathbf{0}$$

we obtain

$$\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im}(u_0)_{m,j})_{,j} + f_i = 0, \ \forall i \in \{1,2,3\},$$

so that

$$\delta J(u_0) = \mathbf{0}.$$

Finally, from such last results and the Legendre transform properties, we have

$$\begin{aligned} & F_1^*(\hat{\sigma}, \hat{Q}, \hat{\hat{Q}}) \\ = \ & -\langle (u_0)_{i,j}, \hat{\sigma}_{ij}\rangle_{L^2} - \langle (u_0)_{i,j}, \hat{Q}_{ij}\rangle_{L^2} + \langle (u_0)_{i,j}, \hat{\hat{Q}}_{ij}\rangle_{L^2} - F_1(u_0, \hat{\sigma}), \end{aligned} \tag{8}$$

$$F_2^*(\hat{\hat{Q}}) = \langle (u_0)_{i,j}, \hat{\hat{Q}}_{ij}\rangle_{L^2} - F_2(u_0),$$

and

$$G^*(\hat{\sigma}, \hat{Q}) = \langle (u_0)_{i,j}, \hat{\sigma}_{ij}\rangle_{L^2} + \langle (u_0)_{i,j}, \hat{Q}_{ij}\rangle_{L^2} - G(u_0).$$

From these results, we obtain

$$\begin{aligned} J^*(\hat{\sigma}, \hat{Q}, \hat{\hat{Q}}) &= -F_1^*(\hat{\sigma}, \hat{Q}, \hat{\hat{Q}}) + F_2^*(\hat{\hat{Q}}) - G^*(\hat{\sigma}, \hat{Q}) \\ &= F_1(u_0, \hat{\sigma}) - F_2(u_0) + G(u_0) \\ &= J(u_0). \end{aligned} \tag{9}$$

Joining the pieces, we have got

$$\begin{aligned} J(u_0) &= \min_{u \in V} \left\{ J(u) + \sum_{i=1}^{3} \frac{K_1}{2} \int_\Omega (\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im} u_{m,j})_{,j} + f_i)^2 \, dx \right\} \\ &= \inf_{\tilde{Q} \in Y^*} \left\{ \sup_{(\sigma, Q) \in B^* \times D^*} J^*(\sigma, Q, \tilde{Q}) \right\} \\ &= J^*(\hat{\sigma}, \hat{Q}, \hat{\hat{Q}}). \end{aligned} \tag{10}$$

The proof is complete.
□

**Remark.** *A similar result is valid if we would define*

$$B^* = \{\sigma \in Y^* \ : \ \|\sigma_{ij}\|_\infty \leq K/8, \ \forall i,j \in \{1,2,3\} \text{ and } \{\sigma_{ij}\} > \varepsilon I_d\}.$$

*This case refers to a positive definite tensor* $\{\sigma_{ij}\}$ *and the previous case to a negative definite one.*

## 3. A closely related primal-dual variational formulation for a similar model

In this section we present a new primal-dual variational formulation for a closely related model of plates.

At this point we start to describe the primal formulation.

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Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set which represents the middle surface of a plate of thickness $h$. The boundary of $\Omega$, which is assumed to be regular (Lipschitzian), is denoted by $\partial\Omega$. The vectorial basis related to the cartesian system $\{x_1, x_2, x_3\}$ is denoted by $(\mathbf{a}_\alpha, \mathbf{a}_3)$, where $\alpha = 1, 2$ (in general Greek indices stand for 1 or 2), and where $\mathbf{a}_3$ is the vector normal to $\Omega$, whereas $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal vectors parallel to $\Omega$. Also, $\mathbf{n}$ is the outward normal to the plate surface.

The displacements will be denoted by

$$\hat{\mathbf{u}} = \{\hat{u}_\alpha, \hat{u}_3\} = \hat{u}_\alpha \mathbf{a}_\alpha + \hat{u}_3 \mathbf{a}_3.$$

The Kirchhoff-Love relations are

$$\begin{aligned} &\hat{u}_\alpha(x_1, x_2, x_3) = u_\alpha(x_1, x_2) - x_3 w(x_1, x_2)_{,\alpha} \\ &\text{and } \hat{u}_3(x_1, x_2, x_3) = w(x_1, x_2). \end{aligned} \tag{11}$$

Here $-h/2 \leq x_3 \leq h/2$ so that we have $u = (u_\alpha, w) \in U$ where

$$\begin{aligned} U &= \Big\{(u_\alpha, w) \in W^{1,2}(\Omega; \mathbb{R}^2) \times W^{2,2}(\Omega), \\ &\quad\; u_\alpha = w = \frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\} \\ &= W_0^{1,2}(\Omega; \mathbb{R}^2) \times W_0^{2,2}(\Omega). \end{aligned}$$

It is worth emphasizing that the boundary conditions here specified refer to a clamped plate.

