GN-OPERATORS

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Abstract. In this article, first we are going to review the concept of ordinary frames , in more general case in measure spaces, namely, gc-frames. We try to develop the use of measure space in describing frames. Then by means of the gc-frames, we shall introduce gn-operators, which we shall show that each trace class operator has a vector-valued integral representation and vice-versa.

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1 Introduction

Frame were first introduced in 1952 by Duffin and Schaeffer [11], reintroduced in 1986 by Daubechies, Grossman, and Meyer [10]. In the last twenty years, the theory of frames have been employed in numerous applications such as filter bank theory [5], sigma-data quantization [4], signal and image processing [6], sensor networks [18], and wireless communications [15]. However, a large number of new applications have emerged.

One of the main virtues of frames is that, given a frame we can get properties of a function and reconstruct it only from the frame coefficients, a sequence of complex numbers[13, 16].

Various generalizations of frame theory have been proposed recently. For example, bounded quasi-projectors [14], frames of subspaces [3, 7], pseudo-frames [17], oblique frames [9, 12], outer frames [2], g-frames [21], and continuous g-frames [1]. All of these generalizations are useful in many applications and can be regarded as special cases of continuous g-frames.

The point of view of the g-frames is based on operator theory. The point of view of the gc-frames is completely different and it is based on reconstruction a quantity by means of a repeated sequence of members of a Hilbert space, which is worthy in applications. However, we shall consider it in more general case, generalization of the ordinary frames according to repeated Lebesgue integrals.

In Section 2, we will introduce gc-frames, which is a generalization of ordinary frames, and we will explain some definitions,examples and theorems related to it. Finally, in Section 3, we will introduce GN operators and explain some definitions and some theorems related to these operators.

Throughout this paper H will be a Hilbert space. Also, (X, μ) will be a measure space and $\{(Y_x, \mu_x)\}_{x\in X}$ will be a class of measure spaces. We shall denote by $\mathcal{M}_H$ the class of all mappings of a measure space to H. Also H_1 will be denote the unit ball of H.

Definition 1.1 *[1] Let $L^2(X,H)$ be the class of all measurable mappings $f : X \to H$ such that*

$$||f||_2^2 = \int_X ||f(x)||^2 d\mu < \infty.$$

By the polar identity we conclude that for each $f, g \in L^2(X,H)$, the mapping $x \mapsto \langle f(x), g(x)\rangle$ of X to $\mathbb{C}$ is measurable, and it can be proved that $L^2(X,H)$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle = \int_X \langle f(x), g(x)\rangle d\mu.$$

We shall write $L^2(X)$ when $H = \mathbb{C}$.

Definition 1.2 *[8] Let $\{f_n\}_{n\in\mathbb{N}}$ be sequence in H. We say that $\{f_n\}_{n\in\mathbb{N}}$ is a frame for H if there exist constants $0 < A \le B < \infty$ such that*

$$A||h||^2 \le \sum_{n\in\mathbb{N}} |\langle f_n, h\rangle|^2 \le B||h||^2, \quad h \in H.$$

The next definition is the continuous version of frames.

Definition 1.3 *[8] Let $f : X \to H$ be weakly measurable (i.e. for each $h \in H$, the mapping $x \mapsto \langle f(x), h\rangle$ is measurable). We say that f is a c-frame (continuous frame) for H if there exist constants $0 < A_f \le B_f < \infty$ such that*

$$A_f||h||^2 \le \int_X |\langle f(x), h\rangle|^2 d\mu \le B_f||h||^2, \quad h \in H.$$

The following Lemmas can be found in operator theory text books [19, 20], which we shall use them.

Definition 1.4 ***(c-frame)****[8] Let $F : X \to H$ be a mapping such that the mapping $x \to \langle h, F(x)\rangle$ of X to $\mathbb{C}$ is measurable (i.e. weakly measurable) for* *each* $h \in H$. F *is called c-frame for* H *if there exists* $0 < A \leq B < \infty$ *such that for any* $h \in H$

$$A\|h\|^2 \leq \int_X |\langle h, F(x)\rangle|^2 \, d\mu \leq B\|h\|^2.$$

Lemma 1.5 *[8] Let* $u : K \to H$ *be a bounded operator with closed range* $\mathcal{R}_u$*.Then there exists a bounded operator* $u^\dagger : H \to K$ *for which*

$$uu^\dagger f = f, \quad f \in \mathcal{R}_u.$$

Also, $u^* : H \to K$ *has closed range and* $(u^*)^\dagger = (u^\dagger)^*$*.*

Lemma 1.6 *[8] Let* $u : K \to H$ *be a bounded surjective operator. Given* $y \in H$ *, the equation* $ux = y$ *has a unique solution of minimal norm, namely,* $x = u^\dagger y$*.*

Lemma 1.7 *[8] Let* u *be a self-adjoint bounded operator on* H*. Let*

$$m_u = \inf_{||h||=1} \langle uh, h\rangle \quad \textit{and} \quad M_u = \sup_{||h||=1} \langle uh, h\rangle.$$

Then, $m_u, M_u \in \sigma(u)$*.*

Definition 1.8 *[8] Let (X,*μ*) is a -finite measure space*

$$l^2(X) = \{\varphi : X \to \mathbb{C}| \quad \varphi \textit{ is measurable and } \quad \|\varphi\|_2 < \infty\},$$

where $\|\varphi\|_2 = (\int_X ||\varphi(x)||^2 d\mu)^{\frac{1}{2}}$*.*

For each $f, g \in l^2(X)$*, the mapping* $x \to \langle f(x), g(x)\rangle$ *of* X *to* $\mathbb{C}$ *is measurable, and it can be proved that* $l^2(X)$ *is a Hilbert space with the inner product defined by*

$$\langle f, g\rangle_{L^2} = \int_X \langle f(x), g(x)\rangle \, d\mu.$$

2 GC-frames

In this section we shall introduce gc-frames which is the generalization of the ordinary frames.

