Characterization of a Nonlinear Equality Composed of Multiple Products of Matrices and their Generalized Inverses

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Abstract. One of matrix equalities composed of multiple products of matrices and their generalized inverses is given by $A_1B_1^-A_2B_2^-\cdots A_kB_k^-A_{k+1} = A$ where A_1, B_1, A_2, B_2, …, A_k, B_k, A_{k+1}, and A are given matrices of appropriate sizes, and B_1^-, B_2^-, $\cdots$, B_k^- are generalized inverses of matrices. The cases for $k = 1, 2$ and their special forms were properly approached in the theory of generalized inverses of matrices. In this note, the author presents an algebraic procedure to derive explicit necessary and sufficient conditions for the equality with $k = 3$ to always hold using certain rank equalities for the block matrices constructed by the given matrices, and then mention a key step of extending the previous work to a general situation.

Keywords: generalized inverse; matrix product; matrix equality; set inclusion; rank equality; block matrix

AMS classifications: 15A09; 15A24; 47A05; 47A50

1 Introduction

Throughout this note, let $\mathbb{C}^{m\times n}$ denote the collections of all $m\times n$ matrices with complex numbers; A^* denote the conjugate transpose; and $r(A)$ denote the rank of A; I_m denote the identity matrix of order m, $[\,A,\ B]$ denote a columnwise partitioned matrix consisting of two submatrices A and B. The Moore–Penrose generalized inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the four Penrose equations

$$(1)\ AXA = A,\quad (2)\ XAX = X,\quad (3)\ (AX)^* = AX,\quad (4)\ (XA)^* = XA, \tag{1}$$

see [10]. Starting with Penrose himself, a matrix X is called a $\{i,\dots,j\}$-generalized inverse of A, denoted by $A^{(i,\dots,j)}$, if it satisfies the ith,…,jth equations in (1). The collection of all $\{i,\dots,j\}$-generalized inverses of A is denoted by $\{A^{(i,\dots,j)}\}$. There are all 15 types of $\{i,\dots,j\}$-generalized inverses of A by definition, but matrix X is called an inner inverse of A if it satisfies $AXA = A$, and is denoted by $A^{(1)} = A^-$.

In this note, the author considers the following general matrix equality

$$A_1B_1^-A_2B_2^-\cdots A_kB_k^-A_{k+1} = A, \tag{2}$$

where A_1, B_1, A_2, B_2, …, A_k, B_k, A_{k+1}, and A are given matrices of appropriate sizes. Apparently, (2) includes many equalities composed of multiple products of matrices and their generalized inverses as its special cases. It should be pointed out that an equality as such does not necessarily hold for different choices of the given matrices. So that we are first interested in establishing necessary and sufficient conditions for the equality to hold for some B_1^-, B_2^-, …, B_k^- or always hold for all B_1^-, B_2^-, …, B_k^-. Recall that generalized inverses of a matrix are not necessarily unique (cf. Lemma 1 below). This fact means that the matrix equality in (2) is in fact certain nonlinear matrix equation involving multiple unknown matrices (cf. [5,6]). Thus it is a challenging task to solve the general matrix equation in (2) due to the noncommutativity of matrix algebra and singularity of the given matrices.

Some concrete situations of (2) were considered in the literature. The author states them with the situation of (2) for $k = 1$:

$$A_1B_1^-A_2 = A. \tag{3}$$

This equality is in fact a linear matrix equation involving the two unknown matrices in B_1^- by Lemma 1 below. So that it is easy to derive necessary and sufficient conditions for the equality to hold and to always hold, respectively; see, e.g., [4, 9, 11, 12] and reference therein.

