*Article*

# Weighted Hermite-Hadamard inequalities for $r-$Times Differentiable Preinvex Functions for $k-$Fractional Integrals

**Fiza Zafar [1,‡*] 0000-0003-0552-2783, Sikander Mehmood [1,‡] and Asim Asiri [2,‡]**

[1] Affiliation 1; Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan 60800, Pakistan
[2] Affiliation 2; Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
* Correspondence: fizazafar@bzu.edu.pk
‡ These authors contributed equally to this work.

**Abstract:** In this paper, we have established some new bounds of Fejér type Hermite-Hadamard inequality for $k-$fractional integrals involving $r-$times differentiable preinvex functions. It is noteworthy that in the past there was no weighted version of left and right sides of the Hermite-Hadamard inequality for k-fractional integrals for generalized convex functions available in literature.

**Keywords:** Hermite-Hadamard-Fejér inequality; $k-$fractional integral; Preinvex function; $r$-times differentiable function

## 1. Introduction

In various disciplines of science, the Hermite-Hadamard inequality for convex functions is studied as it develops a link between convex function theory and integral inequalities. Many generalizations of convex functions have been discovered recently. Researchers have also shown a lot of interest for generalizing this concept for preinvex functions.

Let $\psi$ be a convex function such that $\psi : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ and $\sigma_a, \sigma_b \in \Omega$ with $\sigma_a < \sigma_b$, then

$$\psi\left(\frac{\sigma_a+\sigma_b}{2}\right) \leq \frac{1}{\sigma_b-\sigma_a}\int_{\sigma_a}^{\sigma_b}\psi(t)dt \leq \frac{\psi(\sigma_a)+\psi(\sigma_b)}{2}. \tag{1}$$

is the well known **Hermite-Hadamard inequality** for convex functions.

The generalization of inequality (1) is given by Fejér [2], as follows:

$$\psi\left(\frac{\sigma_a+\sigma_b}{2}\right)\int_{\sigma_a}^{\sigma_b}\phi(t)dt \leq \int_{\sigma_a}^{\sigma_b}\phi(t)\psi(t)dt \leq \frac{\psi(\sigma_a)+\psi(\sigma_b)}{2}\int_{\sigma_a}^{\sigma_b}\phi(t)dt \tag{2}$$

Where $\phi : [\sigma_a;\sigma_b] \to \mathbb{R}$ is a nonnegative, integrable function and symmetric about $t = \frac{\sigma_a+\sigma_b}{2}$.

Due to wide application of fractional calculus and Hermite-Hadamard inequalities in different fields of sciences, researchers are working on $k-$fractional integrals for extending work on Hermite-Hadamard type inequalities

In [11], Sarikaya et al. proposed the following inequalities as follows.

**Theorem 1.** *Let $\psi : [\sigma_a, \sigma_b] \subseteq \mathbb{R} \to \mathbb{R}$ with $0 \leq \sigma_a < \sigma_b$ and $\psi \in L[\sigma_a, \sigma_b]$ be a convex function. If $\psi$ is positive on $[\sigma_a, \sigma_b]$, then the following inequalities hold:*

$$\psi\left(\frac{\sigma_a+\sigma_b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\sigma_b-\sigma_a)^\alpha}\left[J^\alpha_{\sigma_b}\psi(\sigma_a)+J^\alpha_{\sigma_a}\psi(\sigma_b)\right] \leq \frac{\psi(\sigma_a)+\psi(\sigma_b)}{2}.$$

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*with* $\beta > 0$.

Here, the symbols $J^{\beta}_{\sigma_a}$ is the left-sided while $J^{\beta}_{\sigma_b}$ be the right-sided Riemann Liouville fractional integrals of the order $\beta \in \mathbb{R}^+$ that are defined in [4]

$$J^{\beta}_{\sigma_a}\psi(t) = \frac{1}{\Gamma(\beta)} \int_{\sigma_a}^{t} (t-s)^{\beta-1}\psi(s)ds, \; 0 \le \sigma_a < t \le \sigma_b,$$

and

$$J^{\beta}_{\sigma_b}\psi(t) = \frac{1}{\Gamma(\beta)} \int_{t}^{\sigma_b} (s-t)^{\beta-1}\psi(s)ds, \; 0 \le \sigma_a \le t < \sigma_b.$$

For $\beta = 1$, the fractional integral becomes the classical integral.

We now give the definition of $k$-fractional integral which is mainly due to [9].

**Definition 1.** *The left sided Riemann Liouville k-fractional integrals of order* $\beta$, $k > 0$ *are defined as:*

$$J^{\beta,k}_{\sigma_a}\psi(t) = \frac{1}{k\Gamma_k(\beta)} \int\limits_{\sigma_a}^{t} (t-s)^{\frac{\beta}{k}-1}\psi(s)ds, \; 0 \le \sigma_a < t \le \sigma_b,$$

*and the right sided Riemann Liouville k-fractional integrals of order* $\beta$, $k > 0$ *are defined as:*

$$J^{\beta,k}_{\sigma_b}\psi(t) = \frac{1}{k\Gamma_k(\beta)} \int\limits_{t}^{\sigma_b} (s-t)^{\frac{\beta}{k}-1}\psi(s)ds, \; 0 \le \sigma_a \le t < \sigma_b,$$

*where* $\psi \in L_1([\sigma_a, \sigma_b])$.

T. Antczak [1] gave the idea of invex sets as:

**Definition 2.** *A set* $\Omega \subseteq \mathbb{R}$ *be an invex w.r.t the map* $\aleph : \Omega \times \Omega \to \mathbb{R}$ *if for every* $\sigma_a, \sigma_b \in \Omega$ *and* $s \in [0,1]$, $\sigma_b + s\aleph(\sigma_a, \sigma_b) \in \Omega$.

The generalization of convex functions is given by Weir and Mond [12].

**Definition 3.** *Let* $\Omega \subseteq \mathbb{R}$ *is an invex set and* $\psi : \Omega \to \mathbb{R}$ *is called a preinvex function w.r.t.* $\aleph$ *if*

$$\psi(\sigma_b + s\aleph(\sigma_a, \sigma_b)) \le tg(\sigma_a) + (1-s)\psi(\sigma_b).$$

$\forall \; \sigma_a, \sigma_b \in \Omega$ and $s \in [0,1]$.

If $\aleph(\sigma_a, \sigma_b) = \sigma_a - \sigma_b$ , then in classical sense , the preinvex functions become convex functions.

The following lemma for $n-$times differentiable preinvex functions is proposed by Sikander et. al (see [8]).