Definition 2.1 *Let* $f : X \to \mathcal{M}_H$*,* $x \mapsto f_x$ *and let for each* $x \in X$*,* $f_x : Y_x \to H$*. We say that* f *is weakly measurable if*
(i) *for each* $x \in X$*,* $f_x : (Y_x, \mu_x) \to H$ *is weakly measurable, and*
(ii) *for each* $h \in H$*, the mapping*

$$X \to \mathbb{C}, \quad x \mapsto \int_{Y_x} \langle f_x(y), h\rangle d\mu_x$$

is measurable.

We are now ready to give the central definition.

Definition 2.2 *We say that $f : X \to \mathcal{M}_H$ is a gc-frame (generalized continuous frame) for H if f is weakly measurable and there exist constants $0 < A_f \leq B_f < \infty$ such that*

$$A_f||h||^2 \leq \int_X |\int_{Y_x} \langle f_x(y), h\rangle d\mu_x|^2 d\mu \leq B_f||h||^2, \quad h \in H.$$

We call A_f and B_f the gc-frame bounds. The gc-frame f is called a tight gc-frame if A_f and B_f can be chosen so that $A_f = B_f$, and a Parseval gc-frame provided that $A_f = B_f = 1$. If just the right hand inequality satisfies then we say that f is a gc-Bessel mapping for H with gc-Bessel bound B_f.

By the following example, each c-frame for H is indeed a gc-frame for H.

Example 2.3 *Let $f : X \to H$ be weakly measurable. Let $g : X \to \mathcal{M}_H$ be a gc-frame for H where (Y, λ) is a finite measure space with $\lambda(Y) = 1$. Let for each $x \in X$ and $y \in Y$, $g_x(y) = f(x)$. Then we get*

$$A_g||h||^2 \leq \int_X |\langle f(x), h\rangle|^2 d\mu \leq B_g||h||^2, \quad h \in H,$$

which is the usual c-frame.

Let $f : X \to \mathcal{M}_H$ be a gc-Bessel mapping for H with gc-Bessel bound B_f. We define the gc-pre-frame mapping $T_f : L^2(X) \to H$ as a vector-valued mapping by

$$\langle T_f(g), h\rangle = \int_X \int_{Y_x} \langle f_x, h\rangle g d\mu_x d\mu, \quad h \in H.$$

Since

$$\begin{aligned} |\int_X \int_{Y_x} \langle f_x, h\rangle g d\mu_x d\mu| &\leq \int_X |\int_{Y_x} \langle f_x, h\rangle d\mu_x||g| d\mu \\ &\leq (\int_X |\int_{Y_x} \langle f_x, h\rangle d\mu_x|^2 d\mu)^{1/2} (\int_X |g|^2 d\mu)^{1/2} \\ &\leq B_f^{1/2} ||g||_2 ||h||, \end{aligned}$$

T_f is well-defined. Since

$$||T_f(g)|| = \sup_{h \in H_1} |\langle T(g), h\rangle| \leq B_f^{1/2}||g||,$$

T_f is a bounded linear mapping. Let $T_f^* : H \to L^2(X)$ be its adjoint. We have

$$\langle T_f^*(h), g\rangle = \langle h, T_f(g)\rangle = \int_X \int_{Y_x} \langle h, f_x\rangle \overline{g} d\mu_x d\mu.$$

Since for each $h \in H$, the mapping

$$\int_{Y_.} \langle h, f_. \rangle d\mu_. : X \to \mathbb{C}, \quad x \mapsto \int_{Y_x} \langle h, f_x \rangle d\mu_x$$

belongs to $L^2(X)$,

$$\langle T_f^*(h), g \rangle = \langle \int_{Y_.} \langle h, f_. \rangle d\mu_., g \rangle.$$

Thus the gc-analysis operator is defined by

$$T_f^* : H \to L^2(X), \quad T_f^*(h) = \int_{Y_.} \langle h, f_. \rangle d\mu_..$$

By composing T_f and T_f^*, we obtain the gc-frame operator

$$S_f : H \to H, \quad S_f(h) = TT^*(h).$$

Let $f : X \to \mathcal{M}_H$ be a gc-frame with frame bounds A_f and B_f. Then

$$A_f I \leq S_f \leq B_f I.$$

Hence S is a positive invertible operator.

Let $f : X \to \mathcal{M}_H$ be a gc-Bessel mapping. Then the gc-pre-frame and gc-frame operators are vector-valued mappings which are defined by

$$T_f(g) = \int_X g(x) \int_{Y_x} f_x d\mu_x d\mu, \quad g \in L^2(X)$$

where

$$\langle T_f(g), h \rangle = \int_X g(x) \int_{Y_x} \langle f_x, h \rangle d\mu_x d\mu, \quad h \in H$$

and

$$S_f(h) = T_f(\int_{Y_.} \langle h, f_. \rangle d\mu_.) = \int_X \int_{Y_x} \langle h, f_x \rangle d\mu_x \int_{Y_x} f_x d\mu_x d\mu, \quad h \in H$$

where

$$\begin{aligned} \langle S(h), h \rangle &= \int_X \int_{Y_x} \langle h, f_x \rangle d\mu_x \int_{Y_x} \langle f_x, h \rangle d\mu_x d\mu \\ &= \int_X | \int_{Y_x} \langle f_x, h \rangle d\mu_x |^2 d\mu \\ &= || \int_{Y_.} \langle h, f_. \rangle d\mu_. ||^2. \end{aligned}$$

We now give a characterization of gc-frames in term of the gc-pre-frame operator. It is not involve any knowledge of the frame bounds.

Lemma 2.4 *Let $f : X \to \mathcal{M}_H$ be a gc-Bessel mapping. Then the following assertions are equivalent:*
(i) The frame operator S_f is invertible .
(ii) The mapping f is a gc-frame.
(iii) The gc-pre-frame operator T_f is surjective.