The situation of (2) for $k = 2$ is given by

$$A_1B_1^-A_2B_2^-A_3 = A, \tag{4}$$

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where $A_1 \in \mathbb{C}^{m_1\times m_2}$, $B_1 \in \mathbb{C}^{m_3\times m_2}$, $A_2 \in \mathbb{C}^{m_3\times m_4}$, $B_2 \in \mathbb{C}^{m_5\times m_4}$, $A_3 \in \mathbb{C}^{m_5\times m_6}$, and $A \in \mathbb{C}^{m_1\times m_6}$. This equality is in fact a nonlinear matrix equation involving four the unknown matrices in B_1^- and B_2^- by Lemma 1 below. Jiang and Tian [7,14] recently considered (4), and established a group of necessary and sufficient conditions for the equality to hold for all B_1^- and B_2^- using the matrix rank methodology and presented lots of applications of this equality in the theory of generalized inverses.

The situation of (2) for $k = 3$ is given by

$$A_1B_1^-A_2B_2^-A_3B_3^-A_4 = A, \tag{5}$$

where $A_1 \in \mathbb{C}^{m_1\times m_2}$, $B_1 \in \mathbb{C}^{m_3\times m_2}$, $A_2 \in \mathbb{C}^{m_3\times m_4}$, $B_2 \in \mathbb{C}^{m_5\times m_4}$, $A_3 \in \mathbb{C}^{m_5\times m_6}$, $B_3 \in \mathbb{C}^{m_7\times m_6}$, $A_4 \in \mathbb{C}^{m_7\times m_8}$, and $A \in \mathbb{C}^{m_1\times m_8}$. This equality is a nonlinear matrix equation involving the six unknown matrices in B_1^-, B_2^-, and B_3^- by Lemma 1 below.

The purpose of this note is to derive a group of necessary and sufficient conditions for (5) to hold for all B_1^-, B_2^-, and B_3^- through the use of rank equalities for the block matrices constructed by the given matrices, and then to mention a key step for deriving conditions under which (2) always holds.

2 Some preliminaries

In this section, the author presents a series of well-known or established results and facts concerning generalized inverses and ranks of matrices, which we shall use to deal with matrix equality problems described above.

Lemma 1 ([10]). *Let $A \in \mathbb{C}^{m\times n}$. Then, the general expression of A^- of A can be written as*

$$A^- = A^\dagger + F_AU + VE_A, \tag{6}$$

where $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$, and $U, V \in \mathbb{C}^{n\times m}$ are arbitrary.

Lemma 2 ([9]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, and $C \in \mathbb{C}^{l\times n}$. Then*

$$r[A,\ B] \geq r(A), \tag{7}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \geq r(B) + r(C). \tag{8}$$

Lemma 3 ([11,13]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$ be given. Then,*

$$\max_{A^-\in\{A^-\}} r(D - CA^-B) = \min\left\{ r[C,\ D],\ \ r\begin{bmatrix} B \\ D \end{bmatrix},\ \ r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) \right\}. \tag{9}$$

Therefore,

$$CA^-B = D \text{ holds for all } A^- \Leftrightarrow [C,\ D] = 0 \ \text{ or } \ \begin{bmatrix} B \\ D \end{bmatrix} = 0 \ \text{ or } \ r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A). \tag{10}$$

Lemma 4 ([7]). *Let A, A_1, B_1, A_2, B_2, and A_3 be as given in (4). Then, following three statements are equivalent:*

(a) *The equality $A_1B_1^-A_2B_2^-A_3 = A$ holds for all B_1^- and B_2^-.*

(b) *The product $A_1B_1^-A_2B_2^-A_3$ is invariant with respect to the choice of B_1^- and B_2^-, and $A = A_1B_1^\dagger A_2B_2^\dagger A_3$.*

(c) *Any one of the following six conditions holds*

  (i) *$A = 0$ and $A_1 = 0$.*

  (ii) *$A = 0$ and $A_2 = 0$.*

  (iii) *$A = 0$ and $A_3 = 0$.*

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(iv) $A = 0$ *and* $r\begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$.

(v) $A = 0$ *and* $r\begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2)$.

(vi) $r\begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r\begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$.

3 Main results

Theorem 1. *Let* A_1, B_1, A_2, B_2, A_3, B_3, A_4, *and* A *be as given in* (5). *Then, the following three statements are equivalent*:

(a) *The equality* $A_1B_1^-A_2B_2^-A_3B_3^-A_4 = A$ *holds for all* B_1^-, B_2^-, *and* B_3^-.

