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Multiple Sums Involving the Terms of a General Second Order Sequence of Numbers

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Abstract

We evaluate the nested sum $\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}x^{a_0}$ where a_n and c are any integers and x is a real or complex variable. Consequently, we evaluate multiple sums involving the terms of a general second order sequence, the Horadam sequence $(W_j(a,b;p,q))$, defined for all non-negative integers j by the recurrence relation $W_0 = a$, $W_1 = b$; $W_j = pW_{j-1} - qW_{j-2}\,(j \geq 2)$; where a, b, p and q are arbitrary complex numbers, with $p \neq 0$, $q \neq 0$.

2010 *Mathematics Subject Classification*: Primary 11B39; Secondary 11B37.
Keywords: Horadam sequence; Fibonacci number; Lucas number; Lucas sequence; sum-mation identity; nested sum; multiple sum

1 Introduction

Let F_j be the j^{th} Fibonacci number. Ivie [4] has shown that

$$\sum_{s=1}^{m}\sum_{r=1}^{s}F_r = F_{m+4} - F_4 - m,$$

$$\sum_{m=1}^{n}\sum_{s=1}^{m}\sum_{r=1}^{s}F_r = F_{n+6} - F_6 - nF_4 - \frac{n(n+1)}{2},$$

and more generally,

$$\sum_{a_{n-1}=1}^{a_n}\sum_{a_{n-2}=1}^{a_{n-1}}\cdots\sum_{a_0=1}^{a_1}F_{a_0} = F_{a_n+2n} - \sum_{j=0}^{n-1}F_{2(n-j)}\binom{a_n+j-1}{j}. \tag{H}$$

In this paper we will extend the study to the Horadam sequence and derive more such sums. Identity (H) is a special case of the more general identity (see Theorem 3)

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{W_{ra_0+s}}{V_r^{a_0}}$$
$$= (-1)^n\frac{W_{r(a_n+2n)+s}}{q^{rn}V_r^{a_n}} - \frac{1}{V_r^{c-1}}\sum_{j=0}^{n-1}(-1)^{n-j}\frac{W_{r(2n-2j+c-1)+s}}{q^{r(n-j)}}\binom{a_n+j-c}{j},$$

 doi:10.20944/preprints202209.0044.v1

in which $(W_j(a,b;p,q))$ is the Horadam sequence [3] defined for all non-negative integers j, by the recurrence relation

$$W_0 = a,\, W_1 = b;\, W_j = pW_{j-1} - qW_{j-2}\, (j \ge 2); \tag{1.1}$$

where a, b, p and q are arbitrary complex numbers, with $p \ne 0$, $q \ne 0$.
Two important cases of (W_j) are the Lucas sequences of the first kind, $(U_j(p,q)) = (W_j(0,1;p,q))$, and of the second kind, $(V_j(p,q)) = (W_j(2,p;p,q))$; so that

$$U_0 = 0,\, U_1 = 1;\, U_j = pU_{j-1} - qU_{j-2}, (j \ge 2);$$

and

$$V_0 = 2,\, V_1 = p;\, V_j = pV_{j-1} - qV_{j-2}, (j \ge 2).$$

The most well-known Lucas sequences are the Fibonacci sequence, $(F_j) = (U_j(1,-1))$ and the sequence of Lucas numbers, $(L_j) = (V_j(1,-1))$.
The special case $p = 1$ is also important, giving the sequence $(w_j(a,b;q)) = (W_j(a,b;1,q))$, with corresponding special Lucas sequences $(u_j(q)) = (U_j(1,q))$ and $(v_j(q)) = (V_j(1,q))$. The particular case $(G_j(a,b)) = (w_j(a,b;-1))$ is the so-called gibonacci sequence, with the Fibonacci and Lucas sequences, $(F_j) = (G_j(0,1))$ and $(L_j) = (G_j(2,1))$, as special cases. Explicitly,

$$G_0 = a,\, G_1 = b;\, G_j = G_{j-1} + G_{j-2}\, (j \ge 2).$$

The Binet formulas for sequences (U_j), (V_j) and (W_j) in the non-degenerate case, $p^2 - 4q > 0$, are

$$U_j = \frac{\tau^j - \sigma^j}{\tau - \sigma} = \frac{\tau^j - \sigma^j}{\Delta}, \qquad V_j = \tau^j + \sigma^j, \qquad W_j = \mathbb{A}\tau^j + \mathbb{B}\sigma^j\,, \tag{BW}$$

with

$$\mathbb{A} = \frac{b - a\sigma}{\tau - \sigma}, \quad \mathbb{B} = \frac{a\tau - b}{\tau - \sigma},$$

where

$$\tau = \tau(p,q) = \frac{p + \sqrt{p^2 - 4q}}{2}, \quad \sigma = \sigma(p,q) = \frac{p - \sqrt{p^2 - 4q}}{2},$$

are the distinct zeros of the characteristic polynomial $x^2 - px + q$ of the Horadam sequence; so that $\tau\sigma = q$ and $\tau + \sigma = p$.
The Binet formulas for the Fibonacci and Lucas numbers are

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta} = \frac{\alpha^j - \beta^j}{\sqrt{5}}, \quad L_j = \alpha^j + \beta^j, \tag{BF}$$

where $\alpha = \tau(1,-1) = (1 + \sqrt{5})/2$ is the golden ratio and $\beta = \sigma(1,-1) = -1/\alpha$.
Extension of the definition of W_n to negative subscripts is provided by writing the recurrence relation as $W_{-n} = (pW_{-n+1} - W_{-n+2})/q$.

