

C*-Algebraic Bieberbach, Robertson, Lebedev-Milin, Zalcman, Krzyz and Corona Conjectures

K. MAHESH KRISHNA

Post Doctoral Fellow

Statistics and Mathematics Unit

Indian Statistical Institute, Bangalore Centre

Karnataka 560 059 India

Email: kmaheshak@gmail.com

Abstract: We study C*-algebraic versions of following conjectures/theorems: (1) Bieberbach conjecture (de Branges theorem) (2) Robertson conjecture (3) Lebedev-Milin conjecture (4) Zalcman conjecture (5) Krzyz conjecture (6) Corona conjecture (Carleson theorem). We prove that the C*-algebraic Bieberbach Conjecture for invertible coefficients is true for second degree C*-algebraic polynomials.

Keywords: C*-algebra; Bieberbach conjecture; de Branges theorem; Robertson conjecture; Lebedev-Milin conjecture; Zalcman conjecture; Krzyz conjecture; Corona conjecture; Riemann mapping theorem

Mathematics Subject Classification (2020): 30C50, 46L05, 30C15, 12D10, 30B10.

1. C*-ALGEBRAIC BIEBERBACH, ROBERTSON AND LEBEDEV-MILIN CONJECTURES

Let $\mathbb{D}(0, 1)$ be the open unit disc in \mathbb{C} centered at 0 of radius 1. In 1916, Bieberbach made the following conjecture which became known as Bieberbach conjecture [9].

Conjecture 1.1. [2, 9, 17, 27, 29, 38, 41, 49, 63, 65, 73, 95, 96, 117] (*Bieberbach Conjecture/de Branges Theorem*) *If the power series*

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0, 1)$ and the function f is injective on $\mathbb{D}(0, 1)$, then

$$(1) \quad |a_n| \leq n, \quad \forall n \geq 2.$$

Inequality (1) is strict except for rotations of Koebe function on $\mathbb{D}(0, 1)$ defined by

$$k(z) := \sum_{n=1}^{\infty} n z^n.$$

Bieberbach himself proved that $|a_2| \leq 2$ [9]. In 1923 Lowner proved that $|a_3| \leq 3$ [77]. In 1955 Garabedian and Schiffer gave a new proof for $|a_3| \leq 3$ [33]. The inequality $|a_4| \leq 4$ was proved by Garabedian and Schiffer in 1955 [34]. A simpler proof for $|a_4| \leq 4$ is later given by Charzynski and Schiffer in 1960 [15]. The inequality $|a_5| \leq 5$ was proved by Pederson and Schiffer in 1972 [92]. The inequality $|a_6| \leq 6$ was proved by Pederson in 1968 [93] as well as by Ozawa in 1969 [90,91]. On the other side, Littlewood in 1925 showed that $|a_n| \leq en$ for all n [75]. In 1957 Nehari showed that asymptotic Bieberbach conjecture implies Littlewood conjecture [86]. In 1982 Hamilton showed that Littlewood conjecture implies asymptotic Bieberbach conjecture [44].

In 1972 FitzGerald proved that $|a_n| < \sqrt{\frac{7}{6}}n$ for all n [26] which improved the bound obtained by Milin in 1965 [83]. In 1976 Horowitz proved that $|a_n| < \left(\frac{209}{140}\right)^{\frac{1}{6}}n$ for all n [52]. Horowitz improved his result in 1978 and obtained $|a_n| < \left(\frac{1659164137}{681080400}\right)^{\frac{1}{14}}n$ for all n [54] (this result was further improved by Hu in 1983 [55]). In 1955 Hayman proved that $\lim_{n \rightarrow \infty} \frac{a_n}{n}$ exists [45]. In 1986 Brown and Tsao showed that Zalcman conjecture implies Bieberbach conjecture [11] (also see [71, 72, 79, 80]) Further, Bieberbach conjecture has been proved for special classes of functions [6, 16, 22, 23, 37, 46, 53, 58, 76, 89, 98, 103]. Finally in 1985, de Branges proved the conjecture in full generality for all n [17]. In 1991 Weinstein gave another proof of Bieberbach conjecture [114] (also see [4, 12, 18, 19, 21, 24, 28, 47, 48, 59, 62, 88, 106–109, 115, 116]). In 1997 Xie proved a generalization of de Branges theorem [116]. It is interesting to note that Bieberbach conjecture for holomorphic mappings on several complex variables fails [73].

Let \mathcal{A} be a unital C^* -algebra. We define the C^* -algebraic open unit disc centered at 0 and of radius 1, denoted as $\mathbb{D}^*(0, 1)$ as the set of all strict contractions in \mathcal{A} , i.e., set of all elements of \mathcal{A} having norm less than 1. Based on Conjecture 1.1 we set following conjectures.

Conjecture 1.2. (*C^* -algebraic Bieberbach Conjecture for general coefficients*) *Let \mathcal{A} be a unital C^* -algebra. If the C^* -algebraic power series*

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then*

$$(2) \quad a_n a_n^* \leq n^2, \quad a_n^* a_n \leq n^2, \quad \forall n \geq 2.$$

Inequality (2) is strict except for rotations of C^ -algebraic Koebe function*

$$k(z) := \sum_{n=1}^{\infty} n z^n$$

on $\mathbb{D}^(0, 1)$.*

Conjecture 1.3. (*C^* -algebraic Bieberbach Conjecture for invertible coefficients*) *Let \mathcal{A} be a unital C^* -algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . If the C^* -algebraic power series*

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then*

$$\frac{1}{\|a_n^{-1}\|} \leq n, \quad \forall a_n \in I(\mathcal{A}).$$

Here are strong forms of Conjecture 1.2 and Conjecture 1.3.

Conjecture 1.4. (*C^* -algebraic Bieberbach Conjecture for general coefficients - strong form*) *Let \mathcal{A} be a unital C^* -algebra. If the C^* -algebraic power series*

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

KRZYZ AND CORONA CONJECTURES

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, the C^* -algebraic power series

$$f'(z) := 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then

$$a_n a_n^* \leq n^2, \quad a_n^* a_n \leq n^2, \quad \forall n \geq 2.$$

Conjecture 1.5. (*C^* -algebraic Bieberbach Conjecture invertible coefficients - strong form*) Let \mathcal{A} be a unital C^* -algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . If the C^* -algebraic power series

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, the C^* -algebraic power series

$$f'(z) := 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then

$$\frac{1}{\|a_n^{-1}\|} \leq n, \quad \forall a_n \in I(\mathcal{A}).$$

Since holomorphic functions (convergent power series inside the disc of convergence) are infinitely differentiable, strong form, namely Conjectures 1.4 and 1.5 reduce to Conjecture 1.3 for complex numbers. We don't know this for C^* -algebraic convergent power series. If this is true, then the strong form is same as general form. Otherwise, note that we can even stronger form of Conjecture 1.4 by imposing second, third, \dots , differentiable conditions. The same comment holds for other strong form of conjectures also. We next formulate two conjectures which are stronger than Conjecture 1.2 and Conjecture 1.3 for polynomials which is based on Proposition in Page 136 in [117].

