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Goldbach Conjecture and the Trichotomy Law

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Abstract

This paper asserts that the falsity of Goldbach Conjecture results in the breakdown of Trichotomy Law. Assuming that Goldbach Conjecture breaks down at an even integer $2n$, where $n \geq 3$, we note that the differences of $2n$ with the odd primes in the interval $(1, n)$ are composite integers. We form a product δ of the collection of the least prime factors of these composite integers. We show that for a base case this product breaks Trichotomy Law. With the inductive hypothesis that such is the case with $2n$, we prove that for all even integers the breakdown of Goldbach Conjecture entails the breakdown of Trichotomy Law.

The Trichotomy Law (TL) says that given any two arbitrary real numbers a and b, exactly one of the following relations holds between them: $a < b, a > b, a = b$ [1]. *A fortiori*, since the set of positive integers $\mathbb{Z}^+$ is a subset of the real numbers $\mathbb{R}$, one and only one of these relations orders any two arbitrary elements in $\mathbb{Z}^+$.

One of the conjectures that Christian Goldbach proposed, with some later day modifications, states that (GC) Every integer $2n$, where $n \geq 3$ is the sum of two odd primes [2]. In this paper, we shall establish that if GC fails, then TL also fails. The consequence of this, as contraposition, is that if we are to commit to TL —if we are to preserve it —then GC has to be true.

Given a positive even integer $2n$, where $n \geq 3$ we form two open intervals $(1, n)$ and $(n, 2n)$. We collect the odd primes in $(1, n)$ and then form a corresponding collection of integers from $(n, 2n)$. These latter integers are the differences of $2n$ with the odd primes in $(1, n)$.

We take as an example the integer 14. This integer expressed as $2n$ gives $n = 7$. We name the open interval $(1, 7)$ on $\mathbb{Z}^+$ as $S_1(14) = \{x : x \in \mathbb{Z}^+$ and $x \in (1, 7)\}$ and to the open interval $(7, 14)$ on $\mathbb{Z}^+$ as $S_2(14) = \{x : x \in \mathbb{Z}^+$ and $x \in (7, 14)\}$. We can see that $S_1(14)$ and $S_2(14)$ are instances of $S_1(2\chi) = \{x : x \in \mathbb{Z}^+$ and $x \in (1, \chi)\}$ and $S_2(2\chi) = \{x : x \in \mathbb{Z}^+$ and $x \in (\chi, 2\chi)\}$ respectively. If we collect all the odd primes in $(1, 7)$, we have the set $\{3, 5\}$. We can call it $\Gamma(14)$. Similarly, we can give the symbol $\Omega(14)$ to the set $\{11, 9\}$. Here again

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$\Gamma(14)$ and $\Omega(14)$ are instances of the general formulations $\Gamma(2\chi) = \{x : x$ is an odd prime and $x \in S_1(2\chi)\}$ and $\Omega(2\chi) = \{x : x = 2\chi - y$ where $y \in \Gamma(2\chi)\}$. We shall call y the supplement of x, if $x \in \Gamma(2\chi) \cup \Omega(2\chi)$ and $x + y = 2\chi$.

We can see that $\Gamma(12) = \Gamma(14)$. Based on this observation we can form a set $(2\chi)_2$ of all even integers 2χ in $\mathbb{Z}^+$, the cardinality of whose $\Gamma(2\chi)$ is 2. The general case is expressed as $(2\chi)_\eta = \{x : x$ is any even integer 2χ in $\mathbb{Z}^+$, such that the cardinality of $\Gamma(2\chi) = \eta\}$. We shall abbreviate the information that the cardinality of $\Gamma(2\chi) = \eta$ as $\Gamma(2\chi)_\eta$. Consider the power set $\wp(\Gamma(14)) = \{\{\}, \{3\}, \{5\}, \{3, 5\}\}$. If we subtract the empty set from this power set, we represent the resultant set as $\wp^*(\Gamma(14)) = \wp(\Gamma(14)) - \{\}$. This forms the set $\{\{3\}, \{5\}, \{3, 5\}\}$, which is a collection of the various ways in which prime factors could be drawn from the set $\Gamma(14)$. From this, we can make sense of the set denoted by $\wp^*(\Gamma(2\chi)_\eta)$ as the set of combinations $C(\eta, r)$ of elements from $\Gamma(2\chi)_\eta$, with r ranging from 1 to η. We would also need a collection of the products of each of the elements of $\wp^*(\Gamma(2\chi)_\eta)$. Reverting to $\wp^*(\Gamma(14))$, we would want a corresponding set $\{3, 5, 3.5\}$. To generalize, if we have, $\wp^*(\Gamma(2\chi)_\eta) = \{\{p_1\}, \{p_2\}, \dots, \{p_\eta\}, \{p_1, p_2\}, \{p_1, p_3\}, \dots, \{p_1, p_\eta\}, \dots, \{p_r, p_\eta\}, \dots, \{p_1, \dots, p_\eta\}\}$, we need a set