2 Preliminary results

Let x be a real or complex variable, a_n an integer and n a positive integer. We wish to evaluate

$$\sum_{a_{n-1}=1}^{a_n} \sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_0=1}^{a_1} x^{a_0};$$

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the basic ingredient in our derivations.
The geometric progression sum

$$\sum_{k=1}^{m} x^k = \frac{x^{m+1} - x}{x - 1}$$

can be arranged as

$$\frac{x-1}{x}\sum_{j=1}^{m} x^j = x^m - 1. \tag{D}$$

Multiply through the identity

$$\frac{x-1}{x}\sum_{a_0=1}^{a_1} x^{a_0} = x^{a_1} - 1 \tag{2.1}$$

by $(x-1)/x$ and sum over a_1, making use of (D) with $m = a_2$, to obtain

$$\begin{aligned}\left(\frac{x-1}{x}\right)^2 \sum_{a_1=1}^{a_2}\sum_{a_0=1}^{a_1} x^{a_0} &= \frac{x-1}{x}\sum_{a_1=1}^{a_2} x^{a_1} - \frac{x-1}{x}\sum_{a_1=1}^{a_2} 1\\ &= x^{a_2} - 1 - \left(\frac{x-1}{x}\right)\sum_{a_1=1}^{a_2} 1.\end{aligned}$$

Multiply through the above by $(x-1)/x$ and sum over a_2, making use of (D) with $m = a_3$. This gives

$$\begin{aligned}\left(\frac{x-1}{x}\right)^3 \sum_{a_2=1}^{a_3}\sum_{a_1=1}^{a_2}\sum_{a_0=1}^{a_1} x^{a_0} &= \frac{x-1}{x}\sum_{a_2=1}^{a_3} x^{a_2} - \frac{x-1}{x}\sum_{a_2=1}^{a_3} 1 - \left(\frac{x-1}{x}\right)^2 \sum_{a_2=1}^{a_3}\sum_{a_1=1}^{a_2} 1\\ &= x^{a_3} - 1 - \frac{x-1}{x}\sum_{a_2=1}^{a_3} 1 - \left(\frac{x-1}{x}\right)^2 \sum_{a_2=1}^{a_3}\sum_{a_1=1}^{a_2} 1.\end{aligned}$$

Continuing the iteration, we find

$$\left(\frac{x-1}{x}\right)^n \sum_{a_{n-1}=1}^{a_n}\sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_0=1}^{a_1} x^{a_0} = x^{a_n} - 1 - \sum_{j=1}^{n-1}\left(\frac{x-1}{x}\right)^j \sum_{a_{n-1}=1}^{a_n}\sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_{n-j}=1}^{a_{n-j+1}} 1. \tag{2.2}$$

Thus, the task of finding $\sum_{a_{n-1}=1}^{a_n}\sum_{a_{n-2}=1}^{a_{n-1}}\cdots\sum_{a_0=1}^{a_1} x^{a_0}$ reduces to that of evaluating $\sum_{a_{n-1}=1}^{a_n}\sum_{a_{n-2}=1}^{a_{n-1}}\cdots\sum_{a_{n-j}=1}^{a_{n-j+1}} 1$ for the $n-1$ values of j. Note that there are exactly j sums in the latter multiple sum. We will prove a lemma and return to (2.2).

Lemma 1. *Let k, m and b_s be non-negative integers and let s be a positive integer. Then*

$$\sum_{j=1}^{m}\binom{j+k-1}{k} = \binom{k+m}{k+1}, \tag{2.3}$$

$$\sum_{b_{s-1}=1}^{b_s}\sum_{b_{s-2}=1}^{b_{s-1}}\cdots\sum_{b_0=1}^{b_1} 1 = \binom{b_s+s-1}{s}. \tag{2.4}$$

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Proof. Both identities will be proved by induction. To prove (2.3) we keep k fixed and carry out an induction on m. Identity (2.3) is obviously true for $m = 0$ and $m = 1$. Assume the truth for $m = r$ to establish the induction hypothesis:

$$P_r : \quad \sum_{j=1}^{r} \binom{j+k-1}{k} = \binom{k+r}{k+1}.$$

We wish to prove that $P_r \implies P_{r+1}$ for r a positive integer.

$$\begin{aligned}\sum_{j=1}^{r+1} \binom{j+k-1}{k} &= \sum_{j=1}^{r} \binom{j+k-1}{k} + \binom{k+r}{k}\\ &= \binom{k+r}{k+1} + \binom{k+r}{k}, \text{ by hypothesis } P_r,\\ &= \binom{k+r+1}{k+1},\end{aligned}$$

where, in the last step, we used Pascal's identity:

$$\binom{s}{k+1} + \binom{s}{k} = \binom{s+1}{k+1}.$$

Thus, $P_r \implies P_{r+1}$.
Since

$$\sum_{b_0=1}^{b_1} 1 = \binom{b_1}{1} = b_1,$$

identity (2.4) holds for $s = 1$. Assume the veracity for $s = k$ to get the induction hypothesis:

$$P_k : \quad \sum_{b_{k-1}=1}^{b_k} \sum_{b_{k-2}=1}^{b_{k-1}} \cdots \sum_{b_0=1}^{b_1} 1 = \binom{b_k+k-1}{k}.$$

We have

$$\begin{aligned}\sum_{b_k=1}^{b_{k+1}} \sum_{b_{k-1}=1}^{b_k} \sum_{b_{k-2}=1}^{b_{k-1}} \cdots \sum_{b_0=1}^{b_1} 1 &= \sum_{b_k=1}^{b_{k+1}} \binom{b_k+k-1}{k}, \text{ by hypothesis } P_k,\\ &= \binom{b_{k+1}+k}{k+1}, \text{ by (2.3) with } m = b_{k+1}.\end{aligned}$$

Thus, $P_k \implies P_{k+1}$. $\square$

Using identity (2.4), we find

$$\sum_{a_{n-1}=1}^{a_n} \sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_{n-j}=1}^{a_{n-j+1}} 1 = \binom{a_n+j-1}{j},$$

which inserted in (2.2), yields

$$\left(\frac{x-1}{x}\right)^n \sum_{a_{n-1}=1}^{a_n} \sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_0=1}^{a_1} x^{a_0} = x^{a_n} - 1 - \sum_{j=1}^{n-1} \left(\frac{x-1}{x}\right)^j \binom{a_n+j-1}{j}. \tag{2.5}$$

We now give a formal proof of (2.5). In section 3 we will consider some of its applications.