Conjecture 1.6. (*C^* -algebraic Bieberbach Conjecture for polynomials - 1*) Let $n \geq 2$. Let \mathcal{A} be a unital C^* -algebra. If the C^* -algebraic polynomial

$$p(z) := z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n, \quad a_k \in \mathcal{A}, \forall 2 \leq k \leq n$$

is injective on $\mathbb{D}^*(0, 1)$, then

$$a_n a_n^* \leq \frac{1}{n^2}, \quad a_n^* a_n \leq \frac{1}{n^2}.$$

Conjecture 1.7. (*C^* -algebraic Bieberbach Conjecture for polynomials - 2*) Let $n \geq 2$. Let \mathcal{A} be a unital C^* -algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . If the C^* -algebraic polynomial

$$p(z) := z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n, \quad a_k \in I(\mathcal{A}) \cup \{0\}, \forall 2 \leq k \leq n-1, a_n \in I(\mathcal{A})$$

is injective on $\mathbb{D}^*(0, 1)$, then

$$\frac{1}{\|a_n^{-1}\|} \leq \frac{1}{n}.$$

Theorem 1.8. Conjecture 1.3 holds for C^* -algebraic polynomials of degree 2.

Proof. Let \mathcal{A} be a unital C^* -algebra, $a \in \mathcal{A}$ be invertible and $p(z) = z + az^2$ be a polynomial over \mathcal{A} which is injective on $\mathbb{D}^*(0, 1)$. Since p is injective and $p(-a^{-1}) = 0 = p(0)$, we must have $-a^{-1} \notin \mathbb{D}^*(0, 1)$. Therefore $\| -a^{-1} \| \geq 1 > 1/2$. \square

In 1936 Robertson formulated (after the failure of Littlewood-Paley conjecture [76] by Feketo and Szego in 1933 [25]) what came to known as Robertson conjecture [99]. This conjecture implies Bieberbach conjecture.

Conjecture 1.9. [99, 117] (*Robertson conjecture*) *If the power series*

$$f(z) := z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad b_{2n-1} \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0, 1)$ and the function f is injective on $\mathbb{D}(0, 1)$, then

$$1 + \sum_{k=1}^n |b_{2k-1}|^2 \leq n, \quad \forall n \geq 2.$$

We now formulate C^* -algebraic Robertson conjectures.

Conjecture 1.10. (*C^* -algebraic Robertson Conjecture for general coefficients*) *Let \mathcal{A} be a unital C^* -algebra. If the C^* -algebraic power series*

$$f(z) := z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad b_{2n-1} \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then*

$$1 + \sum_{k=1}^n b_{2k-1} b_{2k-1}^* \leq n, \quad 1 + \sum_{k=1}^n b_{2k-1}^* b_{2k-1} \leq n, \quad \forall n \geq 2.$$

Conjecture 1.11. (*C^* -algebraic Robertson Conjecture for invertible coefficients*) *Let \mathcal{A} be a unital C^* -algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . If the C^* -algebraic power series*

$$f(z) := z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad b_{2n-1} \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then*

$$\left\| \frac{1}{\left(1 + \sum_{k=1}^n b_{2k-1} b_{2k-1}^*\right)^{-1}} \right\| \leq n, \quad \left\| \frac{1}{\left(1 + \sum_{k=1}^n b_{2k-1}^* b_{2k-1}\right)^{-1}} \right\| \leq n, \quad \forall n \geq 2.$$

Conjecture 1.12. (*C^* -algebraic Robertson Conjecture for general coefficients - strong form*) *Let \mathcal{A} be a unital C^* -algebra. If the C^* -algebraic power series*

$$f(z) := z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad b_{2n-1} \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$, the C^* -algebraic power series*

$$f'(z) := 1 + \sum_{n=2}^{\infty} (2n-1) b_{2n-1} z^{2n-2}$$

KRZYZ AND CORONA CONJECTURES

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then

$$1 + \sum_{k=1}^n b_{2k-1} b_{2k-1}^* \leq n, \quad 1 + \sum_{k=1}^n b_{2k-1}^* b_{2k-1} \leq n, \quad \forall n \geq 2.$$

Conjecture 1.13. (*C*-algebraic Robertson Conjecture for invertible coefficients - strong form*) Let \mathcal{A} be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . If the C*-algebraic power series

$$f(z) := z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \quad b_{2n-1} \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, the C*-algebraic power series

$$f'(z) := 1 + \sum_{n=2}^{\infty} (2n-1) b_{2n-1} z^{2n-2}$$

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then

$$\left\| \frac{1}{\left(1 + \sum_{k=1}^n b_{2k-1} b_{2k-1}^*\right)^{-1}} \right\| \leq n, \quad \left\| \frac{1}{\left(1 + \sum_{k=1}^n b_{2k-1}^* b_{2k-1}\right)^{-1}} \right\| \leq n, \quad \forall n \geq 2.$$

In 1970's Lebedev and Milin conjectured an inequality which became known as Lebedev-Milin conjecture [84]. This conjecture implies Robertson conjecture.

Conjecture 1.14. [84, 117] (*Lebedev-Milin Conjecture*) Let the power series

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0, 1)$ and the function f is injective on $\mathbb{D}(0, 1)$. Let the sequence $\{c_n\}_{n=1}^{\infty}$ of complex numbers be defined as the coefficients of the power series

$$\log \left(\frac{f(z)}{z} \right) = \sum_{n=1}^{\infty} c_n z^n, \quad \forall z \in \mathbb{D}(0, 1).$$

Then for all $n = 1, 2, \dots$,

$$(3) \quad \sum_{m=1}^n \sum_{k=1}^m \left(k |c_k|^2 - \frac{4}{k} \right) \leq 0.$$

Observe that, as is well-known, Equation (3)'s sum

$$\sum_{m=1}^n \sum_{k=1}^m \left(k |c_k|^2 - \frac{4}{k} \right) = \sum_{k=1}^n (n+1-k) \left(k |c_k|^2 - \frac{4}{k} \right), \quad \forall n = 1, 2, \dots$$

which can be proved by induction. We formulate C*-algebraic versions of Lebedev-Milin conjectures as follows.

Conjecture 1.15. (*C*-algebraic Lebedev-Milin Conjecture for general coefficients*) Let \mathcal{A} be a unital C*-algebra. Assume the following.

(i) *The C^* -algebraic power series*

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$.*

(ii) *The C^* -algebraic power series*

$$f'(z) := 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$.*

(iii) *f is injective on $\mathbb{D}^*(0, 1)$.*

(iv) *There exists a C^* -algebraic power series*

$$g(z) := \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in \mathcal{A}, \forall n \geq 0$$

converging (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$ such that*

$$f(z) = e^{g(z)} z, \quad \forall z \in \mathbb{D}^*(0, 1).$$

Then for all $n = 1, 2, \dots$,

$$\sum_{m=1}^n \sum_{k=1}^m \left(k c_k c_k^* - \frac{4}{k} \right) \leq 0, \quad \sum_{m=1}^n \sum_{k=1}^m \left(k c_k^* c_k - \frac{4}{k} \right) \leq 0.$$

Conjecture 1.16. (*C^* -algebraic Lebedev-Milin Conjecture for invertible coefficients*) *Let \mathcal{A} be a unital C^* -algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . Assume the following.*

(i) *The C^* -algebraic power series*

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$.*

(ii) *The C^* -algebraic power series*

$$f'(z) := 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$.*

(iii) *f is injective on $\mathbb{D}^*(0, 1)$.*

(iv) *There exists a C^* -algebraic power series*

$$g(z) := \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 0$$

converging (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$ such that*

$$f(z) = e^{g(z)} z, \quad \forall z \in \mathbb{D}^*(0, 1).$$

KRZYZ AND CORONA CONJECTURES

Then for all $n = 1, 2, \dots$,

$$\left\| \left[\sum_{m=1}^n \sum_{k=1}^m \left(kc_k c_k^* - \frac{4}{k} \right) \right]^{-1} \right\| \leq 0, \quad \left\| \left[\sum_{m=1}^n \sum_{k=1}^m \left(kc_k^* c_k - \frac{4}{k} \right) \right]^{-1} \right\| \leq 0.$$

Conjecture 1.17. (*C*-algebraic Lebedev-Milin Conjecture for general coefficients - strong form*) Let \mathcal{A} be a unital C*-algebra. Assume the following.

