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An Unhedgeable Black-Scholes-Merton Implicit Option? (*)

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Abstract: In this paper we focus on an implicit assumption in the BSM framework that limits the scope of market network connections to seeking gains in the currency basis, i.e. on trading strategies between the numeraire and the stock and between the numeraire and the option, separately. We relax this assumption and derive the equivalent of the standard BSM approach under a more general market network framework in order to assess its implications. In doing so, we find that it is not possible to hedge on implicit option that allows one to directly trade the option and stock. This represents a potential challenge to the BSM framework, since the missing market network connection provides a potentially useful mechanism for risk-bearing portfolio managers to alter their portfolios

Keywords: Black-Scholes-Merton; hedging; market network; option valuation; portfolio optimization; risk-bearing portfolio managers

1. Introduction

Many portfolio managers purchase risk-bearing allocations of assets in order to maximize returns while reducing risk, with proceeds reinvested in the portfolio. For example, one popular strategy involves selecting a calibrated portfolio allocation of specific assets according to an objective function and constraints, followed by periodic rebalancing, i.e., by selling those assets whose market value has grown past their target percentage of the portfolio's total value and purchasing those that have fallen behind. Maeso and Martellini (2020) provide empirical evidence on the effectiveness of this approach over strictly "buy-and-hold" strategies, irrespective of prior estimates and objective function specifications that might bias portfolio construction. Evaluation of the performance of such strategies emphasizes the returns basis on a risky portfolio, while the quantity basis of individual securities provides control variables for management.

This approach is rather different from the vast majority of research on financial derivatives. This is clearly illustrated by the seminal work of Black and Scholes (1973) and Merton (1973) (BSM, hereafter) and the large body of literature it inspired [for some early contributions see, for example, Cox, Ross, and Rubinstein (1979), Cox, Ingersoll, and Ross (1985), Madan, Milne, and Shefrin (1989), and Merton (1998)]. This literature establishes the central role of the BSM approach in addressing questions concerning the behavior of contingent claims, derivatives, options, and related constructs.

The BSM approach focuses on valuation and risk-management in the cash basis. Option contracts have well-defined values at specific times or states of the world, which are a function of the underlying variables specified in the contract, such as the price of a given stock. In order to understand the behavior of the values of option contracts, behavior of underlying stock prices is specified which, provides a mechanism for linking fluctuations in the value of an option to fluctuations in the prices of the underlying stock.

In the BSM framework, stochastic elements are eliminated through dynamic hedging, by considering a specific weighted portfolio of an option and an underlying stock. Since such portfolios are riskless under BSM assumptions, the values of their constituents are determined by what might be obtained by investing the portfolio in a risk-free asset. Should equivalence fail, a trader might extract from the difference an instantaneous risk-

free profit, an arbitrage, thereby restoring the BSM relation. In the absence of arbitrage, the values of option contracts and other contingent claims can be computed from this fixed relation, which takes the form of a deterministic partial differential equation.

Over the years, the BSM framework has received its share of criticisms. Many question fundamental model assumptions, such as liquidity [see, for example, Schonbucher and Wilmott (2000)], market completeness [see, for example, Dumas and Lyasoff (2012)], and transaction costs [see, for example, Dewynne, Whalley, and Wilmot (1994), Leland (1985), Kennedy, Forsyth, and Vetzal (2009)]. In addition, the assumption of mesokurtosis has created a significant debate among both practitioners and theoreticians [see, for example, Haug and Taleb (2014)].

Others have attempted to resolve some of the above criticisms and noted empirical incongruities, such as the implied volatility smile and surface, by expanding structural assumptions concerning the behavior of the underlying asset price. Heston, (1993), for example, employs a stochastic volatility framework in order to value bond and currency options. Heston and Nandi (1997) derive a closed-form solution should volatility display GARCH properties. Kennedy, Forsyth, and Vetzal (2009) attempt to incorporate jump diffusions in the underlying with transaction costs.

This paper differs from this literature in that it attempts to place the BSM approach in a more general market framework while retaining its remaining core assumptions. It focuses on an implicit assumption about market structure and on the effects of relaxing such assumption on the internal consistency of the BSM approach and the scope of its applicability. Our approach recognizes and speaks to the fact that most portfolio managers purchase risk-bearing allocations of assets in order to attempt to maximize returns while reducing risk, with proceeds reinvested in the portfolio.