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**Lemma 1.** *Let $\Omega \subseteq [0,\infty)$ be an open invex subset with respect to $\aleph : \Omega \times \Omega \to \mathbb{R}$ . Suppose $\psi : \Omega \to \mathbb{R}$ is a function such that $\psi^{(r)}$ exists on $\Omega$ and $\psi^{(r)}$ is integrable on $[\sigma_a, \sigma_a + \aleph(\sigma_b,\sigma_a)]$ for $r \in \mathbb{N}$, $r \geq 1$, then for every $\sigma_a, \sigma_b \in \Omega$ with $\aleph(\sigma_b,\sigma_a) > 0$, the following equality holds:*

$$\begin{aligned}
&\frac{\psi(\sigma_a) + \psi(\sigma_a + \aleph(\sigma_b,\sigma_a))}{2} \\
&-\frac{\Gamma(p+1)}{2(\aleph(\sigma_b,\sigma_a))^{p}}\left[J^{p,k}_{\sigma_a}\psi(\sigma_a + \aleph(\sigma_b,\sigma_a)) + J^{p,k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))^-}\psi(\sigma_a)\right] \\
=\;&\sum_{\kappa=1}^{r-1}\frac{\Gamma(p+1)(\aleph(\sigma_b,\sigma_a))^{\kappa}}{2\,\Gamma(p+\kappa+1)}\left[(-1)^{\kappa-1}\psi^{(\kappa)}(\sigma_a + \aleph(\sigma_b,\sigma_a)) - \psi^{(\kappa)}(\sigma_a)\right] \\
&-\frac{(\aleph(\sigma_b,\sigma_a))^{r}\,\Gamma(p+1)}{2\,\Gamma(p+r)} \\
&\times\int_0^1\left[(1-s)^{p+r-1} + (-1)^{r}s^{p+r-1}\right]\psi^{(r)}(\sigma_a + s\aleph(\sigma_b,\sigma_a))ds,
\end{aligned}$$

*where $p > 0$ and $r \geq 1$*

In this paper, we have developed new Fejér type Hermite-Hadamard identities for higher order differentiable generalized convex functions for $k-$fractional integrals. Then, we have developed both left and right hand side of weighted Hermite-Hadamard inequalities.

## 2. Main Results

In the main section, we make the assumption $\|\phi\|_\infty = \sup_{t\in[\sigma_a,\sigma_a+\aleph(\sigma_b,\sigma_a)]}|\phi(t)|$, where $\phi : [\sigma_a;\sigma_a + \aleph(\sigma_b,\sigma_a)] \to \mathbb{R}$ is a continuous function, $\psi^{(r)}$ is the $r$-th derivative of $\psi$ w.r.t. variable $s$ and $L[\sigma_a,\sigma_b]$ is the collection of all real-valued Riemann integrable functions defined on the interval $[\sigma_a,\sigma_b]$.

**Lemma 2.** *Let $\Omega \subseteq \mathbb{R}$ be an open invex set and $\aleph$ such that $\aleph : \Omega \times \Omega \to \mathbb{R}$ be a mapping. Suppose, $\psi : \Omega \to \mathbb{R}$ is a differentiable mapping such that $\psi^{(r)} \in L[\sigma_a, \sigma_a + \aleph(\sigma_b,\sigma_a)]$ where $\aleph(\sigma_b,\sigma_a) > 0$. If $w : [\sigma_a, \sigma_a + \aleph(\sigma_b,\sigma_a)] \to [0,\infty)$ is an integrable mapping, then $\forall\, \sigma_a, \sigma_b \in \Omega$, then we have the following equality*:

$$\begin{aligned}
&\sum_{m=0}^{r-1}\frac{\psi^{(m)}(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^{m}\aleph(\sigma_b,\sigma_a)^{\frac{p}{k}+r-m}} \\
&\times\left[(-1)^{r-m-1}J^{p,k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))^-}\phi(\sigma_a) + (-1)^{r+1}J^{p,k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))^+}\phi(\sigma_a + \aleph(\sigma_b,\sigma_a))\right] \\
&+(-1)^{r}\frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{p}{k}+r}} \\
&\times\left[J^{p,k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))^-}(gf)(\sigma_a) + J^{p,k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))^+}(gf)(\sigma_a + \aleph(\sigma_b,\sigma_a))\right] \\
=\;&\frac{1}{k\Gamma_k(p)}\int_0^1 w(s)\psi^{(r)}(\sigma_a + s\aleph(\sigma_b,\sigma_a))ds,
\end{aligned} \tag{3}$$

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where

$$w(s)=\begin{cases}\underbrace{\int_0^s\int_0^s\cdots\int_0^s}_{r\ \ integrals}u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r,\ s\in[0,\frac{1}{2}).\\ \underbrace{\int_1^s\int_1^s\cdots\int_1^s}_{r\ \ integrals}(1-u)^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r,\ s\in[\frac{1}{2},1].\end{cases}$$

**Proof.** Consider

$$\begin{aligned}&\int_0^1 w(s)\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds\\ =&\int_0^{\frac{1}{2}}\left(\underbrace{\int_0^s\int_0^s\cdots\int_0^s}_{r\ \ integrals}u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r\right)\\ &\times\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds\\ &+\int_{\frac{1}{2}}^1\left(\underbrace{\int_1^s\int_1^s\cdots\int_1^s}_{r\ \ integrals}(1-u)^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r\right)\\ &\times\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds\\ =&\ I_1+I_2\end{aligned}$$

From the first integral, we have

$$\begin{aligned}I_1=&\int_0^{\frac{1}{2}}\left(\underbrace{\int_0^s\int_0^s\cdots\int_0^s}_{r\ \ integrals}u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r\right)\\ &\times\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds\\ =&\frac{1}{\aleph(\sigma_b,\sigma_a)}\left(\underbrace{\int_0^s\int_0^s\cdots\int_0^s}_{r\ \ integrals}u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r\right)\\ &\times\psi^{(r-1)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))\Big|_0^{\frac{1}{2}}\\ &-\frac{1}{\aleph(\sigma_b,\sigma_a)}\int_0^{\frac{1}{2}}\left(\underbrace{\int_0^s\int_0^s\cdots\int_0^s}_{r-1\ \ integrals}u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^{r-1}\right)\\ &\times\psi^{(r-1)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds\end{aligned}$$

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$$
\begin{aligned}
I_1 = {} & \frac{\psi^{(r-1)}(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{\aleph(\sigma_b,\sigma_a)} \\
& \times \left( \underbrace{\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \dots \int_0^{\frac{1}{2}}}_{r\ \ integrals} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a))(ds)^r \right) \\
& - \frac{1}{\aleph(\sigma_b,\sigma_a)} \int_0^{\frac{1}{2}} \left( \underbrace{\int_0^{s} \int_0^{s} \dots \int_0^{s}}_{r-1\ \ integrals} u^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + u\aleph(\sigma_b,\sigma_a))(du)^{r-1} \right) \\
& \times \psi^{(r-1)}(\sigma_a + s\aleph(\sigma_b,\sigma_a))ds.
\end{aligned}
$$

55 After generalization, we obtain

$$
\begin{aligned}
I_1 = {} & \frac{\psi^{(r-1)}(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{\aleph(\sigma_b,\sigma_a)} \left( \underbrace{\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \dots \int_0^{\frac{1}{2}}}_{r\ \ integrals} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a))(ds)^r \right) \\
& - \frac{\psi^{(r-2)}(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^2} \left( \underbrace{\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \dots \int_0^{\frac{1}{2}}}_{r-1\ \ integrals} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a))(ds)^{r-1} \right) \\
& \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
& \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
& \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
& + (-1)^{r-2} \frac{\psi'(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{r-1}} \left( \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a))(ds)^2 \right) \\
& + (-1)^{r-1} \frac{\psi(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{r}} \left( \int_0^{\frac{1}{2}} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a))ds \right) \\
& + (-1)^{r} \frac{1}{(\aleph(\sigma_b,\sigma_a))^{r}} \left( \int_0^{\frac{1}{2}} s^{\frac{\text{þ}}{k}-1} \psi(\sigma_a + s\aleph(\sigma_b,\sigma_a)) \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a)) \right) ds.
\end{aligned}
$$