Proof: $(i) \Rightarrow (ii)$ Let S_f be invertible. Since

$$\inf_{h\in H_1} \langle T_f T_f^*(h), h\rangle \in \sigma(S),$$

$\inf_{h\in H_1} \langle T_f T_f^*(h), h\rangle > 0$. Hence f is a gc-frame for H.
$(ii) \Rightarrow (iii)$ Let f be a gc-frame for H with gc-frame lower bound A_f. Since

$$A_f ||h||^2 \le ||T_f^*(h)||^2, \quad h \in H$$

T_f is surjective.
$(iii) \Rightarrow (i)$ Let T_f be surjective. Then there exists $A > 0$ such that

$$A||h|| \le ||T_f^*(h)||, \quad h \in H.$$

Thus f is a gc-frame for H. So S_f is invertible. □

The following Theorem indicates a relation between operators and compositions with gc-frames.

Lemma 2.5 *Let K be a Hilbert space and $f : X \to \mathcal{M}_H$ be a gc-Bessel mapping for H with gc-pre-frame operator T_f and upper bound B_f. Let $u : H \to K$ be a bounded linear mapping. Then*
(i) $uf : X \to \mathcal{M}_K$, $x \mapsto uf_x$ is a gc-Bessel mapping for K with gc-pre-frame operator $T_{uf} = uT_f$ and gc-frame operator $S_{uf} = uS_f u^*$.
(ii) If f is a gc-frame for H then uf is a gc-frame for K if and only if u is surjective.

Proof: (i) It is clear that $uf : X \to \mathcal{M}_K$ is weakly measurable. Since

$$\int_X |\int_{Y_x} \langle uf_x, k\rangle d\mu_x|^2 d\mu \le ||u||^2 B_f ||k||^2, \quad k \in K,$$

uf is a gc-Bessel mapping. Let $g \in L^2(X)$. For each $k \in H$

$$\begin{aligned}\langle T_{uf}(g), k\rangle &= \int_X \int_{Y_x} g(x)\langle uf_x, k\rangle d\mu_x d\mu \\ &= \int_X \int_{Y_x} g(x)\langle f_x, u^*(k)\rangle d\mu_x d\mu \\ &= \langle uT_f(g), k\rangle.\end{aligned}$$

So $T_{uf} = uT_f$. Also, we have

$$S_{uf} = T_{uf} T_{uf}^* = T_{uf}(T_f^* u^*) = uTT_f^* u^* = uS_f u^*.$$

(ii) It is clear by the Lemma 2.4.

□

Let f be a gc-frame for H with gc-frame operator S_f. Then $S_f^{-1}f$ is a gc-frame for H with gc-frame operator

$$S_{s^{-1}f} = S_f^{-1}S_fS_f^{-1} = S_f^{-1}.$$

So for any $h \in H$ we have the following retrieval formulas

$$h = S_fS_f^{-1}(h) = \int_X \int_{Y_x} \langle h, S_f^{-1}f_x\rangle d\mu_x \int_{Y_x} f_x d\mu_x d\mu,$$

and

$$\begin{aligned} h = S_f^{-1}S_f(h) &= S_{S^{-1}f}(S_f(h)) \\ &= \int_X \int_{Y_x} \langle S_f(h), S_f^{-1}f_x\rangle d\mu_x \int_{Y_x} S_f^{-1}f_x d\mu_x d\mu \\ &= \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Y_x} S_f^{-1}f_x d\mu_x d\mu. \end{aligned}$$

In the next Theorem, $T_f^\dagger$ will be the pseudo-inverse of T_f .

Theorem 2.6 *Let $f : X \to \mathcal{M}_H$ be a gc-frame for H. Then we have:*
(i) Let $g \in L^2(X)$ and $h = T_f(g)$. Then

$$||g||_2^2 = \int_X |\int_{Y_x} \langle h, S_f^{-1}f_x\rangle d\mu_x|^2 d\mu + \int_X |g(x) - \int_{Y_x} \langle h, S_f^{-1}f_x\rangle d\mu_x|^2 d\mu.$$

(ii) For each $h \in H$, $T_f^\dagger(h) = \int_{Y_.} \langle h, S_f^{-1}f_.\rangle d\mu_.$.
(iii) $||T_f^\dagger||^{-2} = ||S_f^{-1}||$.

Proof: (i) Since

$$\begin{aligned} T_f(g - \int_{Y_.} \langle h, S_f^{-1}f_.\rangle d\mu_.) &= h - T_f(T^*_{S_f^{-1}f})(h) \\ &= h - T_f(T_f^*S_f^{-1})(h) = 0, \end{aligned}$$

$$g - \int_{Y_.} \langle h, S_f^{-1}f_.\rangle d\mu_. \in \ker(T_f) = (\mathcal{R}(T_f^*))^\perp.$$

Since $\int_{Y_.} \langle h, S_f^{-1}f_.\rangle d\mu_. \in \mathcal{R}(T_f^*)$,

$$||g||_2^2 = ||g - \int_{Y_.} \langle h, S_f^{-1}f_.\rangle d\mu_.||_2^2 + ||\int_{Y_.} \langle h, S_f^{-1}f_.\rangle d\mu_.||_2^2.$$

(ii) Since by the Lemma 1.6, $T_f^\dagger(h)$ is the unique solution of minimal norm of $T_f(g) = h$, so

$$\int_X |g(x) - \int_{Y_x} \langle h, S_f^{-1} f_x\rangle d\mu_x|^2 d\mu = 0.$$

Hence, $g = \int_{Y_.} \langle h, S_f^{-1} f_.\rangle d\mu_. = T_f^\dagger(h)$.
(iii) Since f is a gc-frame for H, by the Lemma 2.5, $S_f^{-1}f$ is also a gc-frame. Therefore

$$\begin{aligned} ||T_f^\dagger||^2 &= \sup_{h\in H_1} \int_X |\int_{Y_x} \langle h, S_f^{-1} f_x\rangle d\mu_x|^2 d\mu \\ &= \sup_{h\in H_1} ||T^*_{S_f^{-1}f}(h)||^2 \\ &= ||T^*_{S_f^{-1}f}||^2 = ||S_f^{-1}||. \end{aligned}$$

□

Christensen [8] proved that every frame in a complex Hilbert space is a multiple of a sum of three orthonormal bases. Now, we shall show that a derived vector-valued integral of a gc-Bessel mapping is a multiple of a sum of three orthonormal bases.