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Lemma 2. *Let n be a positive integer. Let x be a real or complex variable. Then,*

$$\left(\frac{x-1}{x}\right)^n \sum_{a_{n-1}=1}^{a_n} \sum_{a_{n-2}=1}^{a_{n-1}} \cdots \sum_{a_0=1}^{a_1} x^{a_0} = x^{a_n} - \sum_{j=0}^{n-1} \left(\frac{x-1}{x}\right)^j \binom{a_n+j-1}{j}. \tag{E}$$

Proof. The proof is by induction on n. The identity is readily verified to be true for $n=1$, giving (2.1). Assume the truth for $n=k$. We have the induction hypothesis:

$$P_k: \quad \left(\frac{x-1}{x}\right)^k \sum_{a_{k-1}=1}^{a_k} \sum_{a_{k-2}=1}^{a_{k-1}} \cdots \sum_{a_0=1}^{a_1} x^{a_0} = x^{a_k} - \sum_{j=0}^{k-1} \left(\frac{x-1}{x}\right)^j \binom{a_k+j-1}{j}.$$

We wish to prove that $P_k \implies P_{k+1}$.
We have

$$\begin{aligned}
&\left(\frac{x-1}{x}\right)^{k+1} \sum_{a_k=1}^{a_{k+1}} \sum_{a_{k-1}=1}^{a_k} \sum_{a_{k-2}=1}^{a_{k-1}} \cdots \sum_{a_0=1}^{a_1} x^{a_0}\\
&\quad = \left(\frac{x-1}{x}\right) \sum_{a_k=1}^{a_{k+1}} \left\{ \left(\frac{x-1}{x}\right)^k \sum_{a_{k-1}=1}^{a_k} \sum_{a_{k-2}=1}^{a_{k-1}} \cdots \sum_{a_0=1}^{a_1} x^{a_0} \right\}\\
&\quad = \left(\frac{x-1}{x}\right) \sum_{a_k=1}^{a_{k+1}} \left\{ x^{a_k} - \sum_{j=0}^{k-1} \left(\frac{x-1}{x}\right)^j \binom{a_k+j-1}{j} \right\}, \text{by induction hypothesis } P_k,\\
&\quad = \frac{x-1}{x} \sum_{a_k=1}^{a_{k+1}} x^{a_k} - \sum_{j=0}^{k-1} \left(\frac{x-1}{x}\right)^{j+1} \sum_{a_k=1}^{a_{k+1}} \binom{a_k+j-1}{j}.
\end{aligned} \tag{2.6}$$

Now,

$$\frac{x-1}{x} \sum_{a_k=1}^{a_{k+1}} x^{a_k} = \frac{x-1}{x} \frac{x^{a_{k+1}+1}-x}{x-1} = x^{a_{k+1}} - 1 \tag{2.7}$$

and

$$\begin{aligned}
\sum_{j=0}^{k-1} \left(\frac{x-1}{x}\right)^{j+1} \sum_{a_k=1}^{a_{k+1}} \binom{a_k+j-1}{j} &= \sum_{j=0}^{k-1} \left(\frac{x-1}{x}\right)^{j+1} \binom{a_{k+1}+j}{j+1}, \text{by (2.3)},\\
&= \sum_{j=1}^{k} \left(\frac{x-1}{x}\right)^j \binom{a_{k+1}+j-1}{j}\\
&= \sum_{j=0}^{k} \left(\frac{x-1}{x}\right)^j \binom{a_{k+1}+j-1}{j} - 1.
\end{aligned} \tag{2.8}$$

Using (2.7) and (2.8) in (2.6), we find

$$P_{k+1}: \quad \left(\frac{x-1}{x}\right)^{k+1} \sum_{a_k=1}^{a_{k+1}} \sum_{a_{k-1}=1}^{a_k} \sum_{a_{k-2}=1}^{a_{k-1}} \cdots \sum_{a_0=1}^{a_1} x^{a_0} = x^{a_{k+1}} - \sum_{j=0}^{k} \left(\frac{x-1}{x}\right)^j \binom{a_{k+1}+j-1}{j}.$$

Thus, $P_k \implies P_{k+1}$. □

3 Main results

In this section, we will apply identity (E) to derive some nested identities involving Horadam numbers.
First we wish to modify identity (E) so that each sum in the multiple sum on the left hand side starts with an arbitrary index, say c. To do this, we replace the sequence of integers (a_i), $i = 0, 1, 2, \ldots, n$ with $(a_i - c + 1)$, $i = 0, 1, 2, \ldots, n$. This gives

$$\left(\frac{x-1}{x}\right)^n \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} x^{a_0} = x^{a_n} - x^{c-1} \sum_{j=0}^{n-1} \left(\frac{x-1}{x}\right)^j \binom{a_j + j - c}{j}. \tag{3.1}$$

Note that the identities in Lemma 1 can also be caused to start from $j = c$ by writing $m - c + 1$ for m and $j - c + 1$ for j, and replacing (b_i), $i = 0, 1, 2, \ldots, s$ with $(b_i - c + 1)$, $i = 0, 1, 2, \ldots, s$, giving

$$\sum_{j=c}^{m} \binom{j - c + k}{k} = \binom{m - c + k + 1}{k + 1}, \tag{3.2}$$

$$\sum_{b_{s-1}=c}^{b_s} \sum_{b_{s-2}=c}^{b_{s-1}} \cdots \sum_{b_0=c}^{b_1} 1 = \binom{b_s + s - c}{s}. \tag{3.3}$$

The identity obtained by setting $c = 0$ in (3.3) provided the motivation for Butler and Karasik [2] to look at nested sums.
Multiplying through (3.1) by $(x/(x-1))^n$ and writing x/y for x and $-x/y$ for x, in turn, we have the following useful versions:

$$\begin{aligned} f(x, y; a_n, n, c) &= \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} \left(\frac{x}{y}\right)^{a_0} \\ &= \left(\frac{x}{x-y}\right)^n \left(\frac{x}{y}\right)^{a_n} - \sum_{j=0}^{n-1} \left(\frac{x}{x-y}\right)^{n-j} \left(\frac{x}{y}\right)^{c-1} \binom{a_n + j - c}{j}, \end{aligned} \tag{A}$$

$$\begin{aligned} g(x, y; a_n, n, c) &= \sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} (-1)^{a_0} \left(\frac{x}{y}\right)^{a_0} \\ &= (-1)^{a_n} \left(\frac{x}{x+y}\right)^n \left(\frac{x}{y}\right)^{a_n} + (-1)^c \sum_{j=0}^{n-1} \left(\frac{x}{x+y}\right)^{n-j} \left(\frac{x}{y}\right)^{c-1} \binom{a_n + j - c}{j}. \end{aligned} \tag{B}$$

Equipped with identities (A) and (B), we are now ready to state the results regarding the nested sums involving Horadam numbers. Theorems 1 and 2 are concerned with Fibonacci and Lucas numbers while Theorems 3–7 address the general Horadam sequence.