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After simplification

$$\begin{aligned} I_1 = {} & \frac{\psi^{(r-1)}(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^{r-1}\aleph(\sigma_b,\sigma_a)} \left( \int_0^{\frac{1}{2}} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a)) ds \right) \\ & - \frac{\psi^{(r-2)}(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^{r-2}(\aleph(\sigma_b,\sigma_a))^2} \left( \int_0^{\frac{1}{2}} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a)) ds \right) \\ & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & + (-1)^{r-2} \frac{\psi'(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2(\aleph(\sigma_b,\sigma_a))^{r-1}} \left( \int_0^{\frac{1}{2}} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a)) ds \right) \\ & + (-1)^{r-1} \frac{\psi(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{r}} \left( \int_0^{\frac{1}{2}} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a)) ds \right) \\ & + (-1)^{r} \frac{1}{(\aleph(\sigma_b,\sigma_a))^{r}} \left( \int_0^{\frac{1}{2}} s^{\frac{\text{þ}}{k}-1} \psi(\sigma_a + s\aleph(\sigma_b,\sigma_a)) \phi(\sigma_a + s\aleph(\sigma_b,\sigma_a)) ds \right). \end{aligned}$$

On substituting $t = \sigma_a + s\aleph(\sigma_b,\sigma_a)$, we get

$$\begin{aligned} I_1 = {} & \frac{\psi^{(r-1)}(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^{r-1}\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+1}} \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1} \phi(t) dt \\ & - \frac{\psi^{(r-2)}(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^{r-2}(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+2}} \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1} \phi(t) dt \\ & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & + (-1)^{r-2} \frac{\psi'(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r-1}} \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1} \phi(t) dt \\ & + (-1)^{r-1} \frac{\psi(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1} \phi(t) dt \\ & + (-1)^{r} \frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} \int_{\sigma_a}^{\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1} \psi(t)\phi(t) dt. \end{aligned}$$

From the definition of $k-$fractional integrals, we have

$$
\begin{aligned}
I_1 &= \frac{\psi^{(r-1)}(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))k\Gamma_k(\text{þ})}{2^{r-1}\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+1}} J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\
&\quad -\frac{\psi^{(r-2)}(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))k\Gamma_k(\text{þ})}{2^{r-2}(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+2}} J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\
&\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad +(-1)^{r-2}\frac{\psi'(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))k\Gamma_k(\text{þ})}{2(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r-1}} J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\
&\quad +(-1)^{r-1}\frac{\psi(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))k\Gamma_k(\text{þ})}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\
&\quad +(-1)^{r}\frac{k\Gamma_k(\text{þ})}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a).
\end{aligned}
$$

After summing the above series, we get

$$
\begin{aligned}
I_1 &= \sum_{m=0}^{r-1}(-1)^{r-m-1}\frac{\psi^{(m)}(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^m\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+r-m}}k\Gamma_k(\text{þ})J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\
&\quad +(-1)^r\frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}}k\Gamma_k(\text{þ})J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a).
\end{aligned}
\tag{4}
$$

Similarly from the second integral, we have

$$
\begin{aligned}
I_2 &= (-1)^{r+1}\sum_{m=0}^{r-1}\frac{\psi^{(m)}(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^m\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+r-m}}k\Gamma_k(\text{þ})J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_{.2},\sigma_a))+}\phi(\sigma_a+\aleph(\sigma_b,\sigma_a)) \\
&\quad +(-1)^r\frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}}k\Gamma_k(\text{þ})J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+}(gf)(\sigma_a+\aleph(\sigma_b,\sigma_a)).
\end{aligned}
\tag{5}
$$

On adding (3)and (4), we obtain the required result. □

**Lemma 3.** *For $k=r=1$, we get Lemma 1 of [7].*

$$
\begin{aligned}
&\frac{\psi(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))}{\aleph(\sigma_b,\sigma_a)^{\text{þ}+1}} \\
&\times\left[J^{\text{þ}}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a)+J^{\text{þ}}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+}\phi(\sigma_a+\aleph(\sigma_b,\sigma_a))\right] \\
&-\frac{1}{(\aleph(\sigma_b,\sigma_a))^{\text{þ}+1}} \\
&\times\left[J^{\text{þ}}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a)+J^{\text{þ}}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+}(gf)(\sigma_a+\aleph(\sigma_b,\sigma_a))\right] \\
=\; &\frac{1}{\Gamma(\text{þ})}\int_0^1 w(s)\psi'(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds,
\end{aligned}
$$

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where

$$w(s)=\begin{cases}\int_0^s u^{\text{þ}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))du, \ s\in[0,\frac{1}{2}).\\ \int_1^s(1-u)^{\text{þ}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))du, \ s\in[\frac{1}{2},1].\end{cases}$$

**Lemma 4.** *If $\psi:[\sigma_a,\sigma_a+\aleph(\sigma_b,\sigma_a)]\to\mathbb{R}$ is an integrable function which is also symmetric about $\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a)$ with $\sigma_a<\sigma_a+\aleph(\sigma_b,\sigma_a)$,then*

$$\begin{aligned}J^{\text{þ},k}_{\sigma_a+}\psi(\sigma_a+\aleph(\sigma_b,\sigma_a)) &= J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\psi(\sigma_a)\\ &= \frac{1}{2}\left[J^{\text{þ},k}_{\sigma_a+}\psi(\sigma_a+\aleph(\sigma_b,\sigma_a))+J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\psi(\sigma_a)\right],\end{aligned} \tag{6}$$

*where $\text{þ}>0$*

**Proof.** Since $\psi$ is symmetric about $\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a)$, we have $\psi(2\sigma_1+\aleph(\sigma_b,\sigma_a)-t)=\psi(t)$, for all $t\in[\sigma_a,\sigma_a+\aleph(\sigma_b,\sigma_a)]$. Taking $2\sigma_1+\aleph(\sigma_b,\sigma_a)-s=t$

$$\begin{aligned}&J^{\text{þ},k}_{\sigma_a+}\psi(\sigma_a+\aleph(\sigma_b,\sigma_a))\\ =&\ \frac{1}{k\Gamma_k(\text{þ})}\int_{\sigma_a}^{\sigma_a+\aleph(\sigma_b,\sigma_a)}[(\sigma_a+\aleph(\sigma_b,\sigma_a)-s]^{\frac{\text{þ}}{k}-1}\psi(s)ds\\ =&\ \frac{1}{k\Gamma_k(\text{þ})}\int_{\sigma_a}^{\sigma_a+\aleph(\sigma_b,\sigma_a)}(t-\sigma_a)^{\frac{\text{þ}}{k}-1}\psi(2\sigma_1+\aleph(\sigma_b,\sigma_a)-t)dt\\ =&\ \frac{1}{k\Gamma_k(\text{þ})}\int_{\sigma_a}^{\sigma_a+\aleph(\sigma_b,\sigma_a)}(t-\sigma_a)^{\frac{\text{þ}}{k}-1}\psi(t)dt\\ =&\ J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\psi(\sigma_a).\end{aligned}$$