Theorem 2.7 *Let $f : X \to \mathcal{M}_H$ be a gc-Bessel mapping with gc-pre-frame operator T_f and $e = \{e_\alpha\}_{\alpha\in X}$ be an orthonormal basis for H. Let $\{\delta_\alpha\}_{\delta\in X}$ be the canonical orthonormal basis for $l^2(X)$. Let $u : H \to l^2(X)$ be the isomorphism which maps e_α to δ_α. Then:*
(i) If $0 < \epsilon < 1$ then there exist oryhonormal bases $e^i = \{e^i_\alpha\}_{\alpha\in X}, i = 1, 2, 3$ for H such that

$$\int_{Y_.} f_. d\mu_. = \frac{||T_f||}{1-\epsilon}(e^1 + e^2 + e^3).$$

(ii) Let $0 < \epsilon < 1$ and $T_{uf} : l^2(X) \to l^2(X)$ be positive. Then there exist orthonormal bases $e^i = \{e^i_\alpha\}_{\alpha\in X}, i = 1, 2$ for H such that

$$\int_{Y_.} f_. d\mu_. = \frac{||T_f||}{2\epsilon}(e^1 + e^2).$$

Proof: (i) If $||T_f|| = 0$ then $T_f^* = 0$. Therefore, for each $h \in H$, $\int_{Y_.} \langle f_., h\rangle d\mu_. = 0$, so (i) is satisfied. Now, let $||T_f|| > 0$. Let $w : H \to H$ be defined by

$$w = \frac{1}{2}I + \frac{1-\epsilon}{2}\frac{T_f u}{||T_f||}.$$

Since $||I - w|| < 1$, w is invertible. So by using the polar decomposition we can write $w = vp$, where v is an unitary and p is a positive operator. But,

$||p|| < 1$, so we can write $p = \frac{1}{2}(z + z^*)$, where z, z^* are unitary operators. Thus

$$T_f u = \frac{||T_f||}{1-\epsilon}(vz + vz^* - I).$$

For each $\alpha \in X$, we have

$$\begin{aligned} T_f u(e_\alpha) &= T_f(\delta_\alpha) \\ &= \int_X \int_{Y_x} \delta_\alpha(x) f_x d\mu_x d\mu \\ &= \int_{Y_\alpha} f_\alpha d\mu_\alpha. \end{aligned}$$

Therefore

$$\int_{Y_.} f_. d\mu_. = T_f u e = \frac{||T_f||}{1-\epsilon}(vze + vz^*e - e).$$

Since, vz and vz^* are unitary operators, vze and vz^*e are orthonormal bases for H. Thus

$$\int_{Y_.} f_. d\mu_. = \frac{||T_f||}{1-\epsilon}(e^1 + e^2 + e^3),$$

where, $e^i = \{e^i_\alpha\}_{\alpha \in X}, i = 1, 2, 3$ are orthonormal bases for H.
(ii) Since, $T_{uf} : l^2(X) \to l^2(X)$ is positive and u is a unitary

$$\begin{aligned} uT_f &= \frac{||T_{uf}||}{2\epsilon}(w + w^*) \\ &= \frac{||T_f||}{2\epsilon}(w + w^*), \end{aligned}$$

where w is a unitary operator. We have

$$\int_{Y_.} f_. d\mu_. = \frac{||T_f||}{2\epsilon}(u^{-1}w + u^{-1}w^*).$$

Thus

$$\int_{Y_.} f_. d\mu_. = \frac{||T_f||}{2\epsilon}(e^1 + e^2),$$

where, $e^i = \{e^i_\alpha\}_{\alpha \in X}, i = 1, 2$ are orthonormal bases for H. □

The following theorem shows that the role of two gc-Bessel mapping can be interchanged .

Theorem 2.8 *Let $f, g : X \to \mathcal{M}_H$ be a gc-Bessel mapping for H with gc-Bessel mapping bounds B_f and B_g such that for each $x \in X$, $f_x : (Y_x, \mu_x) \to H$ and $g_x : (Z_x, \lambda_x) \to H$. Then the following assertions are equivalent:*
(i) For each $h \in H$, $h = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} g_x d\lambda_x d\mu$.
(ii) For each $h \in H$, $h = \int_X \int_{Z_x} \langle h, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu$.
(iii) For each $h, k \in H$, $\langle h, k\rangle = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} \langle g_x, k\rangle d\lambda_x d\mu$.
(iv) For each $h \in H$, $||h||^2 = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} \langle g_x, h\rangle d\lambda_x d\mu$.