We define the operator $\Lambda : U \to Y \times Y$, where $Y = Y^* = L^2(\Omega; \mathbb{R}^{2\times 2})$, by

$$\Lambda(u) = \{\gamma(u), \kappa(u)\},$$

$$\gamma_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{w_{,\alpha} w_{,\beta}}{2},$$

$$\kappa_{\alpha\beta}(u) = -w_{,\alpha\beta}.$$

The constitutive relations are given by

$$N_{\alpha\beta}(u) = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u), \tag{12}$$

$$M_{\alpha\beta}(u) = h_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}(u), \tag{13}$$

where: $\{H_{\alpha\beta\lambda\mu}\}$ and $\left\{h_{\alpha\beta\lambda\mu} = \frac{h^2}{12} H_{\alpha\beta\lambda\mu}\right\}$, are symmetric positive definite fourth order tensors. From now on, we denote $\{\overline{H}_{\alpha\beta\lambda\mu}\} = \{H_{\alpha\beta\lambda\mu}\}^{-1}$ and $\{\overline{h}_{\alpha\beta\lambda\mu}\} = \{h_{\alpha\beta\lambda\mu}\}^{-1}$.

Furthermore $\{N_{\alpha\beta}\}$ denote the membrane force tensor and $\{M_{\alpha\beta}\}$ the moment one. The plate stored energy, represented by $(G \circ \Lambda) : U \to \mathbb{R}$ is expressed by

$$(G \circ \Lambda)(u) = \frac{1}{2}\int_\Omega N_{\alpha\beta}(u)\gamma_{\alpha\beta}(u)\, dx + \frac{1}{2}\int_\Omega M_{\alpha\beta}(u)\kappa_{\alpha\beta}(u)\, dx \tag{14}$$

and the external work, represented by $F : U \to \mathbb{R}$, is given by

$$F(u) = \langle w, P\rangle_{L^2} + \langle u_\alpha, P_\alpha\rangle_{L^2}, \tag{15}$$

where $P, P_1, P_2 \in L^2(\Omega)$ are external loads in the directions $\mathbf{a}_3$, $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively. The potential energy, denoted by $J : U \to \mathbb{R}$ is expressed by:

$$J(u) = (G \circ \Lambda)(u) - F(u)$$

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More explicitly, recalling that

$$\gamma_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{1}{2} w_{,\alpha} w_{,\beta},$$

we have

$$\begin{aligned}
& J(u) \\
= & \frac{1}{2} \int_\Omega H_{\alpha\beta\lambda\mu} \left( \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{1}{2} w_{,\alpha} w_{,\beta} \right) \left( \frac{u_{\lambda,\mu} + u_{\mu,\lambda}}{2} + \frac{1}{2} w_{,\lambda} w_{,\mu} \right) dx \\
& + \frac{1}{2} \int_\Omega h_{\alpha\beta\lambda\mu} w_{,\alpha\beta} w_{,\lambda\mu} \, dx - \langle u_\alpha, P_\alpha \rangle_{L^2} - \langle w, P \rangle_{L^2} \\
= & -\langle \gamma_{\alpha\beta}(u), N_{\alpha\beta} \rangle_{L^2} + \frac{1}{2} \int_\Omega H_{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}(u) \gamma_{\lambda\mu}(u) \, dx \\
& + \frac{1}{2} \int_\Omega h_{\alpha,\beta,\lambda,\mu} w_{,\alpha\beta} w_{,\lambda\mu} \, dx + \left\langle \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{1}{2} w_{,\alpha} w_{,\beta}, N_{\alpha\beta} \right\rangle_{L^2} \\
& - \langle u_\alpha, P_\alpha \rangle_{L^2} - \langle w, P \rangle_{L^2} \\
\geq & \inf_{v \in Y} \left\{ \langle v_{\alpha\beta}, N_{\alpha\beta} \rangle_{L^2} + \frac{1}{2} \int_\Omega H_{\alpha\beta\lambda\mu} v_{\alpha\beta} v_{\lambda\mu} \, dx \right\} \\
& + \inf_{\{u_\alpha\} \in \left[W_0^{1,2}(\Omega)\right]^2} \left\{ \frac{1}{2} \int_\Omega h_{\alpha\beta\lambda\mu} w_{,\alpha\beta} w_{,\lambda\mu} \, dx \right. \\
& \left. + \left\langle \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{1}{2} w_{,\alpha} w_{,\beta}, N_{\alpha\beta} \right\rangle_{L^2} - \langle u_\alpha P_\alpha \rangle_{L^2} - \langle w, P \rangle_{L^2} \right\} \\
\geq & -\frac{1}{2} \int_\Omega \overline{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} \, dx \\
& + \frac{1}{2} \int_\Omega h_{\alpha,\beta,\lambda,\mu} w_{,\alpha\beta} w_{,\lambda\mu} \, dx + \left\langle \frac{1}{2} w_{,\alpha} w_{,\beta}, N_{\alpha\beta} \right\rangle_{L^2} - \langle w, P \rangle_{L^2} \\
= & J_1^*(u, N),
\end{aligned} \tag{16}$$