This paper is organized as follows. Section 2 derives the basic BSM as a reference point. Section 3 introduces assumption implicit in the BSM framework and assesses its reasonableness and possible implications. Section 4 relaxes this assumption and derives the equivalent of the BSM under a more general and practically relevant market network. Finally, Section 5 discuss some potential implications and applications of the results.

2. Black-Scholes-Merton: The Classical Model

2.1. The Case of Vanilla Options

As defined by BSM, the value of a vanilla option contract V , is a function of two underlying state variables, the stock price S and the amount of time remaining to option expiry, t , i.e., $V \equiv V(S, t)$. Two factors drive stock prices: a continuous-time Brownian random element with a specified volatility and a deterministic growth factor. Accordingly, any randomness in the option price originates in the stock price. Under these conditions, a deterministic partial differential equation provides the solution for the option price. With the stochastic element removed, option prices depend exclusively on known constants and a single unknown volatility parameter.

In what follows, and without loss of generality, we define two elementary contracts by their standard payoff terms. For a given strike price, E , a European call option allows the holder to purchase the underlying stock at strike price E at expiry. If $S_T \geq E$, the underlying stock can be purchased and immediately resold at the market rate. Accordingly, the final valuation condition of a call option at expiry T is

$$(1) \quad V_T^{call}(S_T) = \max[S_T - E, 0].$$

Similarly, a European put option gives the holder the right to sell a stock for a fixed strike price E at expiry, yielding a final valuation of

$$(2) \quad V_T^{put}(S_T) = \max[E - S_T, 0].$$

To derive its classical results, the BSM approach includes further simplifying assumptions such as negligible transaction costs, the existence of a risk-free borrowing and lending rate, no restrictions on the volume of borrowing or lending, no restrictions on assuming long or short positions in the underlying asset, and, no arbitrage opportunities.

2.2. The Evolution of the Value of a Derivative

Let W_t be a Weiner process over time t , with $E[dW_t] = 0$, and $Var[dW_t] = dt$. Due to its scaling, over any single time increment, this process yields a deterministic invariant, a measure of the passage of time and the magnitude of variation

$$(3) \quad (dW_t)^2 = dt.$$

Let's now consider $V \equiv V(S, t)$, the value of a derivative contract on an asset with price S at time t , where S is given by a lognormal random walk,

$$(4) \quad dS = \mu S dt + \sigma S dW_t,$$

which merely states that, aside from a trend, an asset's value changes proportionally to its magnitude. Given the characteristics of dW_t , parameter σ describes the volatility of percentage changes in the asset price in the currency numeraire basis.

Since the stochastic integral converges over finite δt , we take the Taylor expansion of dV to approximate the behavior of an option price as the price of the underlying changes and the time-to-expiry advances:

$$(5) \quad dV_t = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS^2) + \dots,$$

where we ignore terms significantly smaller than dt . Using (4) we can obtain:

$$(6) \quad (dS^2) = (\mu^2 S^2 dt^2 + 2\mu\sigma S dt dW_t + \sigma^2 S^2 dW_t^2).$$

The first term in (6) cancels in a straightforward manner as dt approaches zero. As for the second term, it can be shown that $E[dt dW_t] = 0$ and $Var[dt dW_t] = dt^3$ which approaches zero quite rapidly as $dt \rightarrow 0$. Finally, for the last term, we take advantage of the deterministic invariant. Accordingly, (6) yields:

$$(7) \quad dS^2 = \sigma^2 S^2 dt,$$

Then, by substituting (4) and (7) into (5) we obtain

$$(8) \quad dV_t = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] dt + \sigma S \frac{\partial V}{\partial S} dW_t.$$

2.3. Optimal Financial Strategy

The next step is to construct a risk-free portfolio Π_t . This typically consists of a long derivative with price dynamic as in (8) and a short amount Δ of the underlying security:

$$(9) \quad \Pi_t = V_t - \Delta S_t.$$

The stochastic total differential $d\Pi_t$ of this risk-free portfolio is

$$(10) \quad d\Pi_t = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} (\mu S dt + \sigma S dW_t) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (\sigma^2 S^2 dt) - \Delta \mu S dt - \Delta \sigma S dW_t.$$