Proof: $(i) \Rightarrow (ii)$ Let $f, k \in H$. We have

$$\begin{aligned} \overline{\langle h,k\rangle} &= \int_X \int_{Y_x} \langle f_x, h\rangle d\mu_x \int_{Z_x} \langle k, g_x\rangle d\lambda_x d\mu \\ &= \langle \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, h\rangle. \end{aligned}$$

Hence, $k = \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu$.
$(ii) \Rightarrow (iii)$ It is clear.
$(iv) \Rightarrow (i)$ Let

$$F(h) = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} g_x d\mu_x d\lambda.$$

It is clear that $F : H \to H$ is linear. Since

$$\begin{aligned} ||F(h)|| &= \sup_{k\in H_1} |\langle F(h), k\rangle| \\ &= \sup_{k\in H_1} |\int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} \langle g_x, k\rangle d\lambda_x d\mu| \\ &\le \sup_{k\in H_1} (\int_X |\int_{Y_x} \langle k, f_x\rangle d\mu_x|^2 d\mu)^{1/2} (\int_X |\int_{Z_x} \langle h, g_x\rangle d\lambda_x|^2 d\mu)^{1/2} \\ &\le B_f^{1/2} B_g^{1/2} ||h||, \end{aligned}$$

$F \in B(H)$. For each $h \in H$ we have

$$\begin{aligned} \langle h,h\rangle &= ||h||^2 \\ &= \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} \langle g_x, h\rangle d\lambda_x d\mu \\ &= \langle \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} g_x d\lambda_x d\mu, h\rangle. \end{aligned}$$

Hence, for each $h \in H$, $h = \int_X \int_{Y_x} \langle h, f_x\rangle d\mu_x \int_{Z_x} g_x d\lambda_x d\mu$.
$(iii) \to (iv)$ It is clear. □

Definition 2.9 *Let $f, g : X \to \mathcal{M}_H$ be a gc-Bessel mapping for H. We say that f, g is a dual pair if one of the assertions of the Theorem 2.8 holds.*

Now, we show that for a dual pair the lower gc-frame condition automatically is satisfied .

Theorem 2.10 *Let $f, g : X \to \mathcal{M}_H$ be a dual pair. Then f and g are gc-frames for H.*

Proof: For each $h \in H$ we have

$$\begin{aligned} ||h||^2 &\le (\int_X |\int_{Y_x} \langle h, f_x\rangle d\mu_x|^2 d\mu)^{1/2} (\int_X |\int_{Z_x} \langle g_x, h\rangle d\lambda_x|^2 d\mu)^{1/2} \\ &\le B_g (\int_X |\int_{Y_x} \langle h, f_x\rangle d\mu_x|^2 d\mu)^{1/2} ||h||. \end{aligned}$$

Thus

$$B_g^{-2}||h||^2 \leq \int_X |\int_{Y_x} \langle h, f_x\rangle d\mu_x|^2 d\mu.$$

So f is a gc-frame for H, and similarly, g is a gc-frame for H. $\square$

The following theorem will indicate all of the dual pairs of a gc-frame.

Theorem 2.11 *Let $f, g : X \to \mathcal{M}_H$ be a gc-frame such that for each $x \in X$, $f_x, g_x : (Y_x, \mu_x) \to H$. Let $h \in H$. Then :*
(i) In the retrieval formula

$$h = \int_X \int_{Y_x} \langle S_f^{-1}(h), f_x\rangle d\mu_x \int_{Y_x} f_x d\mu_x d\mu,$$

$\int_{Y_\cdot} \langle S_f^{-1}(h), f_\cdot\rangle d\mu_\cdot$ has least norm among all retrieval formulas.
(ii) For each $h \in H$, $h = \int_X \int_{Y_x} \langle h, g_x\rangle \int_{Y_x} f_x d\mu_x$ if and only if there exists a gc-Bessel mapping $l : X \to \mathcal{M}_H$ such that $g = S_f^{-1}f + l$ where for each $k \in H$, $\int_{Y_\cdot} \langle k, l_\cdot\rangle d\mu_\cdot \in \ker(T_f)$.

Proof: (i) Let $q \in L^2(X)$ and $h = \int_X q(x) \int_{Y_x} f_x d\mu_x d\mu$. Then for each $k \in H$

$$\begin{aligned}\langle h, k\rangle &= \int_X \int_{Y_x} \langle S_f^{-1}(h), f_x\rangle d\mu_x \int_{Y_x} \langle f_x, k\rangle d\mu_x d\mu \\ &= \int_X q(x) \int_{Y_x} \langle f_x, k\rangle d\mu_x d\mu.\end{aligned}$$

Therefore

$$\langle \int_X \int_{Y_x} (\langle S_f^{-1}(h), f_x\rangle - q(x)) d\mu_x \int_{Y_x} f_x d\mu_x d\mu, k\rangle$$

$$= \int_X \int_{Y_x} (\langle S_f^{-1}(h), f_x\rangle - q(x)) d\mu_x \int_{Y_x} \langle f_x, k\rangle d\mu_x d\mu = 0.$$

So $T_f(\int_{Y_\cdot} \langle S_f^{-1}(h), f_\cdot\rangle d\mu_\cdot - q) = 0$. Hence, $\int_{Y_\cdot} \langle S_f^{-1}(h), f_\cdot\rangle d\mu_\cdot - q \in \ker(T_f)$. Since $L^2(X) = \ker(T_f) \oplus \mathcal{R}T_f^*$,

$$||q||_2^2 = ||\int_{Y_\cdot} \langle S_f^{-1}(h), f_\cdot\rangle d\mu_\cdot - q||^2 + ||\int_{Y_\cdot} \langle S_f^{-1}(h), f_\cdot\rangle d\mu_\cdot||^2,$$

and (i) is proved.
(ii) Let for each $h \in H$, $h = \int_X \int_{Y_x} \langle h, g_x\rangle d\mu_x \int_{Y_x} f_x d\mu_x d\mu$. Let $g - S_f^{-1}f = l$. By the Theorem (2.8), for each $h, k \in H$

$$\begin{aligned}\langle \int_{Y_\cdot} \langle k, l_\cdot\rangle d\mu_\cdot, \int_{Y_\cdot} \langle h, f_\cdot\rangle d\mu_\cdot\rangle &= \langle \int_{Y_\cdot} \langle k, g_\cdot\rangle d\mu_\cdot, \int_{Y_\cdot} \langle h, f_\cdot\rangle d\mu_\cdot\rangle \\ &- \langle \int_{Y_\cdot} \langle k, S_f^{-1} f_\cdot\rangle d\mu_\cdot, \int_{Y_\cdot} \langle h, f_\cdot\rangle d\mu_\cdot\rangle \\ &= \langle k, h\rangle - \langle k, h\rangle = 0.\end{aligned}$$