(i) The C*-algebraic power series

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$.

(ii) The C*-algebraic power series

$$f'(z) := 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$.

(iii) f is injective on $\mathbb{D}^*(0, 1)$.

(iv) There exists a C*-algebraic power series

$$g(z) := \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in \mathcal{A}, \forall n \geq 0$$

converging (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ such that

$$f(z) = e^{g(z)} z, \quad \forall z \in \mathbb{D}^*(0, 1).$$

(v) The C*-algebraic power series

$$g'(z) := \sum_{n=1}^{\infty} c_n n z^{n-1}$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$.

Then for all $n = 1, 2, \dots$,

$$\sum_{m=1}^n \sum_{k=1}^m \left(kc_k c_k^* - \frac{4}{k} \right) \leq 0, \quad \sum_{m=1}^n \sum_{k=1}^m \left(kc_k^* c_k - \frac{4}{k} \right) \leq 0.$$

Conjecture 1.18. (*C*-algebraic Lebedev-Milin Conjecture for invertible coefficients - strong form*) Let \mathcal{A} be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . Assume the following.

(i) The C*-algebraic power series

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$.

(ii) The C*-algebraic power series

$$f'(z) := 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$.

(iii) f is injective on $\mathbb{D}^*(0, 1)$.

(iv) There exists a C^* -algebraic power series

$$g(z) := \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 0$$

converging (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ such that

$$f(z) = e^{g(z)} z, \quad \forall z \in \mathbb{D}^*(0, 1).$$

(v) The C^* -algebraic power series

$$g'(z) := \sum_{n=1}^{\infty} c_n n z^{n-1}$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$.

Then for all $n = 1, 2, \dots$,

$$\left\| \frac{1}{\left[\sum_{m=1}^n \sum_{k=1}^m \left(k c_k c_k^* - \frac{4}{k} \right) \right]^{-1}} \right\| \leq 0, \quad \left\| \frac{1}{\left[\sum_{m=1}^n \sum_{k=1}^m \left(k c_k^* c_k - \frac{4}{k} \right) \right]^{-1}} \right\| \leq 0.$$

By assuming Lebedev-Milin conjecture one proves the Robertson conjecture using Lebedev-Milin inequalities stated below.

Theorem 1.19. [63, 73] (*Lebedev-Milin Inequalities*) *Let the power series*

$$f(z) := 1 + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 1$$

converges for all $z \in \mathbb{D}(0, 1)$ and let g be the power series defined by

$$e^{f(z)} = \sum_{n=0}^{\infty} b_n z^n =: g(z), \quad \forall z \in \mathbb{D}(0, 1).$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} |b_n|^2 &\leq e^{\sum_{n=0}^{\infty} n |a_n|^2}, \\ \frac{1}{n+1} \sum_{k=0}^n |b_k|^2 &\leq e^{\frac{1}{n+1} \sum_{k=1}^n (n+1-k) (k |a_k|^2 - \frac{4}{k})}, \quad \forall n = 1, 2, \dots, \\ |b_n|^2 &\leq e^{\sum_{k=1}^n (k |a_k|^2 - \frac{4}{k})}, \quad \forall n = 1, 2, \dots \end{aligned}$$

Based on Theorem 1.19 we set the following conjecture.

Conjecture 1.20. *Let \mathcal{A} be a unital C^* -algebra. Let the C^* -algebraic power series*

$$f(z) := 1 + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 1$$

converges for all $z \in \mathbb{D}^*(0, 1)$ and assume that there exists a C^* -algebraic power series g such that

$$e^{f(z)} = \sum_{n=0}^{\infty} b_n z^n =: g(z), \quad \forall z \in \mathbb{D}^*(0, 1).$$

KRZYZ AND CORONA CONJECTURES

Then

$$\begin{aligned} \sum_{n=0}^{\infty} b_n b_n^* &\leq e^{\sum_{n=0}^{\infty} n a_n a_n^*}, \\ \frac{1}{n+1} \sum_{k=0}^n b_k b_k^* &\leq e^{\frac{1}{n+1} \sum_{k=1}^n (n+1-k)(k a_k a_k^* - \frac{4}{k})}, \quad \forall n = 1, 2, \dots, \\ b_n b_n^* &\leq e^{\sum_{k=1}^n (k |a_k a_k^* - \frac{4}{k})}, \quad \forall n = 1, 2, \dots \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} b_n^* b_n &\leq e^{\sum_{n=0}^{\infty} n a_n^* a_n}, \\ \frac{1}{n+1} \sum_{k=0}^n b_k^* b_k &\leq e^{\frac{1}{n+1} \sum_{k=1}^n (n+1-k)(k a_k^* a_k - \frac{4}{k})}, \quad \forall n = 1, 2, \dots, \\ b_n^* b_n &\leq e^{\sum_{k=1}^n (k |a_k^* a_k - \frac{4}{k})}, \quad \forall n = 1, 2, \dots \end{aligned}$$

We next formulate C^* -algebraic versions of Gronwall area theorem [42].

Conjecture 1.21. (*C^* -algebraic Gronwall Area Conjecture for general coefficients*) Let \mathcal{A} be a unital C^* -algebra. Assume the following.

(i) The C^* -algebraic power series

$$f(z) := z + \sum_{n=0}^{\infty} b_n \frac{1}{z^n}, \quad b_n \in \mathcal{A}, \forall n \geq 0$$

converges (in norm of \mathcal{A}) for all $z \in \Delta := \{x \in \mathcal{A} : \|x\| > 1\}$.

(ii) The C^* -algebraic power series

$$f'(z) := 1 + \sum_{n=1}^{\infty} b_n \frac{-n}{z^{n-1}}$$

converges (in norm of \mathcal{A}) for all $z \in \Delta$.

(iii) f is injective on Δ .

(iv) $\lim_{z \rightarrow \infty} f(z) = \infty$.

(v) $\lim_{z \rightarrow \infty} f'(z) = 1$.

Then

$$\sum_{n=1}^{\infty} n b_n b_n^* \leq 1, \quad \sum_{n=1}^{\infty} n b_n^* b_n \leq 1.$$

Conjecture 1.22. (*C^* -algebraic Gronwall Area Conjecture for invertible coefficients*) Let \mathcal{A} be a unital C^* -algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . Assume the following.

(i) The C^* -algebraic power series

$$f(z) := z + \sum_{n=0}^{\infty} b_n \frac{1}{z^n}, \quad b_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 0$$

converges (in norm of \mathcal{A}) for all $z \in \Delta := \{x \in \mathcal{A} : \|x\| > 1\}$.

(ii) *The C*-algebraic power series*

$$f'(z) := 1 + \sum_{n=1}^{\infty} b_n \frac{-n}{z^{n-1}}$$

converges (in norm of \mathcal{A}) for all $z \in \Delta$.