(b) *The product* $A_1B_1^-A_2B_2^-A_3B_3^-A_4$ *is invariant with respect to the choice of* B_1^- *and* B_2^-, *and* B_3^-, *and* $A_1B_1^{\dagger}A_2B_2^{\dagger}A_3B_3^{\dagger}A_4 = A$.

(c) *Any one of the following ten conditions holds*

(i) $A = 0$ *and* $A_1 = 0$.

(ii) $A = 0$ *and* $A_2 = 0$.

(iii) $A = 0$ *and* $A_3 = 0$.

(iv) $A = 0$ *and* $A_4 = 0$.

(v) $A = 0$ *and* $r\begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$.

(vi) $A = 0$ *and* $r\begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2)$.

(vii) $A = 0$ *and* $r\begin{bmatrix} 0 & A_3 \\ A_4 & B_3 \end{bmatrix} = r(B_3)$.

(viii) $A = 0$ *and* $r\begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r\begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$.

(ix) $A = 0$ *and* $r\begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & B_3 & 0 \end{bmatrix} = r\begin{bmatrix} A_3 & B_2 \\ B_3 & 0 \end{bmatrix} = r(B_2) + r(B_3)$.

(x) $r\begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & B_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3)$.

Proof. The equivalence of (a) and (b) follows from the definition of the Moore–Penrose inverse of a matrix and its uniqueness.

Replacing A_3 with $A_3B_4^-A_5$ in Lemma 4(c), we first see that the product $A_1B_1^-A_2B_2^-A_3B_3^-A_4$ is invariant with respect to the choice of B_1^- and B_2^- if and only if any one of the following six conditions holds

(i) $A = 0$ and $A_1 = 0$.

(ii) $A = 0$ and $A_2 = 0$.

(iii) $A = 0$ and $A_3B_3^-A_4 = 0$.

(iv) $A = 0$ and $r\begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$.

(v) $A = 0$ and $r\begin{bmatrix} 0 & A_2 \\ A_3B_3^-A_4 & B_2 \end{bmatrix} = r(B_2)$.

(vi) $r\begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3B_3^-A_4 & B_2 & 0 \end{bmatrix} = r\begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$.

Further by (10), the equality $A_3B_3^-A_4 = 0$ in (iii) of the above six equalities holds for all B_3^- if and only if

$$A_3 = 0 \quad \text{or} \quad A_4 = 0 \quad \text{or} \quad r\begin{bmatrix} 0 & A_3 \\ A_4 & B_3 \end{bmatrix} = r(B_3). \tag{11}$$

By (9),

$$\begin{aligned}
\max_{B_3^-} r\begin{bmatrix} 0 & A_2 \\ A_3B_3^-A_4 & B_2 \end{bmatrix} &= \max_{B_3^-} r\left(\begin{bmatrix} 0 & A_2 \\ 0 & B_2 \end{bmatrix} + \begin{bmatrix} 0 \\ A_3 \end{bmatrix} B_3^-[A_4,\, 0]\right) \\
&= \min\left\{ r\begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix},\ r\begin{bmatrix} 0 & A_2 \\ 0 & B_2 \\ A_4 & 0 \end{bmatrix},\ r\begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & -B_3 & 0 \end{bmatrix} - r(B_3) \right\} \\
&= \min\left\{ r\begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix},\ r\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} + r(A_4),\ r\begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & B_3 & 0 \end{bmatrix} - r(B_3) \right\}.
\end{aligned}$$