□

**Lemma 5.** *Let $\Omega\subseteq\mathbb{R}$ be an open invex set and $\aleph$ such that $\aleph:\Omega\times\Omega\to\mathbb{R}$ be a mapping. Suppose, $\psi:\Omega\to\mathbb{R}$ is a differentiable mapping such that $\psi^{(r)}\in L[\sigma_a,\sigma_a+\aleph(\sigma_b,\sigma_a)]$ where $\aleph(\sigma_b,\sigma_a)>0$. If $w:[\sigma_a,\sigma_a+\aleph(\sigma_b,\sigma_a)]\to[0,\infty)$ is an integrable mapping, then $\forall\,\sigma_a,\sigma_b\in\Omega$, then we have the following equality:*

$$\begin{aligned}&\sum_{m=0}^{r-1}\left[\frac{(-1)^{r+1}\psi^{(m)}(\sigma_a)+(-1)^{r-m-1}\psi^{(m)}(\sigma_a+\aleph(\sigma_b,\sigma_a))}{2(\aleph(\sigma_b,\sigma_a))^{\text{þ}+r-m}}\right]\\ &\times\left[J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a)+J^{\text{þ},k}_{\sigma_a+}\phi(\sigma_a+\aleph(\sigma_b,\sigma_a))\right]\\ &+(-1)^r\left[\frac{J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a)+J^{\text{þ},k}_{\sigma_a+}(gf)(\sigma_a+\aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{\text{þ}+r}}\right]\\ =&\ \frac{1}{k\Gamma_k(\text{þ})}\int_0^1 w(s)\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds,\end{aligned} \tag{7}$$

where

$$\begin{aligned}w(s)=&\underbrace{\int_0^s\int_0^s...\int_0^s}_{r\ \ integrals}u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r\\ &+\underbrace{\int_1^s\int_1^s..\int_1^s}_{r\ \ integrals}(1-u)^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r, s\in[0,1].\end{aligned}$$

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**Proof.** Consider

$$
\begin{aligned}
&\int_0^1 w(s)\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds \\
&= \int_0^1\left(\underbrace{\int_0^s\int_0^s\cdots\int_0^s}_{r\ \ integrals} u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r\right) \\
&\quad\times\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds \\
&\quad+\int_0^1\left(\underbrace{\int_1^s\int_1^s\cdot\cdot\int_1^s}_{r\ \ integrals}(1-u)^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r\right) \\
&\quad\times\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds \\
&= I_1+I_2
\end{aligned}
$$

From the first integral, we have

$$
\begin{aligned}
I_1 &= \int_0^1\left(\underbrace{\int_0^s\int_0^s\cdots\int_0^s}_{r\ \ integrals} u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r\right) \\
&\quad\times\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds \\
&= \frac{1}{\aleph(\sigma_b,\sigma_a)}\left(\underbrace{\int_0^s\int_0^s\cdots\int_0^s}_{r\ \ integrals} u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^r\right) \\
&\quad\times\psi^{(r-1)}(\sigma_a+s\aleph(\sigma_b,\sigma_a)\Big|_0^1 \\
&\quad-\frac{1}{\aleph(\sigma_b,\sigma_a)}\int_0^1\left(\underbrace{\int_0^s\int_0^s\cdots\int_0^s}_{r-1\ \ integrals} u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^{r-1}\right) \\
&\quad\times\psi^{(r-1)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \frac{\psi^{(r-1)}(\sigma_a+\aleph(\sigma_b,\sigma_a))}{\aleph(\sigma_b,\sigma_a)}\left(\underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{r\ \ integrals} s^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+s\aleph(\sigma_b,\sigma_a))(ds)^r\right) \\
&\quad-\frac{1}{\aleph(\sigma_b,\sigma_a)}\int_0^1\left(\underbrace{\int_0^s\int_0^s\cdots\int_0^s}_{r-1\ \ integrals} u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))(du)^{r-1}\right) \\
&\quad\times\psi^{(r-1)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds.
\end{aligned}
$$

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On generalizing the result, we have

$$
\begin{aligned}
I_1 \quad = \quad & \frac{\psi^{(r-1)}(\sigma_a + (\sigma_b, \sigma_a))}{\aleph(\sigma_b, \sigma_a)} \left( \underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{r \ \ integrals} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))(ds)^r \right) \\
& - \frac{\psi^{(r-2)}(\sigma_a + \aleph(\sigma_b, \sigma_a))}{(\aleph(\sigma_b, \sigma_a))^2} \left( \underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{r-1 \ \ integrals} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))(ds)^{r-1} \right) \\
& + \frac{\psi^{(r-3)}(\sigma_a + \aleph(\sigma_b, \sigma_a))}{(\aleph(\sigma_b, \sigma_a))^3} \left( \underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{r-2 \ \ integrals} s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))(ds)^{r-2} \right) \\
& \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
& \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
& \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
& + (-1)^{r-2} \frac{\psi'(\sigma_a + \aleph(\sigma_b, \sigma_a))}{(\aleph(\sigma_b, \sigma_a))^{r-1}} \left( \int_0^1 \int_0^1 s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))(ds)^2 \right) \\
& + (-1)^{r-1} \frac{\psi(\sigma_a + \aleph(\sigma_b, \sigma_a))}{(\aleph(\sigma_b, \sigma_a))^r} \left( \int_0^1 s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))ds \right) \\
& + (-1)^r \frac{1}{(\aleph(\sigma_b, \sigma_a))^r} \left( \int_0^1 s^{\frac{\text{þ}}{k}-1} \psi(\sigma_a + s\aleph(\sigma_b, \sigma_a)) \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a)) \right) ds.
\end{aligned}
$$

After simplification

$$
\begin{aligned}
I_1 \quad = \quad & \frac{\psi^{(r-1)}(\sigma_a + \aleph(\sigma_b, \sigma_a))}{\aleph(\sigma_b, \sigma_a)} \left( \int_0^1 s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))ds \right) \\
& - \frac{\psi^{(r-2)}(\sigma_a + \aleph(\sigma_b, \sigma_a))}{(\aleph(\sigma_b, \sigma_a))^2} \left( \int_0^1 s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))ds \right) \\
& + \frac{\psi^{(r-3)}(\sigma_a + \aleph(\sigma_b, \sigma_a))}{(\aleph(\sigma_b, \sigma_a))^3} \left( \int_0^1 s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))ds \right) \\
& \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
& \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
& \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
& + (-1)^{r-2} \frac{\psi'(\sigma_a + \aleph(\sigma_b, \sigma_a))}{(\aleph(\sigma_b, \sigma_a))^{r-1}} \left( \int_0^1 s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))ds \right) \\
& + (-1)^{r-1} \frac{\psi(\sigma_a + \aleph(\sigma_b, \sigma_a))}{(\aleph(\sigma_b, \sigma_a))^r} \left( \int_0^1 s^{\frac{\text{þ}}{k}-1} \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))ds \right) \\
& + (-1)^r \frac{1}{(\aleph(\sigma_b, \sigma_a))^r} \left( \int_0^1 s^{\frac{\text{þ}}{k}-1} \psi(\sigma_a + s\aleph(\sigma_b, \sigma_a)) \phi(\sigma_a + s\aleph(\sigma_b, \sigma_a))ds \right).
\end{aligned}
$$