(iii) *f is injective on Δ .*

(iv) $\lim_{z \rightarrow \infty} f(z) = \infty$.

(v) $\lim_{z \rightarrow \infty} f'(z) = 1$.

Then

$$\left\| \frac{1}{\left(\sum_{n=1}^{\infty} n b_n b_n^* \right)^{-1}} \right\| \leq 1, \quad \left\| \frac{1}{\left(\sum_{n=1}^{\infty} n b_n^* b_n \right)^{-1}} \right\| \leq 1.$$

We now recall Zalcman conjecture which, as mentioned earlier, implies Bieberbach conjecture.

Conjecture 1.23. [11] *(Zalcman Conjecture) If the power series*

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0, 1)$ and the function f is injective on $\mathbb{D}(0, 1)$, then

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \quad \forall n \geq 2.$$

In 1998 Ma proposed a generalization of Conjecture 1.23 [80].

Conjecture 1.24. [80] *(Ma Conjecture) If the power series*

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0, 1)$ and the function f is injective on $\mathbb{D}(0, 1)$, then

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1), \quad \forall n, m \geq 2.$$

We next formulate C*-algebraic versions of Zalcman conjecture and Ma conjecture as follows.

Conjecture 1.25. *(C*-algebraic Zalcman Conjecture) Let \mathcal{A} be a unital C*-algebra. If the C*-algebraic power series*

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then*

$$(a_n^2 - a_{2n-1})(a_n^2 - a_{2n-1})^* \leq (n-1)^4, \quad (a_n^2 - a_{2n-1})^*(a_n^2 - a_{2n-1}) \leq (n-1)^4, \quad \forall n \geq 2.$$

Conjecture 1.26. *(C*-algebraic Zalcman Conjecture - strong form) Let \mathcal{A} be a unital C*-algebra. If the C*-algebraic power series*

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

KRZYZ AND CORONA CONJECTURES

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, the C^* -algebraic power series

$$f'(z) := 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then

$$(a_n^2 - a_{2n-1})(a_n^2 - a_{2n-1})^* \leq (n-1)^4, \quad (a_n^2 - a_{2n-1})^*(a_n^2 - a_{2n-1}) \leq (n-1)^4, \quad \forall n \geq 2.$$

Conjecture 1.27. (*C^* -algebraic Ma Conjecture*) Let \mathcal{A} be a unital C^* -algebra. If the C^* -algebraic power series

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then

$$\begin{aligned} (a_n a_m - a_{n+m-1})(a_n a_m - a_{n+m-1})^* &\leq (n-1)^2 (m-1)^2, \\ (a_n a_m - a_{n+m-1})^*(a_n a_m - a_{n+m-1}) &\leq (n-1)^2 (m-1)^2, \quad \forall n, m \geq 2. \end{aligned}$$

Conjecture 1.28. (*C^* -algebraic Ma Conjecture - strong form*) Let \mathcal{A} be a unital C^* -algebra. If the C^* -algebraic power series

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, the C^* -algebraic power series

$$f'(z) := 1 + \sum_{n=2}^{\infty} a_n n z^{n-1}$$

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then

$$\begin{aligned} (a_n a_m - a_{n+m-1})(a_n a_m - a_{n+m-1})^* &\leq (n-1)^2 (m-1)^2, \\ (a_n a_m - a_{n+m-1})^*(a_n a_m - a_{n+m-1}) &\leq (n-1)^2 (m-1)^2, \quad \forall n, m \geq 2. \end{aligned}$$

In 1948 Goodman formulated the version of Bieberbach conjecture for p -valent functions [35].

Conjecture 1.29. [35, 36, 43, 78] (*Goodman Conjecture*) Let $p \in \mathbb{N}$. If the power series

$$f(z) := \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 1$$

converges for all $z \in \mathbb{D}(0, 1)$ and the function f takes each value almost p times on $\mathbb{D}(0, 1)$, then

$$|a_n| \leq \sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} |a_k|, \quad \forall n > p.$$

We state C^* -algebraic versions of Conjecture 1.29 as follows.

Conjecture 1.30. (*C*-algebraic Goodman Conjecture for general coefficients*) Let $p \in \mathbb{N}$ \mathcal{A} be a unital C*-algebra. If the C*-algebraic power series

$$f(z) := \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 1$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f takes each value at most p times on $\mathbb{D}^*(0, 1)$, then

$$a_n a_n^* \leq \left(\sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k \right) \left(\sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k \right)^*,$$

$$\forall n > p$$

and

$$a_n^* a_n \leq \left(\sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k \right)^* \left(\sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k \right),$$

$$\forall n > p.$$

Conjecture 1.31. (*C*-algebraic Goodman Conjecture for invertible coefficients*) Let $p \in \mathbb{N}$, \mathcal{A} be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . If the C*-algebraic power series

$$f(z) := \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 1$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f takes each value at most p times on $\mathbb{D}^*(0, 1)$, then

$$\frac{1}{\|a_n^{-1}\|} \leq \frac{1}{\left\| \left(\sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k \right)^{-1} \right\|}, \quad \forall n > p.$$

Conjecture 1.32. (*C*-algebraic Goodman Conjecture for general coefficients - strong form*) Let $p \in \mathbb{N}$ \mathcal{A} be a unital C*-algebra. If the C*-algebraic power series

$$f(z) := \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 1$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, the C*-algebraic power series

$$f'(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 1$$

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f takes each value at most p times on $\mathbb{D}^*(0, 1)$, then

$$a_n a_n^* \leq \left(\sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k \right) \left(\sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k \right)^*,$$

$$\forall n > p$$

KRZYZ AND CORONA CONJECTURES

and

$$a_n^* a_n \leq \left(\sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k \right)^* \left(\sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k \right),$$

$$\forall n > p.$$

Conjecture 1.33. (*C*-algebraic Goodman Conjecture for invertible coefficients - strong form*) Let $p \in \mathbb{N}$, \mathcal{A} be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . If the C*-algebraic power series

$$f(z) := \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 1$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, the C*-algebraic power series

$$f'(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 1$$

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f takes each value at most p times on $\mathbb{D}^*(0, 1)$, then

$$\frac{1}{\|a_n^{-1}\|} \leq \left\| \left(\sum_{k=1}^p \frac{2k(n+p)!}{(p-k)!(p+k)!(n-p-1)(n^2-k^2)} a_k \right)^{-1} \right\|, \quad \forall n > p.$$

Next we wish to state C*-algebraic version of Koebe distortion theorem. First we recall the result.

Theorem 1.34. [73, 117] (*Koebe Distortion Theorem*) If the power series

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 2$$

converges for all $z \in \mathbb{D}(0, 1)$ and the function f is injective on $\mathbb{D}(0, 1)$, then

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}, \quad \forall z \in \mathbb{D}(0, 1) \setminus \{0\}$$

and

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}, \quad \forall z \in \mathbb{D}(0, 1) \setminus \{0\}.$$

Conjecture 1.35. (*C*-algebraic Koebe Distortion Conjecture*) Let \mathcal{A} be a unital C*-algebra. If the C*-algebraic power series

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 2$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, the C*-algebraic power series

$$f'(z) := 1 + \sum_{n=2}^{\infty} n a_n z^{n-1}$$

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and the function f is injective on $\mathbb{D}^*(0, 1)$, then

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \quad \forall z \in \mathbb{D}^*(0, 1) \setminus \{0\}$$

and

$$\frac{1 - \|z\|}{(1 + \|z\|)^3} \leq \|f'(z)\| \leq \frac{1 + \|z\|}{(1 - \|z\|)^3}, \quad \forall z \in \mathbb{D}^*(0, 1) \setminus \{0\}.$$

2. C*-ALGEBRAIC KRZYZ CONJECTURE

A conjecture similar to that of Bieberbach conjecture is Krzyz conjecture.