So that the rank equality $r\begin{bmatrix} 0 & A_2 \\ A_3B_3^-A_4 & B_2 \end{bmatrix} = r(B_2)$ in (v) of the above six equalities holds for all B_3^- if and only if

$$r\begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2) \quad \text{or} \quad r\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} + r(A_4) = r(B_2) \quad \text{or} \quad r\begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & B_3 & 0 \end{bmatrix} = r(B_2) + r(B_3), \tag{12}$$

where the second rank equality means $A_4 = 0$ by noting that $r\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \geq r(B_2)$. By (9),

$$\begin{aligned}
\max_{B_3^-} r\begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3B_3^-A_4 & B_2 & 0 \end{bmatrix} &= \max_{B_3^-} r\left(\begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ A_3 \end{bmatrix} B_3^-[A_4,\, 0,\, 0]\right) \\
&= \min\left\{ r\begin{bmatrix} -A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \end{bmatrix},\ r\begin{bmatrix} -A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix},\ r\begin{bmatrix} -A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & -B_3 & 0 & 0 \end{bmatrix} - r(B_3) \right\} \\
&= \min\left\{ r\begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \end{bmatrix},\ r\begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix},\ r\begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & B_3 & 0 & 0 \end{bmatrix} - r(B_3) \right\}.
\end{aligned}$$

So that the rank equality in (vi) of the above six equalities holds for all B_3^- if and only if

$$r\begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2) \quad \text{or} \quad r\begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) \tag{13}$$

or

$$r\begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & B_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3). \tag{14}$$

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Note from (7) and (8) that

$$r\begin{bmatrix} A & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \end{bmatrix} \geq r(A) + r\begin{bmatrix} 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} \geq r\begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} \geq r(B_1) + r(B_2),$$

$$r\begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix} \geq r\begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \end{bmatrix} \geq r(A) + r\begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} \geq r(B_1) + r(B_2),$$

$$r\begin{bmatrix} A & 0 & A_1 \\ 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix} \geq r\begin{bmatrix} 0 & A_2 & B_1 \\ 0 & B_2 & 0 \\ A_4 & 0 & 0 \end{bmatrix} = r(A_4) + r\begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} \geq r(B_1) + r(B_2).$$

Hence, the first equality in (13) is equivalent to

$$A = 0 \ \text{ and } \ r\begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2); \tag{15}$$

the second equality in (13) implies

$$A = 0 \text{ and } A_4 = 0. \tag{16}$$

Combining the conditions in (i), (ii) and (iv) of the proof with (11)–(16) leads to the ten statements in (c). □

Below, the author presents some direct consequences of the above theorem for different choices of A_1, B_1, A_2, B_2, A_3, B_3,, A_4, and A in (5).

Corollary 1. *Let A_1, B_1, A_2, B_2, A_3, B_3, and A_4 be as given in (5). Then, the following three statements are equivalent:*

(a) *The equality $A_1B_1^-A_2B_2^-A_3B_3^-A_4 = 0$ holds for all B_1^-, B_2^-, and B_3^-.*

(b) *The product $A_1B_1^-A_2B_2^-A_3B_3^-A_4$ is invariant with respect to the choice of B_1^- and B_2^-, and B_3^-, and $A_1B_1^\dagger A_2B_2^\dagger A_3B_3^\dagger A_4 = 0$.*

(c) *Any one of the following ten conditions holds*

(i) $A_1 = 0$.

(ii) $A_2 = 0$.

(iii) $A_3 = 0$.

(iv) $A_4 = 0$.

(v) $r\begin{bmatrix} 0 & A_1 \\ A_2 & B_1 \end{bmatrix} = r(B_1)$.

(vi) $r\begin{bmatrix} 0 & A_2 \\ A_3 & B_2 \end{bmatrix} = r(B_2)$.

(vii) $r\begin{bmatrix} 0 & A_3 \\ A_4 & B_3 \end{bmatrix} = r(B_3)$.

(viii) $r\begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix} = r\begin{bmatrix} A_2 & B_1 \\ B_2 & 0 \end{bmatrix} = r(B_1) + r(B_2)$.

(ix) $r\begin{bmatrix} 0 & 0 & A_2 \\ 0 & A_3 & B_2 \\ A_4 & B_3 & 0 \end{bmatrix} = r\begin{bmatrix} A_3 & B_2 \\ B_3 & 0 \end{bmatrix} = r(B_2) + r(B_3)$.