Theorem 1. *Let a_n, s and c be any integers and let n be a positive integer. Then,*

$$\sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} F_{3a_0+s} = \frac{F_{2n+3a_n+s}}{2^n} - \sum_{j=0}^{n-1} \frac{F_{2(n-j)+3(c-1)+s}}{2^{n-j}} \binom{a_n + j - c}{j}, \tag{F1a}$$

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1} L_{3a_0+s} = \frac{L_{2n+3a_n+s}}{2^n} - \sum_{j=0}^{n-1}\frac{L_{2(n-j)+3(c-1)+s}}{2^{n-j}}\binom{a_n+j-c}{j}. \tag{F1b}$$

Proof. Consider identity (A). Simplify both sides of

$$f(\alpha^3,1;a_n,n,c) \mp f(\beta^3,1;a_n,n,c),$$

using the Binet formulas (BF). □

Theorem 2. *Let a_n, s and c be any integers and let n be a positive integer. Then,*

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{a_0}F_{3a_0+s} = (-1)^{a_n}\frac{F_{n+3a_n+s}}{2^n} + (-1)^c\sum_{j=0}^{n-1}\frac{F_{n-j+3(c-1)+s}}{2^{n-j}}\binom{a_n+j-c}{j}, \tag{F2a}$$

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{a_0}L_{3a_0+s} = (-1)^{a_n}\frac{L_{n+3a_n+s}}{2^n} + (-1)^c\sum_{j=0}^{n-1}\frac{L_{n-j+3(c-1)+s}}{2^{n-j}}\binom{a_n+j-c}{j}. \tag{F2b}$$

Proof. Consider identity (B). Simplify both sides of

$$g(\alpha^3,1;a_n,n,c) \mp g(\beta^3,1;a_n,n,c),$$

using the Binet formulas (BF). □

Theorem 3. *Let r, s, c and a_n be any integers and let n be a positive integer. Then,*

$$\begin{aligned}\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{W_{ra_0+s}}{V_r^{a_0}} \\ = (-1)^n\frac{W_{r(a_n+2n)+s}}{q^{rn}V_r^{a_n}} - \frac{1}{V_r^{c-1}}\sum_{j=0}^{n-1}(-1)^{n-j}\frac{W_{r(2n-2j+c-1)+s}}{q^{r(n-j)}}\binom{a_n+j-c}{j}.\end{aligned} \tag{F3}$$

Proof. Refer to identity (A). Use the Binet formulas (BW) to simplify both sides of

$$\mathbb{A}\tau^s f(\tau^r,V_r;a_n,n,c) + \mathbb{B}\sigma^s f(\sigma^r,V_r;a_n,n,c).$$

□

The restricted Horadam sequence version of (F3) is

$$\begin{aligned}\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{w_{ra_0+s}}{v_r^{a_0}} \\ = (-1)^n\frac{w_{r(a_n+2n)+s}}{q^{rn}v_r^{a_n}} - \frac{1}{v_r^{c-1}}\sum_{j=0}^{n-1}(-1)^{n-j}\frac{w_{r(2n-2j+c-1)+s}}{q^{r(n-j)}}\binom{a_n+j-c}{j}.\end{aligned}$$

In particular,

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1} w_{a_0+s} = (-1)^n\frac{w_{a_n+2n+s}}{q^n} - \sum_{j=0}^{n-1}(-1)^{n-j}\frac{w_{2n-2j+c-1+s}}{q^{n-j}}\binom{a_n+j-c}{j}.$$

The gibonacci version of (F3) is

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{G_{ra_0+s}}{L_r^{a_0}} = (-1)^{n(r-1)}\frac{G_{r(a_n+2n)+s}}{L_r^{a_n}} - \frac{1}{L_r^{c-1}}\sum_{j=0}^{n-1}(-1)^{(n-j)(r-1)}G_{r(2n-2j+c-1)+s}\binom{a_n+j-c}{j};$$

a particular case of which $(r=1,\ s=0)$ is

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1} G_{a_0} = G_{a_n+2n} - \sum_{j=0}^{n-1} G_{2(n-j)}\binom{a_n+j-c}{j};$$

of which (H) is a special case $(c=1)$.

Theorem 4. *Let r, s, c and a_n be any integers and let n be a positive integer. Then,*

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{(-1)^{a_0}W_{2ra_0+s}}{q^{ra_0}} = (-1)^{a_n}\frac{W_{r(2a_n+n)+s}}{q^{ra_n}V_r^n} + \frac{(-1)^c}{q^{r(c-1)}}\sum_{j=0}^{n-1}\frac{W_{r(n-j+2c-2)+s}}{V_r^{n-j}}\binom{a_n+j-c}{j}. \tag{F4}$$

Proof. Refer to (B) and simplify

$$\mathbb{A}\tau^s g(\tau^r,\sigma^r;a_n,n,c) + \mathbb{B}\sigma^s g(\sigma^r,\tau^r;a_n,n,c).$$

□

The gibonacci version of (F4) is

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}(-1)^{(r-1)a_0}G_{2ra_0+s} = (-1)^{(r-1)a_n}\frac{G_{r(2a_n+n)+s}}{L_r^n} + (-1)^{r(c-1)+c}\sum_{j=0}^{n-1}\frac{G_{r(n-j+2c-2)+s}}{L_r^{n-j}}\binom{a_n+j-c}{j}.$$

In particular,

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1} G_{2a_0+s} = G_{2a_n+n+s} - \sum_{j=0}^{n-1} G_{n-j+2c-2+s}\binom{a_n+j-c}{j}.$$

For the proof of Theorems 5 and 6, we require the identities stated in Lemma 3.