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78 After substituting $t = \sigma_a + s\aleph(\sigma_b, \sigma_a)$, we have

$$\begin{aligned} I_1 &= \frac{\psi^{(r-1)}(\sigma_a + \aleph(\sigma_b,\sigma_a))}{\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+1}} \int_{\sigma_a}^{\sigma_a+\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1}\phi(t)dt \\ &\quad - \frac{\psi^{(r-2)}(\sigma_a + \aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+2}} \int_{\sigma_a}^{\sigma_a+\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1}\phi(t)dt \\ &\quad + \frac{\psi^{(r-3)}(\sigma_a + \aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+3}} \int_{\sigma_a}^{\sigma_a+\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1}\phi(t)dt \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\quad + (-1)^{r-2}\frac{\psi'(\sigma_a + \aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r-1}} \int_{\sigma_a}^{\sigma_a+\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1}\phi(t)dt \\ &\quad + (-1)^{r-1}\frac{\psi(\sigma_a + \aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} \int_{\sigma_a}^{\sigma_a+\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1}\phi(t)dt \\ &\quad + (-1)^{r}\frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} \int_{\sigma_a}^{\sigma_a+\aleph(\sigma_b,\sigma_a)} (t-\sigma_a)^{\frac{\text{þ}}{k}-1}\psi(t)\phi(t)dt. \end{aligned}$$

79 Using the definition of $k-$fractional integral

$$\begin{aligned} I_1 &= \frac{\psi^{(r-1)}(\sigma_a + \aleph(\sigma_b,\sigma_a))k\Gamma_k(\text{þ})}{\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+1}} J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\ &\quad - \frac{\psi^{(r-2)}(\sigma_a + \aleph(\sigma_b,\sigma_a))k\Gamma_k(\text{þ})}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+2}} J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\ &\quad + \frac{\psi^{(r-3)}(\sigma_a + \aleph(\sigma_b,\sigma_a))k\Gamma_k(\text{þ})}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+3}} J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\quad + (-1)^{r-2}\frac{\psi'(\sigma_a + \aleph(\sigma_b,\sigma_a))k\Gamma_k(\text{þ})}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r-1}} J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\ &\quad + (-1)^{r-1}\frac{\psi(\sigma_a + \aleph(\sigma_b,\sigma_a))k\Gamma_k(\text{þ})}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\ &\quad + (-1)^{r}\frac{k\Gamma_k(\text{þ})}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a). \end{aligned}$$

80 After adding the above series, we have

$$\begin{aligned} I_1 &= \sum_{m=0}^{r-1}(-1)^{r-m-1}\frac{\psi^{(m)}(\sigma_a + \aleph(\sigma_b,\sigma_a))}{\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+r-m}} k\Gamma_k(\text{þ}) J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) \\ &\quad + (-1)^{r}\frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} k\Gamma_k(\text{þ}) J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a). \end{aligned} \tag{8}$$

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From the second integral

$$\begin{aligned} I_2 &= (-1)^{r+1}\sum_{m=0}^{r-1}\frac{\psi^{(m)}(\sigma_a)}{\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+r-m}}k\Gamma_k(\text{þ})J^{\text{þ},k}_{\sigma_a^+}\phi(\sigma_a+\aleph(\sigma_b,\sigma_a)) \\ &\quad +(-1)^r\frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}}k\Gamma_k(\text{þ})J^{\text{þ},k}_{\sigma_a^+}(gf)(\sigma_a+\aleph(\sigma_b,\sigma_a)). \end{aligned} \tag{9}$$

Upon adding (8)and (9) and utilizing lemma 4, we obtain the required result. □

For $k = r = 1$, we get Lemma 3 of [7].

$$\begin{aligned} &\left[\frac{\psi(\sigma_a)+\psi(\sigma_a+\aleph(\sigma_b,\sigma_a))}{2(\aleph(\sigma_b,\sigma_a))^{\text{þ}+1}}\right] \\ &\times\left[J^{\text{þ}}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))^-}\phi(\sigma_a)+J^{\text{þ}}_{\sigma_a^+}\phi(\sigma_a+\aleph(\sigma_b,\sigma_a))\right] \\ &-\left[\frac{J^{\text{þ}}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))^-}(gf)(\sigma_a)+J^{\text{þ}}_{\sigma_a^+}(gf)(\sigma_a+\aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{\text{þ}+1}}\right] \\ = &\ \frac{1}{\Gamma(\text{þ})}\int_0^1 w(s)\psi'(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds, \end{aligned}$$

where

$$\begin{aligned} w(s) &= \int_0^s u^{\text{þ}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))du \\ &+\int_1^s(1-u)^{\text{þ}-1}\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))du, s\in[0,1]. \end{aligned}$$

**Theorem 2.** *Let $\Omega\subseteq\mathbb{R}$ be an open invex set and $\aleph$ be a function such that $\aleph:\Omega\times\Omega\to\mathbb{R}^{\text{þ}}$. Suppose, $\psi:\Omega\to\mathbb{R}$ is a differentiable mapping such that $\psi^{(r)}\in L[\sigma_a,\sigma_a+\aleph(\sigma_b,\sigma_a)]$ where $\aleph(\sigma_b,\sigma_a)>0$. If there is an integral mapping such that $\phi:[\sigma_a,\sigma_a+\aleph(\sigma_b,\sigma_a)]\to[0,\infty)$ and it is also symmetric with respect to $\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a)$. Let $\left|\psi^{(r)}\right|$ be a preinvex function on $\Omega$, then $\forall\ \sigma_a,\sigma_b\in\Omega$, we have the following inequality:*

$$\begin{aligned} &\Bigg|\sum_{m=0}^{r-1}\frac{\psi^{(m)}(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^m\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+r-m}} \\ &\times\left[(-1)^{r-m-1}J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))^-}\phi(\sigma_a)+(-1)^{r+1}J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))^+}\phi(\sigma_a+\aleph(\sigma_b,\sigma_a))\right] \\ &+(-1)^r\frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} \\ &\times\left[J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))^-}(gf)(\sigma_a)+J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))^+}(gf)(\sigma_a+\aleph(\sigma_b,\sigma_a))\right]\Bigg| \\ \le &\ \frac{\left[\left|\psi^{(r)}(\sigma_a)\right|+\left|\psi^{(r)}(\sigma_b)\right|\right]\|\phi\|_\infty k^r\Gamma(\text{þ})}{2^{\frac{\text{þ}}{k}+r}(\text{þ}+k(-1+r))(\text{þ}+kr)\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)},\quad \textit{when } r \textit{ is even integer} \\ \le &\ \frac{\|\phi\|_\infty\left[\left|\psi^{(r)}(\sigma_a)\right|-\left|\psi^{(r)}(\sigma_b)\right|\right]k^{r+1}\Gamma(\text{þ})}{2^{\frac{\text{þ}}{k}+r}(\text{þ}+k(-1+r))(\text{þ}+kr)(\text{þ}+k+kr)\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)},\quad \textit{when } r \textit{ is even integer} \end{aligned}$$