Hence, for each $k \in H$, $\int_{Y_.} \langle k, l_. \rangle d\mu_. \in (\mathcal{R}T_f^*)^{\perp} = kerT_f$. Now, let $g = S_f^{-1} f + l$ where for each $k \in H$, $\int_{Y_.} \langle k, l_. \rangle d\mu_. \in ker(T_f)$. We have

$$\begin{aligned}
& \int_X \int_{Y_x} \langle h, f_x \rangle d\mu_x \int_{Y_x} \langle g_x, k \rangle d\mu_x d\mu \\
= & \int_X \int_{Y_x} \langle h, f_x \rangle d\mu_x \int_{Y_x} \langle S_f^{-1} f_x + l_x, k \rangle d\mu_x d\mu \\
= & \langle \int_{Y_.} \langle h, f_. \rangle d\mu_., \int_{Y_.} \langle k, S_f^{-1} f_. + l_. \rangle d\mu_. \rangle \\
= & \langle \int_{Y_.} \langle h, f_. \rangle d\mu_., \int_{Y_.} \langle k, S_f^{-1} f_. \rangle d\mu_. \rangle \\
+ & \langle \int_{Y_.} \langle h, f_. \rangle d\mu_., \int_{Y_.} \langle k, l_. \rangle d\mu_. \rangle \\
= & \langle \int_{Y_.} \langle h, f_. \rangle d\mu_., \int_{Y_.} \langle k, S_f^{-1} f_. \rangle d\mu_. \rangle \\
= & \int_X \int_{Y_x} \langle h, f_x \rangle d\mu_x \int_{Y_x} \langle S^{-1} f_x, k \rangle d\mu_x d\mu \\
= & \langle h, k \rangle.
\end{aligned}$$

Thus by the Theorem 2.8, for each $h \in H$,

$$h = \int_X \int_{Y_x} \langle h, g_x \rangle d\mu_x \int_{Y_x} f_x d\mu_x d\mu.$$

$\square$

3 GN-operators

Let $\phi \in B(K, H)$. The operator ϕ is called an n-operator of K to H, if there exist families $\{x_\alpha\}_{\alpha \in I} \subseteq H, \{y_\alpha\}_{\alpha \in I} \subseteq K$ such that $\sum_{\alpha \in I} ||x_\alpha|| ||y_\alpha|| < \infty$ and for each $k \in K$,

$$\phi(k) = \sum_{\alpha \in I} \langle k, y_\alpha \rangle x_\alpha.$$

Each n-operator on H is a trace class operator and vise versa, which in that case

$$tr(|\phi|)$$

$$= \inf\{\sum_{\alpha \in I} ||x_\alpha||.||y_\alpha|| : \quad \forall h \in H, \quad \phi(h) = \sum_{\alpha \in I} \langle h, y_\alpha \rangle x_\alpha, \quad \sum_{\alpha} ||x_\alpha|| ||y_\alpha|| < \infty\}.$$

In this section, we shall generalized the concept of n-operators according to the gc-frames.

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Definition 3.1 *We shall denote by $L^2(X, \mathcal{M}_H)$ the class of all $f : X \to \mathcal{M}_H$ such that :*
(i) For each $x \in X$, $||f_x|| : Y_x \to \mathbb{C}$ is measurable.
(ii) The mapping $\int_{Y_.} ||f_.||d\mu_. : X \to \mathbb{C}$, $x \mapsto \int_{Y_x} ||f_x||d\mu_x$ is measurable and

$$\int_X (\int_{Y_x} ||f_x||d\mu_x)^2 d\mu < \infty.$$

It is clear that each $f \in L^2(X, \mathcal{M}_H)$ is a gc-Bessel mapping for H.

Definition 3.2 *Let $\phi \in B(K, H)$. We say that ϕ is a gn-operator of K into H if there exist $f \in L^2(X, \mathcal{M}_H)$ and $g \in L^2(X, \mathcal{M}_K)$ with $f_x : (Y_x, \mu_x) \to H$ and $g_x : (Z_x, \lambda_x) \to K$ such that*

$$\phi(k) = \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad k \in K.$$

We say that ϕ is σ-finite gn-operator if all of the measure spaces are σ-finite . Also, we say that ϕ is a gn-operator on H if $H = K$.
For each gn-operator ϕ we define its gn-norm by $||\phi||_{gn} = \inf M$, where M is the class of all $\int_X \int_{Y_x} ||f_x||d\mu_x \int_{Z_x} ||g_x||d\lambda_x d\mu$ such that $f \in L^2(X, \mathcal{M}_H)$ and $g \in L^2(X, \mathcal{M}_K)$ and

$$\phi(k) = \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad k \in K.$$

The following Lemma indicates a relation between operator norms and gn-norms.

Lemma 3.3 *Let $\phi \in B(K, H)$ be a gn-operator. Then $||\phi|| \leq ||\phi||_{gn}$.*

Proof: Let $f \in L^2(X, \mathcal{M}_H)$, $g \in L^2(X, \mathcal{M}_K)$ and

$$\phi(k) = \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad k \in K.$$

For each $k \in K$, we have

$$\begin{aligned} ||\phi(k)|| &= ||\int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu|| \\ &= \sup_{h \in H_1} |\int_X \int_{Z_x} \langle k, g_x\rangle \lambda_x \int_{Y_x} \langle f_x, h\rangle d\mu_x d\mu| \\ &\leq ||k|| \int_X \int_{Z_x} ||g_x||d\lambda_x \int_{Y_x} ||f_x||d\mu_x d\mu. \end{aligned}$$

So $||\phi|| \leq ||\phi||_{gn}$. □

The next Lemma shows that compositions of gn-operators and operators on Hilbert spaces are gn-operators, and it will indicates the representations of the compositions according to the representations of the gn-operators.