$$T(2\chi) = \{p_1, p_2, \dots, p_\eta, p_1p_2, p_1p_3, \dots, p_1p_\eta, \dots, p_rp_\eta, \dots, p_1 \dots p_\eta\}$$

Equivalently,

$$T(2\chi) = \{x : x = \prod_{(y \in z) \& (z \in \wp^*(\Gamma(2\chi)_\eta)} y\}$$

Central to our proof is a choice set which we shall call a *route*. We illustrate it for the case $2n = 14$ and then draw a general picture based on it.

	3	5	3.5
11	$< 11, 3 >$	$< 11, 5 >$	$< 11, 3.5 >$
9	$< 9, 3 >$	$< 9, 5 >$	$< 9, 3.5 >$

Table 1: $\Omega(14) \times T(14)$

The array of ordered pairs in Table 1 is formed by the Cartesian product $\Omega(14) \times T(14)$. A route in the above array would be a path that goes through the rows of ordered pairs passing through one and only one of the ordered pair entries in each row. There are 3×3 ways of choosing one element from each row of ordered pairs. So, there are nine routes associated with the integer 14. The same number of routes would be there for the integer 12. The routes associated with the integer 14 are $R_1 = \{< 11, 3 >, < 9, 3 >\}, R_2 = \{< 11, 3 >, < 9, 5 >\}, R_3 = \{< 11, 3 >, < 9, 3.5 >\}, R_4 = \{< 11, 5 >, < 9, 3 >\}, R_5 = \{< 11, 5 >, < 9, 5 >\}, R_6 = \{< 11, 5 >, < 9, 3.5 >\}, R_7 = \{< 11, 3.5 >, < 9, 3 >\}, R_8 = \{< 11, 3.5 >, < 9, 5 >\}, R_9 = \{< 11, 3.5 >, < 9, 3.5 >\}$.

If $O_i = \{o_i\} \times T(2\chi)$, where o_i is the supplement of prime p_i in $\Gamma(2\chi)_\eta$ and $P = \{< x_1, x_2, x_3, \dots, x_i, \dots, x_\eta >: x_1 \in O_1, x_2 \in O_2, x_3 \in O_3, \dots, x_i \in$

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$O_i, \ldots, x_\eta \in O_\eta\}$, then we call any element R_i of P a route of 2χ. We have just one more set to define. $L(R_i(2\chi)) = \{x : x$ is the least prime factor of each o_i in the route R_i of $2\chi\}$.

In the case of the integer 14, none of the routes *obtain*. By this, we mean that in these routes there are ordered pairs in which the second entry are not factors of the first entry. For example in $R_1 = \{< 11, 3 >, < 9, 3 >\}$, the first ordered pair has the second entry 3 which is not a factor of the first entry, 11. If a route has at least one element such that the second entry is not a factor of the first entry then we say that the route does not obtain. If there exist an even integer $2n$ for which GC breaks down, then each one of the supplements of the primes in $\Gamma(2n)$ is a composite integer. In that scenario, there will be at least one route in which $R_i(2n)$ i.e., the R_i of $2n$ obtains. In such cases, we can define a product

$$\delta(R_i(2\chi)) = \prod_{x \in L(R_i(2\chi))} x$$

Now, to the statement and proof of our assertion.