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**Proof.** Applying modulus on both sides of (3)

$$\begin{aligned}
&\Bigg|\sum_{m=0}^{r-1}\frac{\psi^{(m)}(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^m\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+r-m}}\\
&\times\Big[(-1)^{r-m-1}J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a)+(-1)^{r+1}J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+}\phi(\sigma_a+\aleph(\sigma_b,\sigma_a))\Big]\\
&+(-1)^r\frac{1}{(\aleph(\sigma_b,\sigma_a))^{\text{þ}+r}}\Big[J^{\text{þ}}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a)+J^{\text{þ}}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+}(gf)(\sigma_a+\aleph(\sigma_b,\sigma_a))\Big]\Bigg|\\
=&\ \frac{1}{k\Gamma_k(\text{þ})}\left|\int_0^{1/2}w(s)\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds+\frac{1}{k\Gamma_k(\text{þ})}\int_{1/2}^1 w(s)\psi^{(r)}(\sigma_a+s\aleph(\sigma_b,\sigma_a))ds\right|.
\end{aligned}$$

From preinvexity of $\left|\psi^{(r)}\right|$ on $\Omega$ and using the fact $\|\phi\|_\infty = sup_{t\in[\sigma_a,\sigma_b]}|\phi(t)|$, we have

$$\begin{aligned}
&\Bigg|\sum_{m=0}^{r-1}\frac{\psi^{(m)}(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^m\aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+r-m}}\\
&\times\Big[(-1)^{r-m-1}J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a)+(-1)^{r+1}J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+}\phi(\sigma_a+\aleph(\sigma_b,\sigma_a))\Big]\\
&+(-1)^r\frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}}\Big[J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a)+J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+}(gf)(\sigma_a+\aleph(\sigma_b,\sigma_a))\Big]\Bigg|\\
\leq&\ \frac{\|\phi\|_\infty}{k\Gamma_k(\text{þ})}\int_0^{1/2}\Bigg(\underbrace{\int_0^s\int_0^s\dots\int_0^s}_{r\ integrals}u^{\frac{\text{þ}}{k}-1}(du)^r\Bigg)\Big[(1-s)\left|\psi^{(r)}(\sigma_a)\right|+s\left|\psi^{(r)}(\sigma_b)\right|\Big]ds\\
&+\frac{\|\phi\|_\infty}{k\Gamma_k(\text{þ})}\int_{1/2}^1\Bigg(\underbrace{\int_1^s\int_1^s\dots\int_1^s}_{r\ integrals}(1-u)^{\frac{\text{þ}}{k}-1}(du)^r\Bigg)\Big[(1-s)\left|\psi^{(r)}(\sigma_a)\right|+s\left|\psi^{(r)}(\sigma_b)\right|\Big]ds\\
=&\ I_1+I_2 \qquad (10)
\end{aligned}$$

From the first term of (10), we have

$$\begin{aligned}
I_1=&\ \frac{\|\phi\|_\infty}{k\Gamma_k(\text{þ})}\int_0^{1/2}\Bigg(\underbrace{\int_0^s\int_0^s\dots\int_0^s}_{r\ integrals}u^{\frac{\text{þ}}{k}-1}(du)^r\Bigg)\Big[(1-s)\left|\psi^{(r)}(\sigma_a)\right|+s\left|\psi^{(r)}(\sigma_b)\right|\Big]ds\\
=&\ \frac{k^{r-2}\|\phi\|_\infty\Gamma(\text{þ})}{\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)}\int_0^{1/2}u^{\frac{\text{þ}}{k}+r-2}\int_u^{1/2}\Big[(1-s)\left|\psi^{(r)}(\sigma_a)\right|+s\left|\psi^{(r)}(\sigma_b)\right|\Big]dtdu\\
=&\ \frac{k^{r-2}\|\phi\|_\infty\Gamma(\text{þ})}{\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)}\int_0^{1/2}u^{\frac{\text{þ}}{k}+r-2}\Bigg[\left|\psi^{(r)}(\sigma_a)\right|\left(\frac{(1-u)^2}{2}-\frac{1}{8}\right)\\
&+\left|\psi^{(r)}(\sigma_b)\right|\left(\frac{1}{8}-\frac{u^2}{2}\right)\Bigg]du.
\end{aligned}$$

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93 Making the change of variable $t = \sigma_a + u\aleph(\sigma_b, \sigma_a)$ for $u \in [0,1]$

$$
\begin{aligned}
I_1 = & \frac{k^{r-2}\|\phi\|_\infty \Gamma(\text{þ})}{\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)} \frac{\left|\psi^{(r)}(\sigma_a)\right|}{\aleph(\sigma_b,\sigma_a)} \\
& \times \int_{\sigma_a}^{\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a)} \left(\frac{1}{2}\left(1-\frac{t-\sigma_a}{\aleph(\sigma_b,\sigma_a)}\right)^2 - \frac{1}{8}\right)\left(\frac{t-\sigma_a}{\aleph(\sigma_b,\sigma_a)}\right)^{\frac{\text{þ}}{k}+r-2} dt \\
& + \frac{k^{r-2}\|\phi\|_\infty \Gamma(\text{þ})}{\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)} \frac{\left|\psi^{(r)}(\sigma_b)\right|}{\aleph(\sigma_b,\sigma_a)} \\
& \times \int_{\sigma_a}^{\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a)} \left(\frac{1}{8} - \frac{1}{2}\left(\frac{t-\sigma_a}{\aleph(\sigma_b,\sigma_a)}\right)^2\right)\left(\frac{t-\sigma_a}{\aleph(\sigma_b,\sigma_a)}\right)^{\frac{\text{þ}}{k}+r-2} dt. \qquad (11)
\end{aligned}
$$

94 From the second term of (10), we have

$$
\begin{aligned}
I_2 = & \frac{\|\phi\|_\infty}{k\Gamma_k(\text{þ})} \int_{1/2}^{1} \left( \underbrace{\int_1^s \int_1^s \dots \int_1^s}_{r\ integrals} (1-u)^{\frac{\text{þ}}{k}-1} (du)^r \right) \left[(1-s)\left|\psi^{(r)}(\sigma_a)\right| + s\left|\psi^{(r)}(\sigma_b)\right|\right] ds \\
= & \frac{(-1)^{r-1}k^{r-2}\|\phi\|_\infty \Gamma(\text{þ})}{\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)} \int_{1/2}^{1} \left(\int_1^s (1-u)^{\frac{\text{þ}}{k}+r-2} du\right) \left[(1-s)\left|\psi^{(r)}(\sigma_a)\right| + s\left|\psi^{(r)}(\sigma_b)\right|\right] ds \\
= & \frac{(-1)^{r-1}k^{r-2}\|\phi\|_\infty \Gamma(\text{þ})}{\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)} \int_{1/2}^{1} (1-u)^{\frac{\text{þ}}{k}+r-2} \left(\int_{1/2}^{u} \left[(1-s)\left|\psi^{(r)}(\sigma_a)\right| + s\left|\psi^{(r)}(\sigma_b)\right|\right] ds\right) du \\
= & \frac{(-1)^{r-1}k^{r-2}\|\phi\|_\infty \Gamma(\text{þ})}{\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)} \int_{1/2}^{1} (1-u)^{\frac{\text{þ}}{k}+r-2} \left[\left|\psi^{(r)}(\sigma_a)\right|\left(\frac{1}{8} - \frac{(1-u)^2}{2}\right)\right. \\
& \left. + \left|\psi^{(r)}(\sigma_b)\right|\left(\frac{u^2}{2} - \frac{1}{8}\right)\right] du.
\end{aligned}
$$