$\forall u \in U, N \in B^*$, where $B^* = B_1^* \cap B_2^*$,

$$B_1^* = \{N \in Y^* \ : \ N_{\alpha\beta,\beta} + P_\alpha = 0, \text{ in } \Omega\},$$

$$B_2^* = \{N \in Y^* \ : \ \|N\|_\infty \leq K\},$$

for an appropriate constant $K > 0$

At this point, we also define

$$V_1 = \{u \in U \ : \ \|w_{,\alpha\beta}\|_\infty \leq K_3, \ \forall \alpha, \beta \in \{1, 2\}\},$$

for an appropriate constant $K_3 > 0$

We highlight the constants $K_3 > 0$ and $K > 0$ must be such that the restrictions which define $B_2^*$ and $V_1$ are not active at a concerning critical point.

Here we present the following primal-dual formulation suitable for an optimization of the original primal variational formulation

$$J_2^*(u, N) = J_1^*(u, N) + \frac{K_1}{2} \int_\Omega \left( (h_{\alpha\beta\lambda\mu} w_{,\lambda\mu})_{,\alpha\beta} - (N_{\alpha\beta} w_{,\beta})_{,\alpha} - P \right)^2 dx.$$

More specifically,

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$$
\begin{aligned}
J_2^*(u,N) \;=\;& -\frac{1}{2}\int_\Omega \overline{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu}\,dx \\
&+\frac{1}{2}\int_\Omega h_{\alpha,\beta,\lambda,\mu}w_{,\alpha\beta}w_{,\lambda\mu}\,dx+\left\langle \frac{1}{2}w_{,\alpha}w_{,\beta},N_{\alpha\beta}\right\rangle_{L^2} \\
&-\langle w,P\rangle_{L^2} \\
&+\frac{K_1}{2}\int_\Omega\left(\left(h_{\alpha\beta\lambda\mu}w_{,\lambda\mu}\right)_{,\alpha\beta}-(N_{\alpha\beta}w_{,\beta})_{,\alpha}-P\right)^2\,dx. \qquad (17)
\end{aligned}
$$

We may observe that for

$$K_1\approx\mathcal{O}\left(\frac{1}{4\|H\|K_3^2}\right)$$

and $K_3>0$ sufficiently small, $J_2^*$ is convex in $u$ and concave in $N$ and on $V_1\times B^*$.

Finally, we may also define $J_3$ by

$$J_3(u)=\sup_{N\in B^*}J_2^*(u,N).$$

We observe that $J_3$ has a large region of convexity around any critical point.

## 4. A duality principle for a related model in phase transitions

In this section we present a duality principle for a related model in phase transition.

Let $\Omega=[0,1]\subset\mathbb{R}$ and consider a functional $J:V\to\mathbb{R}$ where

$$J(u)=\frac{1}{2}\int_\Omega((u')^2-1)^2\,dx+\frac{1}{2}\int_\Omega u^2\,dx,$$

and where

$$V=\{u\in W^{1,4}(\Omega)\;:\;u(0)=0\text{ and }u(1)=1/2\}.$$

A global optimum point is not attained for $J$ so that the problem of finding a global minimum for $J$ has no solution.

Anyway, one question remains, how the minimizing sequences behave close the infimum of $J$.

From the Ekeland variational principle the equation

$$\delta J(u_0)\approx\mathbf{0},$$

may be approximately satisfied by points for which $J$ is arbitrarily close to its infimum.

We intend to use duality theory to approximately solve such a global optimization problem.