Lemma 3.4 *If $\phi \in B(K,H)$ is a gn-operator, $v \in B(K)$ and $u \in B(H)$ then $u\phi v$ is a gn-operator.*

Proof: Let

$$\phi(k) = \int_X \int_{Z_x} \langle k, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad k \in K.$$

By the Theorem (2.5), for each $k \in K$, we have

$$\begin{aligned} u\phi v(k) &= u\int_X \int_{Z_x} \langle v(k), g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu \\ &= u\int_X \int_{Z_x} \langle k, v^* g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu \\ &= \int_X \int_{Z_x} \langle k, v^* g_x\rangle d\lambda_x \int_{Y_x} u f_x d\mu_x d\mu. \end{aligned}$$

Thus $u\phi v \in B(K,H)$ is a gn-operator. □

Now, we shall show relationships among trace class operators, n-operators and gn-operators.

Theorem 3.5 *The following assertions are satisfied:*
(i) If $\phi \in B(K,H)$ is an n-operator then ϕ is a σ-finite gn-operator.
(ii) Let ϕ be a σ-finite gn-operator on H. If

$$\phi(h) = \int_X \int_{Z_x} \langle h, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad h \in H$$

then

$$tr(\phi) = \int_X \int_{Z_x \times Y_x} \langle f_x, g_x\rangle d(\lambda_x \times \mu_x) d\mu.$$

(iii) If ϕ is a gn-operator on H then ϕ is a trace class operator and $||\phi||_{gn} = tr(|\phi|)$.

Proof: (i) Let I be a countable index set and

$$l^2(I) = \{\{x_\alpha\}_{\alpha\in I} | x_\alpha \in \mathbb{C}, \sum_{\alpha\in I} |x_\alpha|^2 < \infty\}$$

Since ϕ is an n-operator, there exist $\{x_\alpha\}_{\alpha\in I} \subseteq H, \{y_\alpha\}_{\alpha\in I} \subseteq K$ such that $\sum_\alpha ||x_\alpha|| ||y_\alpha|| < \infty$ and

$$\phi(k) = \sum_\alpha \langle k, y_\alpha\rangle x_\alpha, \quad k \in K.$$

Without less of generality we can suppose that for each $\alpha \in I$, $x_\alpha \neq 0$ and $y_\alpha \neq 0$. Since $\{||x_\alpha|| ||y_\alpha||\}_{\alpha\in I} \in l^2(I)$, there exists $\{a_\alpha\}_{\alpha\in I}, \{b_\alpha\}_{\alpha\in I} \in l^2(I)$ such that $||x_\alpha|| ||y_\alpha|| = a_\alpha \overline{b_\alpha}$. Thus

$$\phi(k) = \sum_\alpha \langle k, y_\alpha\rangle x_\alpha = \sum_\alpha \langle k, \frac{a_\alpha y_\alpha}{||y_\alpha||}\rangle \frac{b_\alpha x_\alpha}{||x_\alpha||}.$$

Let $g = \{a_\alpha y_\alpha / ||y_\alpha||\}_{\alpha \in I}$ and $f = \{b_\alpha x_\alpha / ||x_\alpha||\}_{\alpha \in I}$. Let $f : I \to \mathcal{M}_H, \alpha \mapsto f_\alpha$ and $g : I \to \mathcal{M}_K, \alpha \mapsto g_\alpha$ be defined by

$$f_\alpha : Y \to H, \quad y \mapsto \frac{b_\alpha x_\alpha}{||x_\alpha||} \quad \text{and} \quad g_\alpha : Y \to K, \quad y \mapsto \frac{a_\alpha y_\alpha}{||y_\alpha||},$$

where (Y, λ) is any measure space with $\lambda(Y) = 1$. Let $X = I$ and μ be the counting measure. Then it is evident that f and g are weakly measurable and for each $h \in H$ and $k \in K$ we have

$$\begin{aligned} \int_X \int_Y \langle k, g_\alpha \rangle d\lambda \int_Y \langle f_\alpha, h \rangle d\lambda d\mu &= \sum_\alpha \langle k, \frac{a_\alpha y_\alpha}{||y_\alpha||} \rangle \langle \frac{b_\alpha x_\alpha}{||x_\alpha||}, h \rangle \\ &= \langle \sum_\alpha \langle k, \frac{a_\alpha y_\alpha}{||y_\alpha||} \rangle \frac{b_\alpha x_\alpha}{||x_\alpha||}, h \rangle \\ &= \langle \phi(k), h \rangle. \end{aligned}$$

So

$$\phi(k) = \int_X \int_Y \langle k, g_\alpha \rangle d\lambda \int_Y f_\alpha d\lambda d\mu.$$

Hence, ϕ is a σ-finite gn-operator.
(ii) Let

$$\phi(h) = \int_X \int_{Z_x} \langle h, g_x \rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad h \in H$$

and let $\{e_\alpha\}$ be an orthonormal basis for H. We have

$$\begin{aligned} tr(\phi) &= \sum_\alpha \langle \int_X \int_{Z_x} \langle e_\alpha, g_x \rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, e_\alpha \rangle \\ &= \sum_\alpha \int_X \int_{Z_x} \langle e_\alpha, g_x \rangle d\lambda_x \int_{Y_x} \langle f_x, e_\alpha \rangle d\mu_x d\mu. \end{aligned}$$