Conjecture 2.1. [5, 7, 81, 94] (**Krzyz Conjecture**) *If the power series*

$$f(z) := \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \forall n \geq 0$$

converges for all $z \in \mathbb{D}(0, 1)$ and $0 < |f(z)| < 1$ on $\mathbb{D}(0, 1)$, then

$$(4) \quad |a_n| \leq \frac{2}{e}, \quad \forall n \geq 1.$$

Inequality (4) is strict except for rotations of functions

$$f_n(z) := e^{\frac{z^n+1}{z^n-1}}.$$

on $\mathbb{D}(0, 1)$, $n \in \mathbb{N}$.

By taking $z = 0$, we see that $|a_0| \leq 1$. In 1953 while answering a problem by Shapiro in **The American Mathematical Monthly**, Robertson proved that $|a_1| \leq \frac{2}{e}$ [104] (also see [97]). In 1977 Hummel, Scheinberg, and Zalcman proved that $|a_2| \leq \frac{2}{e}$ and $|a_3| \leq \frac{2}{e}$ [56]. In 1987 Brown proved that $|a_4| \leq \frac{2}{e}$ [10]. In 2003 Samaris proved that $|a_5| \leq \frac{2}{e}$ [102]. In 1978 Horowitz proved that there exists $0 < c < 1$ such that $|a_n| \leq c$ for all n [51]. In fact, $c = 1 - \frac{1}{3\pi} + \frac{4}{\pi} \sin\left(\frac{1}{12}\right)$. In 2021, Agler and McCarthy obtained a connection between Conjecture 2.1 and the entropy conjecture for polynomials with zeros on the standard unit circle group [1]. An extension of Krzyz conjecture has been formulated by Samaris in 2001 [101]. We state C*-algebraic versions of Krzyz conjecture as follows.

Conjecture 2.2. (*C*-algebraic Krzyz Conjecture for general coefficients*) *Let \mathcal{A} be a unital C*-algebra. If the C*-algebraic power series*

$$f(z) := \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 0$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^(0, 1)$ and $0 < \|f(z)\| < 1$ on $\mathbb{D}^*(0, 1)$, then*

$$(5) \quad a_n a_n^* \leq \frac{4}{e^2}, \quad a_n^* a_n \leq \frac{4}{e^2}, \quad \forall n \geq 1.$$

Inequality (5) is strict except for rotations of C-algebraic functions*

$$f_n(z) := e^{\frac{z^n+1}{z^n-1}}.$$

on $\mathbb{D}^(0, 1)$, $n \in \mathbb{N}$.*

Conjecture 2.3. (*C*-algebraic Krzyz Conjecture for invertible coefficients*) *Let \mathcal{A} be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . If the C*-algebraic power series*

$$f(z) := \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 0$$

KRZYZ AND CORONA CONJECTURES

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$ and $0 < \|f(z)\| < 1$ on $\mathbb{D}^*(0, 1)$, then

$$\frac{1}{\|a_n^{-1}\|} \leq \frac{2}{e}, \quad \forall a_n \in I(\mathcal{A}), n \geq 1.$$

Conjecture 2.4. (*C*-algebraic Krzyz Conjecture for general coefficients - strong form*) Let \mathcal{A} be a unital C*-algebra. If the C*-algebraic power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}, \forall n \geq 0$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, $0 < \|f(z)\| < 1$ on $\mathbb{D}^*(0, 1)$ and the C*-algebraic power series

$$f'(z) := \sum_{n=1}^{\infty} a_n n z^{n-1}$$

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, then

$$a_n a_n^* \leq \frac{4}{e^2}, \quad a_n^* a_n \leq \frac{4}{e^2}, \quad \forall n \geq 0.$$

Conjecture 2.5. (*C*-algebraic Krzyz Conjecture for invertible coefficients - strong form*) Let \mathcal{A} be a unital C*-algebra and let $I(\mathcal{A})$ be the set of all invertible elements in \mathcal{A} . If the C*-algebraic power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in I(\mathcal{A}) \cup \{0\}, \forall n \geq 0$$

converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, $0 < \|f(z)\| < 1$ on $\mathbb{D}^*(0, 1)$ and the C*-algebraic power series

$$f'(z) := \sum_{n=1}^{\infty} a_n n z^{n-1}$$

also converges (in norm of \mathcal{A}) for all $z \in \mathbb{D}^*(0, 1)$, then

$$\frac{1}{\|a_n^{-1}\|} \leq \frac{2}{e}, \quad \forall a_n \in I(\mathcal{A}), n \geq 1.$$

We end this section by asking a problem which corresponds to **Riemann Mapping Theorem** in one complex variable [31, 32, 39, 40, 61, 74, 82, 85, 112, 113]. We set the following notion of analyticity and conformality.

Definition 2.6. Let \mathcal{A} be a unital C*-algebra and Ω be an open set in \mathcal{A} (in the norm topology). A map $f : \Omega \rightarrow \mathcal{A}$ is said to be C*-algebraic holomorphic or analytic if for each $a \in \Omega$, there exists a C*-algebraic power series which converges in the norm around a , i.e., there exists a real $r > 0$, a C*-algebraic disc $\mathbb{D}^*(a, r) := \{z \in \mathcal{A} : \|z - a\| < r\} \subseteq \Omega$ and a sequence $\{a_n\}_{n=0}^{\infty} \subseteq \mathcal{A}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n, \quad \forall z \in \mathbb{D}^*(a, r)$$

where the series converges in the norm of \mathcal{A} .

Definition 2.7. Let \mathcal{A} be a unital C*-algebra and Ω_1, Ω_2 be open sets in \mathcal{A} . We say that Ω_1 and Ω_2 are C*-algebraic conformal or C*-algebraic biholomorphic to each other if there

is a bijective C^* -algebraic holomorphic function $f : \Omega_1 \rightarrow \Omega_2$ such that $f^{-1} : \Omega_2 \rightarrow \Omega_1$ is C^* -algebraic holomorphic.

Problem 2.8. (C^* -algebraic Riemann Mapping Problem) Let A be a unital C^* -algebra. Classify open subsets of A which is C^* -algebraic biholomorphic to $\mathbb{D}^*(0, 1)$.

3. C^* -ALGEBRAIC CORONA CONJECTURE

Everything started from the paper of Carleson published in 1962 [14].

Theorem 3.1. [13, 14, 66, 87] (*Corona Conjecture/Carleson Theorem*) Let $n \in \mathbb{N}$ and $f_1, \dots, f_n : \mathbb{D}(0, 1) \rightarrow \mathbb{C}$ be bounded analytic functions such that

$$\sum_{j=1}^n |f_j(z)|^2 \geq \delta, \quad \forall z \in \mathbb{D}(0, 1)$$

for some $\delta > 0$. Then there are bounded analytic functions $g_1, \dots, g_n : \mathbb{D}(0, 1) \rightarrow \mathbb{C}$ such that

$$\sum_{j=1}^n f_j(z)g_j(z) = 1, \quad \forall z \in \mathbb{D}(0, 1).$$

Other proofs of Theorem 3.1 were given by Wolff [3, 30, 64], Slodkowski [105], Berndtsson and Ransford [8], Hormander [50], Kelleher and Taylor [60] and Jones [57]. In 1980 Rosenblum [100] and Tolokonnikov [110] proved Theorem 3.1 for countably many functions. History of Theorem 3.1 is beautifully presented in [20]. In 2007 Trent gave an algorithm to produce g_1, \dots, g_n in Theorem 3.1 whenever f_1, \dots, f_n are polynomials [111].