Theorem 1. *The falsity of GC results in the breakdown of TL.*

Proof. We shall attempt an inductive proof on the cardinality of $\Gamma(2\chi)$. For the base case, we have $\Gamma(2\chi)_1$. Let $2m \in (2\chi)_1$. Let the sole member of $\Gamma(2m)_1$ be p_i and its supplement o_i. Suppressing a mathematical fact, let us assume that GC breaks down at *all* routes of *every* element of $(2\chi)_1$. This makes GC break down at $2m$, which in turn shows that o_i and m are composites. Let R_i be a route of $2m$.

Since o_i is an odd composite, p_i is its prime factor. So, $\delta(R_i(2m)) = p_i$. Is $2m = 2\delta(R_i(2m))$? If so, $m = p_i$. This identity counters the assumption that GC breaks down at $2m$; that $p_i \in (1, m)$. So, $2m \neq 2\delta(R_i(2m))$. Is $2m < 2\delta(R_i(2m))$ then? If so, we have the absurd situation that $m < p_i$. Hence, $2m \not< 2\delta(R_i(2m))$. Since two of the options of TL do not obtain, the remaining order, $2m > 2\delta(R_i(2m))$ has to obtain. But, this cannot be so. Since, $p_i + o_i = 2m$, and $p_i | o_i$, we have $p_i | m$. If $m > p_i$, then $m \geq 2p_i$. If so, then Bertrand's postulate [3] dictates that cardinality of $(\Gamma(2\chi)) > 1$, contrary to the assumption made on the cardinality of the set involved. So, $m \not> p_i$. Therefore, $2m \not> 2\delta(R_i(2m))$. We then see that at $\Gamma(2\chi)_1$, the breakdown of GC has, as a consequence, the breakdown of TL.

The inductive hypothesis we assume here is (IH) Given any arbitrary route R_i of any arbitrary element $2n$ of $(2\chi)_\eta$, if GC breaks down in R_i, then TL breaks down.

Now consider any arbitrary $2n \in (2\chi)_{\eta+1}$. Let $R_i(2n)$ be any arbitrary route of $2n$ which renders GC false. Then $\delta(R_i(2n))$ exists. Two cases arise. Either $\delta(R_i(2n))$ has as factors only the primes from $\Gamma(2\chi)_\eta$, which is a proper subset of $\Gamma(2\chi)_{\eta+1}$ i.e., non of its factors belong to $\Gamma(2\chi)_{\eta+1} - \Gamma(2\chi)_\eta$. Or, it has as factors primes which belong to $\Gamma(2\chi)_{\eta+1} - \Gamma(2\chi)_\eta$ i.e., $p_{\eta+1}$, is a factor of $\delta(R_i(2n))$.

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If $\delta(R_i(2n))$ does not involve $p_{\eta+1}$ as a factor, then it is equal to one of the δ's from a route of an even integer that belongs to $(2\chi)_\eta$. But TL breaks down in $(2\chi)_\eta$ according to the assumption of IH.

The prime number $p_{\eta+1}$ can be a factor of $\delta(R_i(2n))$ *iff* there is an element of $\Omega(2n)$, say o_v whose sole factor is this prime. This is because if some other prime lesser than it were to be a factor of o_v, then $p_{\eta+1}$ does not have the property of being the smallest prime factor of o_v. So, $o_v = (p_{\eta+1})^\beta$ for some $\beta \geq 2$. This would mean that o_v is greater than $2n$. This follows from Bertrand's postulate according to which, there has to be at least a prime between $n/2$ and n. Whether there is one, or more than one prime, since $p_{\eta+1}$ is the greatest prime amongst them, we have $n/2 < p_{\eta+1} < n$. From this $o_v > 2n$ follows. But by definition of $\Omega(2n)$, $o_v < 2n$. Neither can it be equal to $2n$. For, if $o_v = 2n$, and since $p_{\eta+1} + o_v = 2n$, $p_{\eta+1} = 0$. This cannot be. Hence, the assumption of the falsity of GC results in the breakdown of TL. □

References

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