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Lemma 3. *Let r and d be any integers. Then,*

$$U_{r+d} - \tau^r U_d = \sigma^d U_r, \tag{L1}$$
$$U_{r+d} - \sigma^r U_d = \tau^d U_r, \tag{L2}$$
$$V_{r+d} - \tau^r V_d = -\sigma^d U_r \Delta, \tag{L3}$$
$$V_{r+d} - \sigma^r V_d = \tau^d U_r \Delta. \tag{L4}$$

Proof. Each identity follows directly from the Binet formulas. For example, to prove (L1), we have

$$\begin{aligned} U_{r+d} - \tau^r U_d &= \frac{\tau^{r+d} - \sigma^{r+d}}{\tau - \sigma} - \tau^r \frac{\tau^d - \sigma^d}{\tau - \sigma} \\ &= \frac{1}{\tau - \sigma}\left(\tau^{r+d} - \sigma^{r+d} - \tau^{r+d} + \tau^r \sigma^d\right) \\ &= \frac{\sigma^d}{\tau - \sigma}\left(\tau^r - \sigma^r\right) = \sigma^d U_r. \end{aligned}$$

□

Theorem 5. *Let r, s, c, d and a_n be any integers; $r \neq 0$, $r + d \neq 0$. Let n be a positive integer. Then,*

$$\begin{aligned} &\sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} \left(\frac{U_d}{U_{r+d}}\right)^{a_0} W_{ra_0+s} \\ &\quad = \frac{(-1)^n U_d^{n+a_n}}{q^{dn} U_r^n U_{r+d}^{a_n}} W_{(r+d)n+ra_n+s} \\ &\qquad - \left(\frac{U_d}{U_{r+d}}\right)^{c-1} \sum_{j=0}^{n-1} \frac{(-1)^{n-j}}{q^{d(n-j)}} \left(\frac{U_d}{U_r}\right)^{n-j} W_{r(n-j+c-1)+d(n-j)+s} \binom{a_n + j - c}{j}. \end{aligned} \tag{F5}$$

Proof. Refer to identity (A) and simplify both sides of

$$\mathbb{A}\tau^s f(\tau^r U_d, U_{r+d}; a_n, n) + \mathbb{B}\sigma^s f(\sigma^r U_d, U_{r+d}; a_n, n),$$

using the Binet formulas and (L1) and (L2). □

Note that when $d = r$, (F5) reduces to (F3).
The gibonacci version of (F5) is

$$\begin{aligned} &\sum_{a_{n-1}=c}^{a_n} \sum_{a_{n-2}=c}^{a_{n-1}} \cdots \sum_{a_0=c}^{a_1} \left(\frac{F_d}{F_{r+d}}\right)^{a_0} G_{ra_0+s} \\ &\quad = \frac{(-1)^{n(d+1)} F_d^{n+a_n}}{F_r^n F_{r+d}^{a_n}} G_{(r+d)n+ra_n+s} \\ &\qquad - \left(\frac{F_d}{F_{r+d}}\right)^{c-1} \sum_{j=0}^{n-1} (-1)^{(n-j)(d+1)} \left(\frac{F_d}{F_r}\right)^{n-j} G_{r(n-j+c-1)+d(n-j)+s} \binom{a_n + j - c}{j}. \end{aligned}$$

The identity stated in Lemma 4 is required in the proof of Theorem 6.

Lemma 4 ([1, Lemma 1])**.** *For integer j,*

$$\mathbb{A}\tau^j - \mathbb{B}\sigma^j = \frac{w_{j+1} - qw_{j-1}}{\Delta}.$$

Theorem 6. *Let r, s, c, d and a_n be any integers; $r \neq 0$. If n is a positive even integer, then,*

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{V_d}{V_{r+d}}\right)^{a_0} W_{ra_0+s}\\
&\quad = \frac{1}{q^{dn}\Delta^n}\left(\frac{V_d}{U_r}\right)^n\left(\frac{V_d}{V_{r+d}}\right)^{a_n} W_{r(n+a_n)+dn+s}\\
&\qquad - \left(\frac{V_d}{V_{r+d}}\right)^{c-1}\frac{1}{\Delta^n}\sum_{j=0}^{(n-2)/2}\frac{\Delta^{2j}}{q^{d(n-2j)}}\left(\frac{V_d}{U_r}\right)^{n-2j} W_{(r+d)(n-2j)+r(c-1)+s}\binom{a_n+2j-c}{2j}\\
&\qquad - \left(\frac{V_d}{V_{r+d}}\right)^{c-1}\frac{1}{\Delta^{n+2}}\sum_{j=1}^{n/2}\left\{\frac{\Delta^{2j}}{q^{d(n-2j+1)}}\left(\frac{V_d}{U_r}\right)^{n-2j+1}\left(W_{(r+d)(n-2j+1)+r(c-1)+s+1} - \right.\right.\\
&\qquad\qquad \left.\left. qW_{(r+d)(n-2j+1)+r(c-1)+s-1}\right)\binom{a_n+2j-1-c}{2j-1}\right\},
\end{aligned} \tag{F6a}$$

while if n is a positive odd integer, then,

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{V_d}{V_{r+d}}\right)^{a_0} W_{ra_0+s}\\
&\quad = \frac{1}{q^{dn}\Delta^{n+1}}\left(\frac{V_d}{U_r}\right)^n\left(\frac{V_d}{V_{r+d}}\right)^{a_n}\left(W_{r(n+a_n)+dn+s+1} - qW_{r(n+a_n)+dn+s-1}\right)\\
&\qquad - \left(\frac{V_d}{V_{r+d}}\right)^{c-1}\frac{1}{\Delta^{n+1}}\sum_{j=0}^{(n-1)/2}\left\{\frac{\Delta^{2j}}{q^{d(n-2j)}}\left(\frac{V_d}{U_r}\right)^{n-2j}\left(W_{(r+d)(n-2j)+r(c-1)+s+1} - \right.\right.\\
&\qquad\qquad \left.\left. qW_{(r+d)(n-2j)+r(c-1)+s-1}\right)\binom{a_n+2j-c}{2j}\right\}\\
&\qquad - \left(\frac{V_d}{V_{r+d}}\right)^{c-1}\frac{1}{\Delta^{n+1}}\sum_{j=1}^{(n-1)/2}\frac{\Delta^{2j}}{q^{d(n-2j+1)}}\left(\frac{V_d}{U_r}\right)^{n-2j+1} W_{(r+d)(n-2j+1)+r(c-1)+s}\binom{a_n+2j-1-c}{2j-1}.
\end{aligned} \tag{F6b}$$