We state C^* -algebraic version of Conjecture 3.1 as follows.

Conjecture 3.2. (C^* -algebraic Corona Conjecture) Let A be a unital C^* -algebra. Let $n \in \mathbb{N}$ and $f_1, \dots, f_n : \mathbb{D}^*(0, 1) \rightarrow A$ be bounded C^* -algebraic holomorphic functions such that

$$\sum_{j=1}^n f_j(z)f_j(z)^* \geq \delta, \quad \forall z \in \mathbb{D}^*(0, 1) \quad \text{and} \quad \sum_{j=1}^n f_j(z)^*f_j(z) \geq \delta, \quad \forall z \in \mathbb{D}^*(0, 1).$$

for some real $\delta > 0$. Then there are bounded C^* -algebraic holomorphic functions $g_1, \dots, g_n, h_1, \dots, h_n : \mathbb{D}^*(0, 1) \rightarrow A$ such that

$$\sum_{j=1}^n f_j(z)g_j(z) = 1, \quad \forall z \in \mathbb{D}^*(0, 1)$$

and

$$\sum_{j=1}^n h_j(z)f_j(z) = 1, \quad \forall z \in \mathbb{D}^*(0, 1).$$

Remark 3.3. Some of the above conjectures can be formulated for unital Banach algebras as well as for Banach $*$ -algebras.

Remark 3.4. (i) C^* -algebraic Sendov Conjecture has been formulated in [68].

(ii) C^* -algebraic Schoenberg Conjecture has been formulated in [69].

(iii) C^* -algebraic Smale Mean Value Conjecture and Dubinin-Sugawa Dual Mean Value Conjecture have been formulated in [70].

(iv) C^* -algebraic Casas-Alvero Conjecture has been formulated in [67].

KRZYZ AND CORONA CONJECTURES

REFERENCES

- [1] Jim Agler and John E. McCarthy. The Krzyż conjecture and an entropy conjecture. *J. Anal. Math.*, 144(1):207–226, 2021.
- [2] O. P. Ahuja. The Bieberbach conjecture and its impact on the developments in geometric function theory. *Math. Chronicle*, 15:1–28, 1986.
- [3] Mats Andersson. *Topics in complex analysis*. Universitext. Springer-Verlag, New York, 1997.
- [4] Richard Askey and George Gasper. Positive Jacobi polynomial sums. II. *Amer. J. Math.*, 98(3):709–737, 1976.
- [5] Farit G. Avkhadiiev and Karl-Joachim Wirths. *Schwarz-Pick type inequalities*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2009.
- [6] Albert Baernstein, II. Bieberbach’s conjecture for tourists. In *Harmonic analysis (Minneapolis, Minn., 1981)*, volume 908 of *Lecture Notes in Math.*, pages 48–73. Springer, Berlin-New York, 1982.
- [7] Catherine Beneteau and Dmitry Khavinson. A survey of certain extremal problems for non-vanishing analytic functions. In *Complex and harmonic analysis*, pages 45–61. DEStech Publ., Inc., Lancaster, PA, 2007.
- [8] B. Berndtsson and T. J. Ransford. Analytic multifunctions, the $\bar{\partial}$ -equation, and a proof of the corona theorem. *Pacific J. Math.*, 124(1):57–72, 1986.
- [9] L. Bieberbach. Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *Sitzungsber. Preuss. Akad. Wiss. Phys-Math. Kl.*, pages 940–955, 1916.
- [10] Johnny E. Brown. Iteration of functions subordinate to schlicht functions. *Complex Variables Theory Appl.*, 9(2-3):143–152, 1987.
- [11] Johnny E. Brown and Anna Tsao. On the Zalcman conjecture for starlike and typically real functions. *Math. Z.*, 191(3):467–474, 1986.
- [12] Xavier Buff. On the Bieberbach conjecture and holomorphic dynamics. *Proc. Amer. Math. Soc.*, 131(3):755–759, 2003.
- [13] Lennart Carleson. An interpolation problem for bounded analytic functions. *Amer. J. Math.*, 80:921–930, 1958.
- [14] Lennart Carleson. Interpolations by bounded analytic functions and the corona problem. *Ann. of Math. (2)*, 76:547–559, 1962.
- [15] Z. Charzynski and M. Schiffer. A new proof of the Bieberbach conjecture for the fourth coefficient. *Arch. Rational Mech. Anal.*, 5:187–193 (1960), 1960.
- [16] J. Clunie and Ch. Pommerenke. On the coefficients of close-to-convex univalent functions. *J. London Math. Soc.*, 41:161–165, 1966.
- [17] Louis de Branges. A proof of the Bieberbach conjecture. *Acta Math.*, 154(1-2):137–152, 1985.
- [18] Louis de Branges. Unitary linear systems whose transfer functions are Riemann mapping functions. In *Operator theory and systems (Amsterdam, 1985)*, volume 19 of *Oper. Theory Adv. Appl.*, pages 105–124. Birkhäuser, Basel, 1986.
- [19] Louis de Branges. Underlying concepts in the proof of the Bieberbach conjecture. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 25–42. Amer. Math. Soc., Providence, RI, 1987.
- [20] Ronald G. Douglas, Steven G. Krantz, Eric T. Sawyer, Sergei Treil, and Brett D. Wicks. A history of the corona problem. In *The corona problem*, volume 72 of *Fields Inst. Commun.*, pages 1–29. Springer, New York, 2014.
- [21] David Drasin, Peter Duren, and Albert Marden, editors. *The Bieberbach conjecture*, volume 21 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1986.
- [22] Peter L. Duren. Coefficients of univalent functions. *Bull. Amer. Math. Soc.*, 83(5):891–911, 1977.
- [23] Gertraud Ehrig. Coefficient estimates concerning the Bieberbach conjecture. *Math. Z.*, 140:111–126, 1974.
- [24] Shalosh B. Ekhad and Doron Zeilberger. A high-school algebra, “formal calculus”, proof of the Bieberbach conjecture [after L. Weinstein]. In *Jerusalem combinatorics '93*, volume 178 of *Contemp. Math.*, pages 113–115. Amer. Math. Soc., Providence, RI, 1994.
- [25] M. Fekete and G. Szego. Eine Bemerkung Über Ungerade Schlichte Funktionen. *J. London Math. Soc.*, 8(2):85–89, 1933.
- [26] Carl H. FitzGerald. Quadratic inequalities and coefficient estimates for schlicht functions. *Arch. Rational Mech. Anal.*, 46:356–368, 1972.
- [27] Carl H. FitzGerald. The Bieberbach conjecture: retrospective. *Notices Amer. Math. Soc.*, 32(1):2–6, 1985.
- [28] Carl H. FitzGerald and Ch. Pommerenke. The de Branges theorem on univalent functions. *Trans. Amer. Math. Soc.*, 290(2):683–690, 1985.