Since

$$\begin{aligned} & \int_X | \int_{Z_x} \langle e_\alpha, g_x \rangle d\lambda_x \int_{Y_x} \langle f_x, e_\alpha \rangle d\mu_x | d\mu \\ \leq & \int_X \int_{Z_x} |\langle e_\alpha, g_x \rangle| d\lambda_x \int_{Y_x} |\langle f_x, e_\alpha \rangle| d\mu_x d\mu \\ \leq & (\int_X (\int_{Z_x} ||g_x|| d\lambda_x)^2 d\mu)^{1/2} (\int_X (\int_{Y_x} ||f_x|| d\mu_x)^2 d\mu)^{1/2} < \infty, \end{aligned}$$

and

$$\int_{Z_x} |\langle e_\alpha, g_x \rangle| d\lambda_x \int_{Y_x} |\langle f_x, e_\alpha \rangle| d\mu_x < \infty, \quad a.e[\mu].$$

Thus

$$\int_{Z_x} |\langle e_\alpha, g_x \rangle| d\lambda_x \int_{Y_x} |\langle f_x, e_\alpha \rangle| d\mu_x = \int_{Z_x \times Y_x} |\langle e_\alpha, g_x \rangle| |\langle f_x, e_\alpha | d(\lambda_x \times \mu_x).$$

Hence

$$\begin{aligned}
&\int_X \sum_\alpha \Big|\int_{Z_x} \langle e_\alpha, g_x\rangle d\lambda_x \int_{Y_x} \langle f_x, e_\alpha\rangle d\mu_x\Big| d\mu \\
\leq\ & \int_X \sum_\alpha \int_{Z_x\times Y_x} |\langle e_\alpha, g_x\rangle||\langle f_x, e_\alpha\rangle| d(\lambda_x \times \mu_x) d\mu \\
\leq\ & \int_X \int_{Z_x\times Y_x} \sum_\alpha |\langle e_\alpha, g_x\rangle||\langle f_x, e_\alpha\rangle| d(\lambda_x \times \mu_x) d\mu \\
\leq\ & \int_X \int_{Z_x\times Y_x} (\sum_\alpha |\langle e_\alpha, g_x\rangle|^2)^{1/2} (\sum_\alpha |\langle f_x, e_\alpha\rangle|^2)^{1/2} d(\lambda_x \times \mu_x) d\mu \\
=\ & \int_X \int_{Z_x} ||g_x|| d\lambda_x \int_{Y_x} ||f_x|| d\mu_x d\mu \\
\leq\ & (\int_X (\int_{Z_x} ||g_x|| d\lambda_x)^2 d\lambda)^{1/2} (\int_X (\int_{Y_x} ||f_x|| d\mu_x)^2 d\mu)^{1/2} < \infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
tr(\phi) &= \int_X \sum_\alpha \int_{Z_x} \langle h, g_x\rangle d\lambda_x \int_{Y_x} \langle f_x, e_\alpha\rangle d\mu_x d\mu \\
&= \int_X \sum_\alpha \int_{Z_x\times Y_x} \langle e_\alpha, g_x\rangle \langle f_x, e_\alpha\rangle d(\lambda_x \times \mu_x) \\
&= \int_X \int_{Z_x\times Y_x} \sum_\alpha \langle e_\alpha, g_x\rangle \langle f_x, e_\alpha\rangle d(\lambda_x \times \mu_x) d\mu \\
&= \int_X \int_{Z_x\times Y_x} \langle f_x, g_x\rangle d(\lambda_x \times \mu_x) d\mu.
\end{aligned}$$

(iii) Let

$$\phi(h) = \int_X \int_{Z_x} \langle h, g_x\rangle d\lambda_x \int_{Y_x} f_x d\mu_x d\mu, \quad h \in H.$$

Let $\phi = u|\phi|$ be the polar decomposition of ϕ. So $|\phi| = u^*\phi$. By the Lemma 3.4, $|\phi|$ is a gn-operator and

$$|\phi|(h) = \int_X \int_{Z_x} \langle h, g_x\rangle d\lambda_x \int_{Y_x} u^* f_x d\mu_x d\mu, \quad h \in H.$$

Let $\{e_j\}_{\in J}$ be an orthonormal basis for H. We have

$$\begin{aligned}
tr(|\phi|) &= \sum_j \langle |\phi|(e_j), e_j\rangle \leq \sum_j \int_X \int_{Z_x} |\langle e_j, g_x\rangle| d\lambda_x \int_{Y_x} |\langle u^* f_x, e_j\rangle| d\mu_x d\mu \\
&\leq \int_X \int_{Z_x} \int_{Y_x} \sum_j |\langle e_j, g_x\rangle||\langle u^* f_x, e_j\rangle| d\lambda_x d\mu_x d\mu
\end{aligned}$$

$$\begin{aligned}
&\leq \int_X \int_{Z_x} (\sum_j |\langle e_j, g_x\rangle|^2)^{1/2} d\lambda_x) \int_{Y_x} (\sum_j |\langle u^* f_x, e_j\rangle|^2)^{1/2} d\mu_x d\mu \\
&= \int_X \int_{Z_x} ||g_x|| d\lambda_x \int_{Y_x} ||f_x|| d\mu_x d\mu.
\end{aligned}$$

Hence

$$tr(|\phi|) \leq ||\phi||_{gn}.$$

Since $||\phi||_{gn} < \infty$, ϕ is a trace class operator. Since

$$tr(|\phi|)$$

$$= \inf\{\sum_{\alpha\in I} ||x_\alpha||.||y_\alpha|| : \quad \forall h \in H, \quad \phi(h) = \sum_{\alpha\in I} \langle h, y_\alpha\rangle x_\alpha, \quad \sum_\alpha ||x_\alpha||||y_\alpha|| < \infty\},$$

thus $||\phi||_{gn} \leq tr(|\phi|)$, so

$$||\phi||_{gn} = tr(|\phi|),$$

and the Theorem is proved. □

The following result can be dedicated by the Theorem 3.5.

Corollary 3.6 *Let $\phi \in B(H)$. Then ϕ is a trace class operator if and only if ϕ is an gn-operator on H.*

Conclusion. Finally, we proved that $\phi \in B(H)$ is a trace class operator if and only if an gn-operator and this can be investigated for other operators as well.

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