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(x) $r\begin{bmatrix} 0 & 0 & 0 & A_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4 & B_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3).$

Corollary 2. *Let $B_1 \in \mathbb{C}^{m_2\times m_1}$, $B_2 \in \mathbb{C}^{m_3\times m_2}$, $B_3 \in \mathbb{C}^{m_4\times m_3}$, and $A \in \mathbb{C}^{m_2\times m_4}$. Then, the following three statements are equivalent:*

(a) *The equality $B_1^- B_2^- B_3^- = A$ holds for all B_1^-, B_2^-, and B_3^-.*

(b) *The product $B_1^- B_2^- B_3^-$ is invariant with respect to the choice of B_1^- and B_2^-, and B_3^-, and $B_1^- B_2^- B_3^- = A$.*

(c) $r\begin{bmatrix} A & 0 & 0 & I_{m_1} \\ 0 & 0 & I_{m_2} & B_1 \\ 0 & I_{m_3} & B_2 & 0 \\ I_{m_4} & B_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3).$

Corollary 3. *Let A_1, B_1, A_2, B_2, A_3, B_3, and A_4 be as given in* (5), *and $A \in \mathbb{C}^{m_8\times m_1}$. Then, the following three statements are equivalent:*

(a) *$\{A_1B_1^-A_2B_2^-A_3B_3^-A_4\} \subseteq \{A^-\}$, namely, $AA_1B_1^-A_2B_2^-A_3B_3^-A_4A = A$ holds for all B_1^-, B_2^-, and B_3^-.*

(b) $A = 0$ *or* $r\begin{bmatrix} A & 0 & 0 & AA_1 \\ 0 & 0 & A_2 & B_1 \\ 0 & A_3 & B_2 & 0 \\ A_4A & B_3 & 0 & 0 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3).$

(c) $A = 0$ *or* $r\begin{bmatrix} 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \\ B_3 & 0 & -A_4AA_1 \end{bmatrix} = r(B_1) + r(B_2) + r(B_3) - r(A).$

4 Conclusions

Constructions and characterizations of equalities composed of matrices and their generalized inverses are basic and classic topics in the current theory of generalized inverses, which include a diversity of issues for discrimination and consideration. As one of such kind of problems, the author proposed a family of equalities for products of matrices and their generalized inverses, and presented various necessary and sufficient conditions for a concrete matrix equality to hold through the skillful use of various equalities and inequalities for ranks of matrices.

As consequences and applications of the preceding results, the author would like to mention some further topics regarding the characterization of matrix equalities as follows:

(I) Corollary 3 can be used to derive necessary and sufficient conditions for some concrete reverse order laws to hold for generalized inverses of multiple matrix products, such as,

$$\{C^-B^-A^-\} \subseteq \{(ABC)^-\}, \ \ \{(CD)^-C(BC)^-B(AB)^-\} \subseteq \{(ABCD)^-\},$$

$$\{(CDE)^-CD(BCD)^-BC(ABC)^-\} \subseteq \{(ABCDE)^-\},$$

etc.

(II) Lemma 4 and Theorem 1 suggest a proper rule of deriving necessary and sufficient conditions for (4) and (5) to hold by certain rank equalities for block matrices composed by the given matrices. In view of this fact, it is easy to figure out that such a block matrix associated with (2) is given by

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 & (-1)^{k+1}A \\ B_1 & A_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & A_k & 0 \\ 0 & 0 & \cdots & B_k & A_{k+1} \end{bmatrix},$$

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while necessary and sufficient conditions for (2) to hold can be represented by certain rank equalities composed of the block matrix and its submatrices.

Moreover, the author believes that more profound and remarkable findings about the matrix equalities for matrices and their generalized inverses can further be constructed with some effort, and they will inform as many readers as possible to familiar with some novel and important issues associated with matrix equalities and to use them to dissect various complex problems in matrix analysis and applications.

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