Proof. First employ the summation identity

$$\sum_{j=0}^{m} f_j = \sum_{j=0}^{\lfloor m/2\rfloor} f_{2j} + \sum_{j=1}^{\lceil m/2\rceil} f_{2j-1}$$

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to write (A) as

$$\begin{aligned} f(x,y;a_n,n,c) &= \sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{x}{y}\right)^{a_0} \\ &= \left(\frac{x}{x-y}\right)^n\left(\frac{x}{y}\right)^{a_n} - \sum_{j=0}^{\lfloor (n-1)/2\rfloor}\left(\frac{x}{x-y}\right)^{n-2j}\left(\frac{x}{y}\right)^{c-1}\binom{a_n+2j-c}{2j} \\ &\quad - \sum_{j=1}^{\lceil (n-1)/2\rceil}\left(\frac{x}{x-y}\right)^{n-2j+1}\left(\frac{x}{y}\right)^{c-1}\binom{a_n+2j-1-c}{2j-1}. \end{aligned} \tag{A2}$$

Using (A2), the Binet formulas and (L3) and (L4),

$$\mathbb{A}\tau^s f(\tau^r V_d, V_{r+d}; a_n, n) + \mathbb{B}\sigma^s f(\sigma^r V_d, V_{r+d}; a_n, n)$$

evaluates to

$$\begin{aligned} &\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{V_d}{V_{r+d}}\right)^{a_0} W_{ra_0+s} \\ &= \frac{1}{q^{dn}}\left(\frac{V_d}{U_r\Delta}\right)^n\left(\frac{V_d}{V_{r+d}}\right)^{a_n}\left(\mathbb{A}\tau^{r(n+a_n)+dn+s} + (-1)^n\mathbb{B}\sigma^{r(n+a_n)+dn+s}\right) \\ &\quad - \left(\frac{V_d}{V_{r+d}}\right)^{c-1}\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\left\{\frac{1}{q^{d(n-2j)}}\left(\frac{V_d}{U_r\Delta}\right)^{n-2j}\left(\mathbb{A}\tau^{(r+d)(n-2j)+r(c-1)+s} + \right.\right. \\ &\qquad\qquad \left.\left. (-1)^n\mathbb{B}\sigma^{(r+d)(n-2j)+r(c-1)+s}\right)\binom{a_n+2j-c}{2j}\right\} \\ &\quad - \left(\frac{V_d}{V_{r+d}}\right)^{c-1}\sum_{j=1}^{\lceil (n-1)/2\rceil}\left\{\frac{1}{q^{d(n-2j+1)}}\left(\frac{V_d}{U_r\Delta}\right)^{n-2j+1}\left(\mathbb{A}\tau^{(r+d)(n-2j+1)+r(c-1)+s} - \right.\right. \\ &\qquad\qquad \left.\left. (-1)^n\mathbb{B}\sigma^{(r+d)(n-2j+1)+r(c-1)+s}\right)\binom{a_n+2j-1-c}{2j-1}\right\}. \end{aligned} \tag{3.4}$$

The stated results now follow from the parity consideration of n in (3.4), the Binet formulas and Lemma 4. □

We now state the gibonacci versions of (F6a) and (F6b) for any integers r, s, c, d and a_n

such that $r \neq 0$. If n is a positive even integer, then,

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{L_d}{L_{r+d}}\right)^{a_0}G_{ra_0+s}\\
&=\frac{1}{5^{n/2}}\left(\frac{L_d}{F_r}\right)^{n}\left(\frac{L_d}{L_{r+d}}\right)^{a_n}G_{r(n+a_n)+dn+s}\\
&\quad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{1}{5^{n/2}}\sum_{j=0}^{(n-2)/2}5^j\left(\frac{L_d}{F_r}\right)^{n-2j}G_{(r+d)(n-2j)+r(c-1)+s}\binom{a_n+2j-c}{2j}\\
&\quad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{(-1)^d}{5^{(n+2)/2}}\sum_{j=1}^{n/2}\left\{5^j\left(\frac{L_d}{F_r}\right)^{n-2j+1}\left(G_{(r+d)(n-2j+1)+r(c-1)+s+1}+\right.\right.\\
&\qquad\left.\left.G_{(r+d)(n-2j+1)+r(c-1)+s-1}\right)\binom{a_n+2j-1-c}{2j-1}\right\},
\end{aligned}$$

while if n is a positive odd integer, then,

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{L_d}{L_{r+d}}\right)^{a_0}G_{ra_0+s}\\
&=\frac{(-1)^d}{5^{(n+1)/2}}\left(\frac{L_d}{F_r}\right)^{n}\left(\frac{L_d}{L_{r+d}}\right)^{a_n}\left(G_{r(n+a_n)+dn+s+1}+G_{r(n+a_n)+dn+s-1}\right)\\
&\quad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{(-1)^d}{5^{(n+1)/2}}\sum_{j=0}^{(n-1)/2}\left\{5^j\left(\frac{L_d}{F_r}\right)^{n-2j}\left(G_{(r+d)(n-2j)+r(c-1)+s+1}+\right.\right.\\
&\qquad\left.\left.G_{(r+d)(n-2j)+r(c-1)+s-1}\right)\binom{a_n+2j-c}{2j}\right\}\\
&\quad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{1}{5^{(n+1)/2}}\sum_{j=1}^{(n-1)/2}5^j\left(\frac{L_d}{F_r}\right)^{n-2j+1}G_{(r+d)(n-2j+1)+r(c-1)+s}\binom{a_n+2j-1-c}{2j-1}.
\end{aligned}$$