Since Δ is a control variable continuously-adjusted by the portfolio manager, by setting $\Delta = \frac{\partial V}{\partial S}$, the stochastic term, dW_t along with the trend component, μ , can be eliminated. As $\frac{\partial V}{\partial S}$ changes with S and t , so will the specific value of Δ , hence the dynamic nature of the hedging strategy. This produces a riskless dynamic portfolio,

$$(11) \quad d\Pi_t = \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (\sigma^2 S^2 dt).$$

If we are to exclude risk-free profits from dynamic arbitrage, this portfolio cannot grow by more than the risk-free rate, r , producing the equilibrium condition

$$(12) \quad d\Pi_t = \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (\sigma^2 S^2 dt) = r \left(V_t - \frac{\partial V}{\partial S} S_t \right) dt,$$

or after some simple manipulations

$$(13) \quad \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} (\sigma^2 S^2) \frac{\partial^2 V}{\partial S^2} - rV = 0,$$

subject to the final condition at expiry and the boundary conditions that the value of the stock cannot drop below zero and the limit of $\frac{\partial V}{\partial S}$ adheres to standard conditions. For example, for a call option, we simply require that $\lim_{S \rightarrow \infty} V(S, t) = 1$, which means that as the stock price approaches infinity, the difference between rates of change in the stock price and the value of the call option becomes vanishingly small.

This is the key BSM partial differential equation, which has a closed-form solution for European contracts and numerical solutions for American contracts. It describes a universe inhabited entirely by arbitrageurs trading three assets, one of which risk-free, denominated in terms of the currency numeraire.

3. Black-Scholes-Merton: An Implicit Assumption

3.1. The Implicit Assumption

The derivation of the central BSM equation, as in (13), implicitly makes an additional assumption, which precludes the price of an option relative to the price of its underlying stock from entering into the option valuation process. The implicit assumption postulates the absence of feedback between variables, t , S_t and V_t . Instead, there is a unidirectional causal chain $(S_t, t) \rightarrow V_t(S_t, t)$.

This implicit assumption means that the BSM approach ignores some aspects of the actual market structure and of the economic incentives of market participants. As the valuation of options contracts are functions of the underlying stock, amidst no-arbitrage arguments, it ignores the fact that the two may be substitutes for the agents whose interactions determine their prices. As such, this assumption has remained largely unaddressed [for an exception involving partial feedback effects from the volume of trading see, Schönbucher and Wilmott (2000)].

By contrast, the nature of exchange provides a mechanism for traders to select the asset basis most relevant to their strategies and to convert their performance metrics accordingly. The appropriate choice of basis will vary by trader and strategy. However, BSM only measures portfolio behavior in a single currency numeraire basis. This could become relevant should speculators or arbitrage-seekers using various other bases enter the BSM universe, seeking risk-adjusted excess returns or employing other trading approaches.

For example, Merton (1971) demonstrates that continuous rebalancing to original portfolio weights is the optimal strategy for risk-bearing, risk-averse investors. Since this cannot take place in practice, due to the restrictions of discrete time, Maeso and Martellini (2020) empirically examine the discrete-time rebalancing premium on a monthly basis and find performance that clearly exceeds the benchmark “buy-and-hold” strategy. This relates directly to option-pricing behavior. Indeed, it is possible to view the discretely implemented BSM delta-hedging strategy as a form of time-varying rebalancing itself. As such, gains attributed to arbitrage strategies may actually be the product of the rebalancing premium phenomenon.

Although seen as an arbitrage strategy, BSM-rebalancing carries the risk that the arbitrageur’s estimate of present and future volatility is incorrect. In this, case the implicit BSM assumption preventing relative price levels from affecting option prices may no longer be innocuous. A rational investor will want to evaluate risk along with relative costs and projected returns in dynamic strategies along with any other possible risk-bearing trading approaches.

As a model framework, the implicit BSM assumption limits available market network connections. It implies that all portfolio adjustments by traders occur according to the solution to the BSM partial differential equation, the final and boundary conditions of which are derived from legal stipulations of the contract alone. Within a universe comprised entirely of arbitrageurs seeking gains in the currency basis, there are no speculators attempting to find risk-adjusted excess return opportunities by alternative strategies, nor are there arbitrageurs in other numeraire bases. In reality, $V_t(S_t, t)$ and S_t could easily compete for capital allocations.