- [29] O. M. Fomenko and G. V. Kuzmina. The last 100 days of the Bieberbach conjecture. *Math. Intelligencer*, 8(1):40–47, 1986.
- [30] T. W. Gamelin. Wolff’s proof of the corona theorem. *Israel J. Math.*, 37(1-2):113–119, 1980.
- [31] P. R. Garabedian. Univalent functions and the Riemann mapping theorem. *Proc. Amer. Math. Soc.*, 61(2):242–244 (1977), 1976.
- [32] P. R. Garabedian. A simple proof of a simple version of the Riemann mapping theorem by simple functional analysis. *Amer. Math. Monthly*, 98(9):824–826, 1991.
- [33] P. R. Garabedian and M. Schiffer. A coefficient inequality for schlicht functions. *Ann. of Math. (2)*, 61:116–136, 1955.
- [34] P. R. Garabedian and M. Schiffer. A proof of the Bieberbach conjecture for the fourth coefficient. *J. Rational Mech. Anal.*, 4:427–465, 1955.
- [35] A. W. Goodman. On some determinants related to p -valent functions. *Trans. Amer. Math. Soc.*, 63:175–192, 1948.
- [36] A. W. Goodman and M. S. Robertson. A class of multivalent functions. *Trans. Amer. Math. Soc.*, 70:127–136, 1951.
- [37] Gerald S. Goodman. On the determination of univalent functions with prescribed initial coefficients. *Arch. Rational Mech. Anal.*, 24:78–81, 1967.
- [38] Ian Graham and Gabriela Kohr. *Geometric function theory in one and higher dimensions*, volume 255 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 2003.
- [39] Jeremy Gray. On the history of the Riemann mapping theorem. *Rend. Circ. Mat. Palermo (2) Suppl.*, (34):47–94, 1994.
- [40] Robert E. Greene and Kang-Tae Kim. The Riemann mapping theorem from Riemann’s viewpoint. *Complex Anal. Synerg.*, 3(1):Paper No. 1, 11, 2017.
- [41] Arcadii Z. Grinshpan. The Bieberbach conjecture and Milin’s functionals. *Amer. Math. Monthly*, 106(3):203–214, 1999.
- [42] T. H. Gronwall. Some remarks on conformal representation. *Ann. of Math. (2)*, 16(1-4):72–76, 1914/15.
- [43] David J. Hallenbeck and Albert E. Livingston. A coefficient estimate for multivalent functions. *Proc. Amer. Math. Soc.*, 54:201–206, 1976.
- [44] D. H. Hamilton. On Littlewood’s conjecture for univalent functions. *Proc. Amer. Math. Soc.*, 86(1):32–36, 1982.
- [45] W. K. Hayman. The asymptotic behaviour of p -valent functions. *Proc. London Math. Soc. (3)*, 5:257–284, 1955.
- [46] W. K. Hayman. Coefficient problems for univalent functions and related function classes. *J. London Math. Soc.*, 40:385–406, 1965.
- [47] J. William Helton and Frederick Weening. Some systems theorems arising from the Bieberbach conjecture. *Internat. J. Robust Nonlinear Control*, 6(1):65–82, 1996.
- [48] Peter Henrici. *Applied and computational complex analysis. Vol. 3 : Discrete Fourier analysis—Cauchy integrals - construction of conformal maps - univalent functions*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1986.
- [49] Finbarr Holland. The Bieberbach conjecture: a summer drama. *Irish Math. Soc. Newsl.*, (13):38–47, 1985.
- [50] Lars Hörmander. Generators for some rings of analytic functions. *Bull. Amer. Math. Soc.*, 73:943–949, 1967.
- [51] Charles Horowitz. Coefficients of nonvanishing functions in H^∞ . *Israel J. Math.*, 30(3):285–291, 1978.
- [52] David Horowitz. A refinement for coefficient estimates of univalent functions. *Proc. Amer. Math. Soc.*, 54:176–178, 1976.
- [53] David Horowitz. Coefficient estimates for univalent polynomials. *J. Analyse Math.*, 31:112–124, 1977.
- [54] David Horowitz. A further refinement for coefficient estimates of univalent functions. *Proc. Amer. Math. Soc.*, 71(2):217–221, 1978.
- [55] Ke Hu. Distortion and coefficient estimation of schlicht functions. *Proc. Amer. Math. Soc.*, 87(3):487–492, 1983.
- [56] J. A. Hummel, Stephen Scheinberg, and Lawrence Zalcman. A coefficient problem for bounded nonvanishing functions. *J. Analyse Math.*, 31:169–190, 1977.
- [57] Peter W. Jones. Estimates for the corona problem. *J. Functional Analysis*, 39(2):162–181, 1980.
- [58] Wilfred Kaplan. Close-to-convex schlicht functions. *Michigan Math. J.*, 1:169–185 (1953), 1952.
- [59] Nicholas D. Kazarinoff. Special functions and the Bieberbach conjecture. *Amer. Math. Monthly*, 95(8):689–696, 1988.
- [60] J. Kelleher and B. A. Taylor. An application of the Corona theorem to some rings of entire functions. *Bull. Amer. Math. Soc.*, 73:246–249, 1967.
- [61] Kunihiko Kodaira. *Complex analysis*, volume 107 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