In particular, for the Fibonacci sequence, we have that if n is a positive even integer, then,

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{L_d}{L_{r+d}}\right)^{a_0}F_{ra_0+s}\\
&=\frac{1}{5^{n/2}}\left(\frac{L_d}{F_r}\right)^{n}\left(\frac{L_d}{L_{r+d}}\right)^{a_n}F_{r(n+a_n)+dn+s}\\
&\quad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{1}{5^{n/2}}\sum_{j=0}^{(n-2)/2}5^j\left(\frac{L_d}{F_r}\right)^{n-2j}F_{(r+d)(n-2j)+r(c-1)+s}\binom{a_n+2j-c}{2j}\\
&\quad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{(-1)^d}{5^{(n+2)/2}}\sum_{j=1}^{n/2}5^j\left(\frac{L_d}{F_r}\right)^{n-2j+1}L_{(r+d)(n-2j+1)+r(c-1)+s}\binom{a_n+2j-1-c}{2j-1},
\end{aligned}$$

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while if n is a positive odd integer, then,

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{L_d}{L_{r+d}}\right)^{a_0}F_{ra_0+s}\\
&\quad=\frac{(-1)^d}{5^{(n+1)/2}}\left(\frac{L_d}{F_r}\right)^n\left(\frac{L_d}{L_{r+d}}\right)^{a_n}L_{r(n+a_n)+dn+s}\\
&\qquad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{(-1)^d}{5^{(n+1)/2}}\sum_{j=0}^{(n-1)/2}5^j\left(\frac{L_d}{F_r}\right)^{n-2j}L_{(r+d)(n-2j)+r(c-1)+s}\binom{a_n+2j-c}{2j}\\
&\qquad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{1}{5^{(n+1)/2}}\sum_{j=1}^{(n-1)/2}5^j\left(\frac{L_d}{F_r}\right)^{n-2j+1}F_{(r+d)(n-2j+1)+r(c-1)+s}\binom{a_n+2j-1-c}{2j-1}.
\end{aligned}$$

As for the sequence of Lucas numbers, we have that if n is a positive even integer, then,

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{L_d}{L_{r+d}}\right)^{a_0}L_{ra_0+s}\\
&\quad=\frac{1}{5^{n/2}}\left(\frac{L_d}{F_r}\right)^n\left(\frac{L_d}{L_{r+d}}\right)^{a_n}L_{r(n+a_n)+dn+s}\\
&\qquad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{1}{5^{n/2}}\sum_{j=0}^{(n-2)/2}5^j\left(\frac{L_d}{F_r}\right)^{n-2j}L_{(r+d)(n-2j)+r(c-1)+s}\binom{a_n+2j-c}{2j}\\
&\qquad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{(-1)^d}{5^{n/2}}\sum_{j=1}^{n/2}5^j\left(\frac{L_d}{F_r}\right)^{n-2j+1}F_{(r+d)(n-2j+1)+r(c-1)+s}\binom{a_n+2j-1-c}{2j-1},
\end{aligned}$$

while if n is a positive odd integer, then,

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{L_d}{L_{r+d}}\right)^{a_0}L_{ra_0+s}\\
&\quad=\frac{(-1)^d}{5^{(n-1)/2}}\left(\frac{L_d}{F_r}\right)^n\left(\frac{L_d}{L_{r+d}}\right)^{a_n}F_{r(n+a_n)+dn+s}\\
&\qquad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{(-1)^d}{5^{(n-1)/2}}\sum_{j=0}^{(n-1)/2}5^j\left(\frac{L_d}{F_r}\right)^{n-2j}F_{(r+d)(n-2j)+r(c-1)+s}\binom{a_n+2j-c}{2j}\\
&\qquad-\left(\frac{L_d}{L_{r+d}}\right)^{c-1}\frac{1}{5^{(n+1)/2}}\sum_{j=1}^{(n-1)/2}5^j\left(\frac{L_d}{F_r}\right)^{n-2j+1}L_{(r+d)(n-2j+1)+r(c-1)+s}\binom{a_n+2j-1-c}{2j-1}.
\end{aligned}$$

Theorem 7. *Let r, s, d, a_n, c be any integers; $r+1\neq d$. If $W_{r+s}\neq 0$ and $W_{r+d}\neq 0$ then,*

$$\begin{aligned}
&\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}q^{a_0}\left(\frac{U_{r-d}}{U_{r-d+1}}\right)^{a_0}\left(\frac{W_{s+d-1}}{W_{s+d}}\right)^{a_0}\\
&\quad=(-1)^nq^{n+a_n}U_{r-d}^n\left(\frac{U_{r-d}}{U_{r-d+1}}\right)^{a_n}\left(\frac{W_{s+d-1}}{W_{s+d}}\right)^{a_n}\left(\frac{W_{s+d-1}}{W_{r+s}}\right)^n\\
&\qquad-q^{c-1}\left(\frac{U_{r-d}}{U_{r-d+1}}\right)^{c-1}\left(\frac{W_{s+d-1}}{W_{s+d}}\right)^{c-1}\sum_{j=0}^{n-1}(-1)^{n-j}q^{n-j}U_{r-d}^{n-j}\left(\frac{W_{s+d-1}}{W_{r+s}}\right)^{n-j}\binom{a_n+j-c}{j}.
\end{aligned}\tag{F7}$$