95 By the change of variable $t = \sigma_a + (1-u)\aleph(\sigma_b, \sigma_a)$

$$
\begin{aligned}
I_2 = & \frac{(-1)^{r-1}k^{r-2}\|\phi\|_\infty \Gamma(\text{þ})}{\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)} \frac{\left|\psi^{(r)}(\sigma_a)\right|}{\aleph(\sigma_b,\sigma_a)} \\
& \times \int_{\sigma_a}^{\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a)} \left(\frac{1}{8} - \frac{1}{2}\left(\frac{t-\sigma_a}{\aleph(\sigma_b,\sigma_a)}\right)^2\right)\left(\frac{t-\sigma_a}{\aleph(\sigma_b,\sigma_a)}\right)^{\frac{\text{þ}}{k}+r-2} dt \\
& + \frac{(-1)^{r-1}k^{r-2}\|\phi\|_\infty \Gamma(\text{þ})}{\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)} \frac{\left|\psi^{(r)}(\sigma_b)\right|}{\aleph(\sigma_b,\sigma_a)} \\
& \times \int_{\sigma_a}^{\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a)} \left(\frac{1}{2}\left(1-\frac{t-\sigma_a}{\aleph(\sigma_b,\sigma_a)}\right)^2 - \frac{1}{8}\right)\left(\frac{t-\sigma_a}{\aleph(\sigma_b,\sigma_a)}\right)^{\frac{\text{þ}}{k}+r-2} dt. \qquad (12)
\end{aligned}
$$

96 **Case (i)** when $r$ is odd integer

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Adding (11) and (12) based on (10), we have

$$\begin{aligned}
&\Bigg| \sum_{m=0}^{r-1} \frac{\psi^{(m)}(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^m \aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+r-m}} \\
&\times \left[ (-1)^{r-m-1} J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-} \phi(\sigma_a) + (-1)^{r+1} J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+} \phi(\sigma_a + \aleph(\sigma_b,\sigma_a)) \right] \\
&+(-1)^r \frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} \left[ J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-} (gf)(\sigma_a) + J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+} (gf)(\sigma_a + \aleph(\sigma_b,\sigma_a)) \right] \Bigg| \\
\leq \; & \frac{\left[ \left|\psi^{(r)}(\sigma_a)\right| + \left|\psi^{(r)}(\sigma_b)\right| \right] \|\phi\|_\infty k^r \Gamma(\text{þ})}{2^{\frac{\text{þ}}{k}+r} (\text{þ} + k(-1+r))(\text{þ}+kr)\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)}
\end{aligned}$$

**Case (ii)** when $r$ is even integer

Adding (11) and (12) based on (10), we have

$$\begin{aligned}
&\Bigg| \sum_{m=0}^{r-1} \frac{\psi^{(m)}(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{2^m \aleph(\sigma_b,\sigma_a)^{\frac{\text{þ}}{k}+r-m}} \\
&\times \left[ (-1)^{r-m-1} J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-} \phi(\sigma_a) + (-1)^{r+1} J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+} \phi(\sigma_a + \aleph(\sigma_b,\sigma_a)) \right] \\
&+(-1)^r \frac{1}{(\aleph(\sigma_b,\sigma_a))^{\frac{\text{þ}}{k}+r}} \left[ J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-} (gf)(\sigma_a) + J^{\text{þ},k}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+} (gf)(\sigma_a + \aleph(\sigma_b,\sigma_a)) \right] \Bigg| \\
\leq \; & \frac{\|\phi\|_\infty \left[ \left|\psi^{(r)}(\sigma_a)\right| - \left|\psi^{(r)}(\sigma_b)\right| \right] k^{r+1} \Gamma(\text{þ})}{2^{\frac{\text{þ}}{k}+r} (\text{þ} + k(-1+r))(\text{þ}+kr)(\text{þ}+k+kr)\Gamma_k(\text{þ})\Gamma(\text{þ}+r-1)}
\end{aligned}$$

□

For $k = r = 1$, we get Theorem 3 of [7].

$$\begin{aligned}
&\Bigg| \frac{\psi(\sigma_a + \frac{1}{2}\aleph(\sigma_b,\sigma_a))}{\aleph(\sigma_b,\sigma_a)^{\text{þ}+1}} \\
&\times \left[ J^{\text{þ}}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-} \phi(\sigma_a) + J^{\text{þ}}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+} \phi(\sigma_a + \aleph(\sigma_b,\sigma_a)) \right] \\
&+ \frac{1}{(\aleph(\sigma_b,\sigma_a))^{\text{þ}+2}} \left[ J^{\text{þ}}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))-} (gf)(\sigma_a) + J^{\text{þ}}_{(\sigma_a+\frac{1}{2}\aleph(\sigma_b,\sigma_a))+} (gf)(\sigma_a + \aleph(\sigma_b,\sigma_a)) \right] \Bigg| \\
\leq \; & \frac{\|\phi\|_\infty}{\Gamma(\text{þ}+2)} \frac{1}{2^{\text{þ}+1}} \left[ |\psi'(\sigma_a)| + |\psi'(\sigma_b)| \right]
\end{aligned}$$

**Theorem 3.** *Let $\Omega \subseteq \mathbb{R}$ be an open invex set and $\aleph$ be a function such that $\aleph : \Omega \times \Omega \to \mathbb{R}^{þ}$. Suppose, $\psi : \Omega \to \mathbb{R}$ is a differentiable mapping such that $\psi^{(r)} \in L[\sigma_a, \sigma_a + \aleph(\sigma_b,\sigma_a)]$ where $\aleph(\sigma_b,\sigma_a) > 0$. If there is an integral mapping such that $\phi : [\sigma_a, \sigma_a + \aleph(\sigma_b,\sigma_a)] \to [0,\infty)$ and*

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*it is also symmetric with respect to* $\sigma_a + \frac{1}{2}\aleph(\sigma_b, \sigma_a)$. *Let* $\left|\psi^{(r)}\right|$ *be a preinvex function on* $\Omega$, *then* $\forall\, \sigma_a, \sigma_b \in \Omega$, *we have the following inequality:*

$$
\begin{aligned}
J &= \Bigg| \sum_{m=0}^{r-1} \left[ \frac{(-1)^{r+1}\psi^{(m)}(\sigma_a) + (-1)^{r-m-1}\psi^{(m)}(\sigma_a + \aleph(\sigma_b, \sigma_a))}{2(\aleph(\sigma_b, \sigma_a))^{\text{þ}+r-m}} \right] \\
&\quad \times \left[ J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) + J^{\text{þ},k}_{\sigma_a+}\phi(\sigma_a + \aleph(\sigma_b, \sigma_a)) \right] \\
&\quad + (-1)^r \left[ \frac{J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a) + J^{\text{þ},k}_{\sigma_a+}(gf)(\sigma_a + \aleph(\sigma_b, \sigma_a))}{(\aleph(\sigma_b, \sigma_a))^{\text{þ}+r}} \right] \Bigg| \\
&\leq \frac{\|\phi\|_\infty}{(\text{þ}+r)\Gamma(\text{þ}+r-1)} \left[ \left|\psi^{(r)}(\sigma_a)\right| + \left|\psi^{(r)}(\sigma_b)\right| \right] \quad ;\textit{ when } r \textit{ is odd integer} \\
&\leq \frac{\|\phi\|_\infty}{(\text{þ}+r)(\text{þ}+r+1)\Gamma(\text{þ}+r-1)} \\
&\quad \times \left[ \left|\psi^{(r)}(\sigma_a)\right| - \left|\psi^{(r)}(\sigma_b)\right| \right] \;;\textit{ when } r \textit{ is even integer}
\end{aligned}
$$