KRZYZ AND CORONA CONJECTURES

- [62] Wolfram Koepf. Power series, Bieberbach conjecture and the de Branges and Weinsten functions. In *Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation*, pages 169–175. ACM, New York, 2003.
- [63] Wolfram Koepf. Bieberbach’s conjecture, the de Branges and Weinsten functions and the Askey-Gasper inequality. *Ramanujan J.*, 13(1-3):103–129, 2007.
- [64] Paul Koosis. *Introduction to H_p spaces*, volume 40 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1980. With an appendix on Wolff’s proof of the corona theorem.
- [65] J. Korevaar. Ludwig Bieberbach’s conjecture and its proof by Louis de Branges. *Amer. Math. Monthly*, 93(7):505–514, 1986.
- [66] Steven G. Krantz. *Geometric function theory: Explorations in complex analysis*. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2006.
- [67] K. Mahesh Krishna. C^* -algebraic Casas-Alvero conjecture. *arXiv:2206.09197v2 [math.FA]*, 19 July, 2022.
- [68] K. Mahesh Krishna. C^* -algebraic Gauss-Lucas Theorem and C^* -algebraic Sendov’s Conjecture. *arXiv:2203.06916v1 [math.OA]*, 14 Mar, 2022.
- [69] K. Mahesh Krishna. C^* -algebraic Schoenberg conjecture. *arXiv:2206.06653v1 [math.OA]*, 14 June, 2022.
- [70] K. Mahesh Krishna. C^* -algebraic Smale mean value conjecture and Dubinin-Sugawa dual mean value conjecture. *arXiv:2206.08154v1 [math.FA]*, 16 June, 2022.
- [71] Samuel L. Krushkal. Proof of the Zalcman conjecture for initial coefficients. *Georgian Math. J.*, 17(4):663–681, 2010.
- [72] Samuel L. Krushkal. Erratum: Proof of the Zalcman conjecture for initial coefficients [Georgian Math. J. 17 (2010), 663–681]. *Georgian Math. J.*, 19(4):777, 2012.
- [73] Prem K. Kythe. *Complex Analysis : Conformal Inequalities and the Bieberbach Conjecture*. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2016.
- [74] Mark Levi. Riemann mapping theorem by steepest descent. *Amer. Math. Monthly*, 114(3):246–251, 2007.
- [75] J. E. Littlewood. On Inequalities in the Theory of Functions. *Proc. London Math. Soc. (2)*, 23(7):481–519, 1925.
- [76] J. E. Littlewood and E. A. C. Paley. A proof that an odd schlicht function has bounded coefficients. *J. London Math. Soc.*, 7(3):167–169, 1932.
- [77] Karl Lowner. Untersuchungen uber schlichte konforme Abbildungen des Einheitskreises. I. *Math. Ann.*, 89(1-2):103–121, 1923.
- [78] Abdallah Lyzzaik and David Styer. Goodman’s conjecture and the coefficients of univalent functions. *Proc. Amer. Math. Soc.*, 69(1):111–114, 1978.
- [79] Wan Cang Ma. The Zalcman conjecture for close-to-convex functions. *Proc. Amer. Math. Soc.*, 104(3):741–744, 1988.
- [80] William Ma. Generalized Zalcman conjecture for starlike and typically real functions. *J. Math. Anal. Appl.*, 234(1):328–339, 1999.
- [81] Maria J. Martin, Eric T. Sawyer, Ignacio Uriarte-Tuero, and Dragan Vukotic. The Krzyz conjecture revisited. *Adv. Math.*, 273:716–745, 2015.
- [82] H. P. McKean. A quick proof of Riemann’s mapping theorem. *Comm. Pure Appl. Math.*, 52(3):405–409, 1999.
- [83] I. M. Milin. A bound for the coefficients of schlicht functions. *Dokl. Akad. Nauk SSSR*, 160:769–771, 1965.
- [84] I. M. Milin. *Univalent functions and orthonormal systems*. Translations of Mathematical Monographs, Vol. 49. American Mathematical Society, Providence, R.I., 1977.
- [85] Tristan Needham. *Visual complex analysis*. The Clarendon Press, Oxford University Press, New York, 1997.
- [86] Zeev Nehari. On the coefficients of univalent functions. *Proc. Amer. Math. Soc.*, 8(2):291–293, 1957.
- [87] D. J. Newman. Some remarks on the maximal ideal structure of H^∞ . *Ann. of Math. (2)*, 70:438–445, 1959.
- [88] N.K. Nikolskii and V.I. Vasyunin. Quasi-orthogonal Hilbert space decompositions and estimates of univalent functions - II. In A.A. Gonchar and E.B Saff, editors, *Progress in Approximation Theory*, volume 19 of *Springer Series in Computational Mathematics*. Springer, New York, 1992.
- [89] Arthur Obrock. An inequality for certain schlicht functions. *Proc. Amer. Math. Soc.*, 17:1250–1253, 1966.
- [90] Mitsuru Ozawa. An elementary proof of the Bieberbach conjecture for the sixth coefficient. *Kodai Math. Sem. Rep.*, 21:129–132, 1969.
- [91] Mitsuru Ozawa. On the Bieberbach conjecture for the sixth coefficient. *Kodai Math. Sem. Rep.*, 21:97–128, 1969.
- [92] R. Pederson and M. Schiffer. A proof of the Bieberbach conjecture for the fifth coefficient. *Arch. Rational Mech. Anal.*, 45:161–193, 1972.
- [93] Roger N. Pederson. A proof of the Bieberbach conjecture for the sixth coefficient. *Arch. Rational Mech. Anal.*, 31:331–351, 1968/69.

- [94] Ronen Peretz. *The Krzyz Conjecture : Theory and Method*. World Scientific Publishers, Singapore, 2021.
- [95] Ch. Pommerenke. The Bieberbach conjecture. *Math. Intelligencer*, 7(2):23–25, 32, 1985.
- [96] Christian Pommerenke. *Univalent functions*. Studia Mathematica/Mathematische Lehrbücher, Band XXV. Vandenhoeck & Ruprecht, Göttingen, 1975. With a chapter on quadratic differentials by Gerd Jensen.
- [97] C. T. Rajagopal. On inequalities for analytic functions. *Amer. Math. Monthly*, 60:693–695, 1953.
- [98] Maxwell O. Reade. On close-to-convex univalent functions. *Michigan Math. J.*, 3:59–62, 1955.
- [99] M. S. Robertson. A remark on the odd schlicht functions. *Bull. Amer. Math. Soc.*, 42(6):366–370, 1936.
- [100] Marvin Rosenblum. A corona theorem for countably many functions. *Integral Equations Operator Theory*, 3(1):125–137, 1980.
- [101] N. Samaris. An extension of Krzyż’s conjecture. *J. Comput. Appl. Math.*, 137(1):13–18, 2001.
- [102] Nikolas Samaris. A proof of Krzyz’s conjecture for the fifth coefficient. *Complex Var. Theory Appl.*, 48(9):753–766, 2003.
- [103] Menahem Schiffer. Univalent functions whose n first coefficients are real. *J. Analyse Math.*, 18:329–349, 1967.
- [104] H. S. Shapiro and M. S. Robertson. Advanced Problems and Solutions: Solutions: 4468. *Amer. Math. Monthly*, 60(2):131–132, 1953.
- [105] Zbigniew Slodkowski. An analytic set-valued selection and its applications to the corona theorem, to polynomial hulls and joint spectra. *Trans. Amer. Math. Soc.*, 294(1):367–377, 1986.
- [106] Norbert Steinmetz. de Branges’ proof of the Bieberbach conjecture. In *General inequalities, 5 (Oberwolfach, 1986)*, volume 80 of *Internat. Schriftenreihe Numer. Math.*, pages 3–16. Birkhäuser, Basel, 1987.
- [107] Pavel G. Todorov. A simple proof of the Bieberbach conjecture. *Serdica*, 19(2-3):204–214, 1993.
- [108] Pavel G. Todorov. A new proof of the final stage of the proof of the Milin, Robertson and Bieberbach conjectures which does not use the Askey and Gasper inequality. *Acad. Roy. Belg. Bull. Cl. Sci. (6)*, 9(7-12):349–359, 1998.
- [109] Pavel G. Todorov. A structural formula of the Weinstein functions used in his proof of the Milin, Robertson and Bieberbach conjectures. *Publ. Inst. Math. (Beograd) (N.S.)*, 70(84):9–18, 2001.
- [110] V. A. Tolokonnikov. Estimates in Carleson’s corona theorem and finitely generated ideals in an algebra H^∞ . *Functional Analysis and Its Applications*, 14(4):320–322, 1980.
- [111] Tavan T. Trent. An algorithm for corona solutions on $H^\infty(D)$. *Integral Equations Operator Theory*, 59(3):421–435, 2007.
- [112] J. L. Walsh. History of the Riemann mapping theorem. *Amer. Math. Monthly*, 80:270–276, 1973.
- [113] Elias Wegert. *Visual complex functions: An introduction with phase portraits*. Birkhäuser/Springer Basel AG, Basel, 2012.
- [114] Lenard Weinstein. The Bieberbach conjecture. *Internat. Math. Res. Notices*, (5):61–64, 1991.
- [115] Herbert S. Wilf. A footnote on two proofs of the Bieberbach-de Branges theorem. *Bull. London Math. Soc.*, 26(1):61–63, 1994.
- [116] Ming-Qin Xie. A generalization of the de Branges theorem. *Proc. Amer. Math. Soc.*, 125(12):3605–3611, 1997.
- [117] Paul Zorn. The Bieberbach conjecture. *Math. Mag.*, 59(3):131–148, 1986.