Proof. Refer to (A) and simplify

$$f(qU_{r-d}W_{s+d-1}, U_{r-d+1}W_{s+d}; a_n, n, c),$$

making use of the following identity [3, Identity 3.15]:

$$W_{r+s} = U_{r-d+1}W_{s+d} - qU_{r-d}W_{s+d-1}.$$

□

The restricted Horadam version ($p = 1$) and the gibonacci version of (F7) are

$$\begin{aligned}\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1} q^{a_0}\left(\frac{u_{r-d}}{u_{r-d+1}}\right)^{a_0}\left(\frac{w_{s+d-1}}{w_{s+d}}\right)^{a_0}\\ = (-1)^n q^{n+a_n} u_{r-d}^n \left(\frac{u_{r-d}}{u_{r-d+1}}\right)^{a_n}\left(\frac{w_{s+d-1}}{w_{s+d}}\right)^{a_n}\left(\frac{w_{s+d-1}}{w_{r+s}}\right)^{n}\\ - q^{c-1}\left(\frac{u_{r-d}}{u_{r-d+1}}\right)^{c-1}\left(\frac{w_{s+d-1}}{w_{s+d}}\right)^{c-1}\sum_{j=0}^{n-1}(-1)^{n-j}q^{n-j}u_{r-d}^{n-j}\left(\frac{w_{s+d-1}}{w_{r+s}}\right)^{n-j}\binom{a_n+j-c}{j},\end{aligned}$$

$$\begin{aligned}\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1} (-1)^{a_0}\left(\frac{F_{r-d}}{F_{r-d+1}}\right)^{a_0}\left(\frac{G_{s+d-1}}{G_{s+d}}\right)^{a_0}\\ = (-1)^{a_n} F_{r-d}^n \left(\frac{F_{r-d}}{F_{r-d+1}}\right)^{a_n}\left(\frac{G_{s+d-1}}{G_{s+d}}\right)^{a_n}\left(\frac{G_{s+d-1}}{G_{r+s}}\right)^{n}\\ + (-1)^{c}\left(\frac{F_{r-d}}{F_{r-d+1}}\right)^{c-1}\left(\frac{G_{s+d-1}}{G_{s+d}}\right)^{c-1}\sum_{j=0}^{n-1}F_{r-d}^{n-j}\left(\frac{G_{s+d-1}}{G_{r+s}}\right)^{n-j}\binom{a_n+j-c}{j}.\end{aligned}$$

Taking $r = 1$, $d = 0$ gives

$$\begin{aligned}\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1} q^{a_0}\left(\frac{w_{s-1}}{w_{s}}\right)^{a_0}\\ = (-1)^n q^{n+a_n}\left(\frac{w_{s-1}}{w_{s}}\right)^{a_n}\left(\frac{w_{s-1}}{w_{s+1}}\right)^{n}\\ - q^{c-1}\left(\frac{w_{s-1}}{w_{s}}\right)^{c-1}\sum_{j=0}^{n-1}(-1)^{n-j}q^{n-j}\left(\frac{w_{s-1}}{w_{s+1}}\right)^{n-j}\binom{a_n+j-c}{j},\end{aligned}$$

$$\begin{aligned}\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1} (-1)^{a_0}\left(\frac{G_{s-1}}{G_{s}}\right)^{a_0}\\ = (-1)^{a_n}\left(\frac{G_{s-1}}{G_{s}}\right)^{a_n}\left(\frac{G_{s-1}}{G_{s+1}}\right)^{n}\\ + (-1)^{c}\left(\frac{G_{s-1}}{G_{s}}\right)^{c-1}\sum_{j=0}^{n-1}\left(\frac{G_{s-1}}{G_{s+1}}\right)^{n-j}\binom{a_n+j-c}{j}.\end{aligned}$$

4 Concluding comments

In this paper we evaluated the following nested sums involving the terms of the Horadam sequence:

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{W_{ra_0+s}}{V_r^{a_0}},\quad \sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\frac{(-1)^{a_0}W_{2ra_0+s}}{q^{ra_0}},$$

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{U_d}{U_{r+d}}\right)^{a_0}W_{ra_0+s},\quad \sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}\left(\frac{V_d}{V_{r+d}}\right)^{a_0}W_{ra_0+s},$$

$$\sum_{a_{n-1}=c}^{a_n}\sum_{a_{n-2}=c}^{a_{n-1}}\cdots\sum_{a_0=c}^{a_1}q^{a_0}\left(\frac{U_{r-d}}{U_{r-d+1}}\right)^{a_0}\left(\frac{W_{s+d-1}}{W_{s+d}}\right)^{a_0}.$$

Explicit results for the special Horadam sequence (w_j) and the gibonacci sequence G_j were presented.
The starting point for each summation in the multiple sums considered was fixed at $a_i = c$, for $i = 0, 1, 2, \ldots, n-1$. It is possible to choose a different lower limit for each sum in the nested sum. The aim would then be to evaluate

$$\sum_{a_{n-1}=c_{n-1}}^{a_n}\sum_{a_{n-2}=c_{n-2}}^{a_{n-1}}\cdots\sum_{a_1=c_1}^{a_2}\sum_{a_0=c_0}^{a_1}x^{a_0}, \tag{S}$$

in which the sequence of integers (c_i), $i = 0, 1, 2, \ldots, n-1$, is such that c_i may be different from c_j if i is different from j. In this case, a similar iteration to what produced (2.2) would give

$$\begin{aligned}&\left(\frac{x-1}{x}\right)^n\sum_{a_{n-1}=c_{n-1}}^{a_n}\sum_{a_{n-2}=c_{n-2}}^{a_{n-1}}\cdots\sum_{a_0=c_0}^{a_1}x^{a_0}\\ &\quad = x^{a_n} - x^{c_{n-1}-1} - \sum_{j=1}^{n-1}\left\{\left(\frac{x-1}{x}\right)^j x^{c_{n-j-1}-1}\sum_{a_{n-1}=c_{n-1}}^{a_n}\sum_{a_{n-2}=c_{n-2}}^{a_{n-1}}\cdots\sum_{a_{n-j}=c_{n-j}}^{a_{n-j+1}}1\right\}.\end{aligned}$$

A suggestion for further research would be the determination of sums of the form

$$\sum_{b_{s-1}=c_{s-1}}^{b_s}\sum_{b_{s-2}=c_{s-2}}^{b_{s-1}}\cdots\sum_{b_1=c_1}^{b_2}\sum_{b_0=c_0}^{b_1}1,$$

which would facilitate the evaluation of (S).

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