At this point we define, $F:V\to\mathbb{R}$ and $G:V\to\mathbb{R}$ by

$$F(u)=\frac{1}{2}\int_\Omega((u')^2-1)^2\,dx+\frac{K}{2}\int_\Omega(u')^2\,dx,$$

and

$$G(u)=-\frac{1}{2}\int_\Omega u^2\,dx++\frac{K}{2}\int_\Omega(u')^2\,dx,$$

so that

$$J(u)=F(u)-G(u),\forall u\in V.$$

Observe that if $K>0$ is large enough, both $F$ and $G$ are convex.

Denoting $Y=Y^*=L^2(\Omega)$ we also define the polar functionals $F^*:Y^*\to\mathbb{R}$ and $G^*:Y^*\to\mathbb{R}$ by

$$F^*(v^*)=\sup_{u\in V}\{\langle u,v^*\rangle_{L^2}-F(u)\},$$

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and

$$G^*(v^*) = \sup_{u \in V}\{\langle u, v^*\rangle_{L^2} - G(u)\}.$$

From the standard Toland result in [12] for D.C. optimization, we may obtain

$$\inf_{u \in U} J(u) = \inf_{u \in U}\{F(u) - G(u)\} = \inf_{v^* \in Y^*}\{G^*(v^*) - F^*(v^*)\}.$$

In fact, we may also obtain

$$\inf_{u \in U} J(u) = \inf_{(u,v^*) \in U \times Y^*}\{G^*(v^*) - \langle u, v^*\rangle_{L^2} + F(u)\}.$$

With such results in mind, we define a primal dual variational formulation for the primal problem, represented by $J_1^* : V \times Y^* \to \mathbb{R}$, where

$$J_1^*(u, v^*) = G^*(v^*) - \langle u, v^*\rangle_{L^2} + F(u).$$

Having defined such a functional, we may obtain numerical results by solving a sequence of convex auxiliary sub-problems, through the following algorithm.

1. Set $K \gg 1$. and $0 < \varepsilon \ll 1$.
2. Choose $u_1 \in V$, such that $\|u_1\|_{1,\infty} \ll K/4$.
3. Set $n = 1$.
4. Calculate $v_n^*$ solution of equation:

$$\frac{\partial J_1^*(u_n, v_n^*)}{\partial v^*} = \mathbf{0},$$

that is

$$\frac{\partial G^*(v_n^*)}{\partial v^*} - u_n = 0,$$

so that

$$v_n^* = \frac{\partial G(u_n)}{\partial u}.$$

5. Calculate $u_{n+1}$ by solving the equation:

$$\frac{\partial J_1^*(u_{n+1}, v_n^*)}{\partial u} = \mathbf{0},$$

that is

$$-v_n^* + \frac{\partial F(u_{n+1})}{\partial u} = 0.$$

6. If $\|u_n - u_{n+1}\| \leq \varepsilon$, then stop, else set $n := n + 1$ and go to item 4.

We have obtained numerical results for $K = 10000000$. For the solution $u_0$ obtained please see Figure 1.

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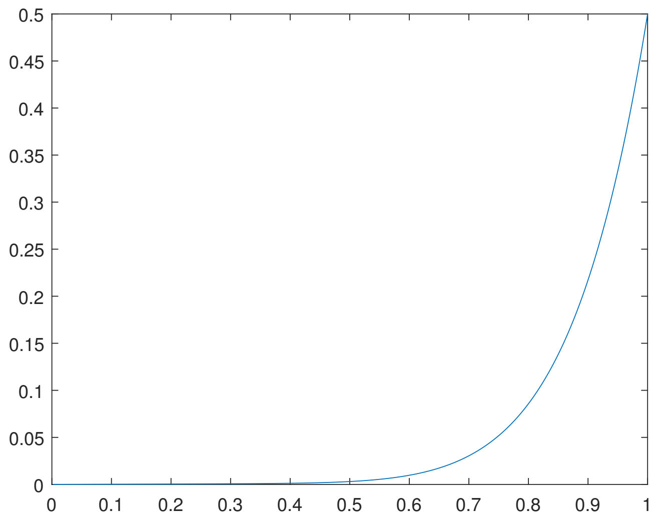


**Figure 1.** solution $u_0(x)$ through the primal dual formulation for a large $K > 0$

## 5. Conclusion

In this article we have developed a convex dual variational formulation suitable for non-convex variational primal formulations.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principle here presented is applied to a model in non-linear elasticity. In a future research, we intend to extend such results for other related models of plates and shells.

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