**Proof.** Applying modulus on both sides of (7),

$$
\begin{aligned}
&\Bigg| \sum_{m=0}^{r-1} \left[ \frac{(-1)^{r+1}\psi^{(m)}(\sigma_a) + (-1)^{r-m-1}\psi^{(m)}(\sigma_a + \aleph(\sigma_b, \sigma_a))}{2(\aleph(\sigma_b, \sigma_a))^{\text{þ}+r-m}} \right] \\
&\times \left[ J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a) + J^{\text{þ},k}_{\sigma_a+}\phi(\sigma_a + \aleph(\sigma_b, \sigma_a)) \right] \\
&+ (-1)^r \left[ \frac{J^{\text{þ},k}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a) + J^{\text{þ},k}_{\sigma_a+}(gf)(\sigma_a + \aleph(\sigma_b, \sigma_a))}{(\aleph(\sigma_b, \sigma_a))^{\text{þ}+r}} \right] \Bigg| \\
= &\Bigg| \frac{1}{k\Gamma_k(\text{þ})} \int_0^1 \Bigg( \underbrace{\int_0^s \int_0^s \ldots \int_0^s}_{r\ \ integrals} u^{\frac{\text{þ}}{k}-1}\phi(\sigma_a + u\aleph(\sigma_b, \sigma_a))(du)^r \\
&+ \underbrace{\int_1^s \int_1^s .. \int_1^s}_{r\ \ integrals} (1-u)^{\frac{\text{þ}}{k}-1}\phi(\sigma_a + u\aleph(\sigma_b, \sigma_a))(du)^r \Bigg) \psi^{(r)}(\sigma_a + s\aleph(\sigma_b, \sigma_a))ds \Bigg|
\end{aligned}
$$

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From preinvexity of $\left|\psi^{(r)}\right|$ on $\Omega$, we have

$$\begin{aligned}
J &= \Bigg|\sum_{\kappa=0}^{r-1}\left[\frac{(-1)^{r+1}\psi^{(\kappa)}(\sigma_a)+(-1)^{r-\kappa-1}\psi^{(\kappa)}(\sigma_a+\aleph(\sigma_b,\sigma_a))}{2(\aleph(\sigma_b,\sigma_a))^{þ+r-\kappa}}\right] \\
&\quad \times\left[J^{þ}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a)+J^{þ}_{\sigma_a+}\phi(\sigma_a+\aleph(\sigma_b,\sigma_a))\right] \\
&\quad +(-1)^r\left[\frac{J^{þ}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a)+J^{þ}_{\sigma_a+}(gf)(\sigma_a+\aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{þ+r}}\right]\Bigg| \\
&= \frac{1}{k\Gamma_k(þ)}\int_0^1\Bigg(\underbrace{\int_0^s\int_0^s\ldots\int_0^s}_{r\ \ integrals}u^{\frac{þ}{k}-1}|\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))|(du)^r \\
&\quad +\underbrace{\int_1^s\int_1^s..\int_1^s}_{r\ \ integrals}(1-u)^{\frac{þ}{k}-1}|\phi(\sigma_a+u\aleph(\sigma_b,\sigma_a))|(du)^r\Bigg)\left((1-s)\left|\psi^{(r)}(\sigma_a)\right|+s\left|\psi^{(r)}(\sigma_b)\right|\right)ds
\end{aligned}$$

After simplification and letting $\|\phi\|_\infty = sup_{t\in[\sigma_a,\sigma_b]}|\phi(t)|$, we have

$$\begin{aligned}
J &\leq \frac{k^{r-2}\|\phi\|_\infty\Gamma(þ)}{\Gamma_k(þ)\Gamma(þ+r-1)}\int_0^1 u^{\frac{þ}{k}+r-2}\left(\int_0^u\left((1-s)\left|\psi^{(r)}(\sigma_a)\right|+s\left|\psi^{(r)}(\sigma_b)\right|\right)ds\right)du \\
&\quad +\frac{(-1)^{r-1}k^{r-2}\|\phi\|_\infty\Gamma(þ)}{\Gamma_k(þ)\Gamma(þ+r-1)}\int_0^1(1-u)^{\frac{þ}{k}+r-2}\left(\int_u^1\left((1-s)\left|\psi^{(r)}(\sigma_a)\right|+s\left|\psi^{(r)}(\sigma_b)\right|\right)ds\right)du
\end{aligned}$$

**Case (i)** $n$ is odd integer
Adding (11) and (12) based on (10), we have

$$J \leq \frac{\|\phi\|_\infty k^{r-1}\Gamma(þ)}{(þ+kr)\Gamma(þ+r-1)\Gamma_k(þ)}\left[\left|\psi^{(r)}(\sigma_a)\right|+\left|\psi^{(r)}(\sigma_b)\right|\right]$$

**Case (ii)** $n$ is even integer
Adding (11) and (12) based on (10), we have

$$J \leq \frac{\|\phi\|_\infty k^{r}\Gamma(þ)}{(þ+kr)(þ+kr+k)\Gamma(þ+r-1)}\left[\left|\psi^{(r)}(\sigma_a)\right|-\left|\psi^{(r)}(\sigma_b)\right|\right]$$

□

For $k=r=1$, we have

$$\begin{aligned}
&\Bigg|\left[\frac{\psi(\sigma_a)+\psi(\sigma_a+\aleph(\sigma_b,\sigma_a))}{2(\aleph(\sigma_b,\sigma_a))^{þ+1}}\right] \\
&\times\left[J^{þ}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}\phi(\sigma_a)+J^{þ}_{\sigma_a+}\phi(\sigma_a+\aleph(\sigma_b,\sigma_a))\right] \\
&-\left[\frac{J^{þ}_{(\sigma_a+\aleph(\sigma_b,\sigma_a))-}(gf)(\sigma_a)+J^{þ}_{\sigma_a+}(gf)(\sigma_a+\aleph(\sigma_b,\sigma_a))}{(\aleph(\sigma_b,\sigma_a))^{þ+1}}\right]\Bigg| \\
\leq\ &\frac{\|\phi\|_\infty}{(þ+1)\Gamma(þ)}\left[|\psi'(\sigma_a)|+|\psi'(\sigma_b)|\right]
\end{aligned}$$

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## 3. Conclusion

The new lower and upper bounds of Fejér type Hermite-Hadamard inequalities for $r-$times differentiable preinvex have been established. We have also given special cases of our results.

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