3.2. Market Networks and Non-Contractual Exchange Behavior

The BSM approach focuses on trading strategies between the numeraire and the stock, and the numeraire and the option, separately. In doing so, it neglects a fundamental opportunity provided by the market structure. Specifically, the option and stock may also be exchanged for one another. Although some traders exchange directly, the price ratios in the numeraire, S_t/V_t and V_t/S_t , provide the exchange rates necessary to perform this operation through the numeraire. This property of exchange is endowed by the market itself. It is an implicit embedded conversion option, which is not described in the contract.

In practice, if one has reason to believe that V_t will rise relative to S_t , one may want to use anticipated profits from the leveraged upward movement of V_t to acquire more of the underlying than could be obtained otherwise. This potential opportunity certainly applies to near-the-money options, where percentage changes in the price of the underlying stock typically translate into much greater percentage changes in the value of an option, a leverage effect known as “gearing.” To apply this strategy, one purchases the option at the initial price in hopes of greater relative (not absolute) price appreciation, then converts the option to stock by selling it rather (than exercising it) and uses the proceeds to invest in more shares of the stock.

If all assets may be exchanged for one another, the state of the market at a given time may be represented as a complete directed graph, one in which all nodes are connected with one another and in which edge connections are bidirectional. Each asset, including the numeraire, may be represented by a unique node in the graph. Rates of exchange between assets are directionally-dependent attributes of edge connections between nodes representing their respective assets. While the exchange opportunity we identify produces a complete directed graph – see Figure 1, the analysis of BSM produces a different, more limited market network – see Figure 2.

The distinction is especially relevant to those whose option purchases are ultimately devoted to taking long or short positions within the underlying portfolio they manage. For such managers, options are not merely a potential arbitrage opportunity, but a means to a very specific end. Since BSM assumes that no such arbitrage exists in the first place, BSM itself points to the importance of this constituency. Accordingly, one might expect that rational traders should want to monitor such exchange rates carefully.

The perspectives of these managers are, generally, absent in the no-arbitrage derivatives literature. Yet, to most fund managers, arbitrage cash flows would be materially no different than influxes of new capital for investment. Ultimately, such managers are responsible for investing the capital entrusted to them in positions that will

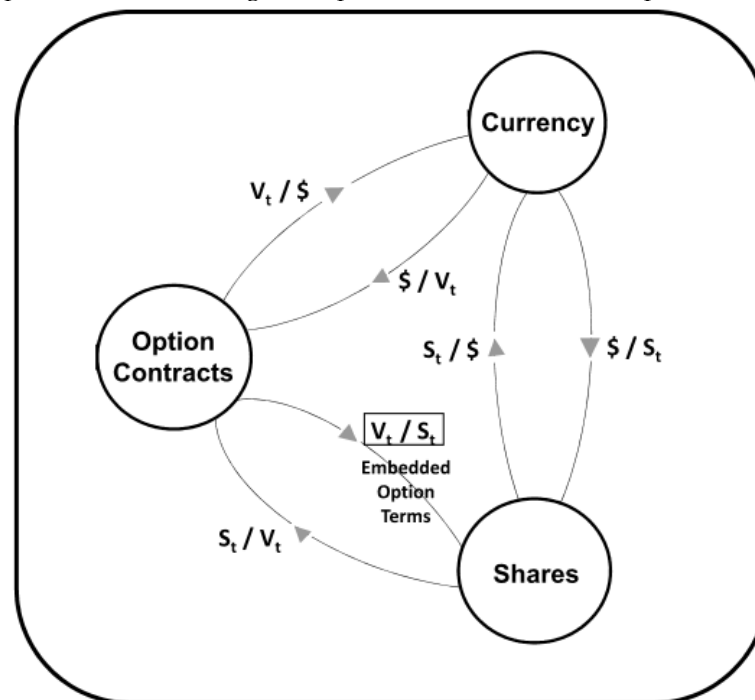


Figure 1. Generalized market connections between assets and implicit exchange rates.

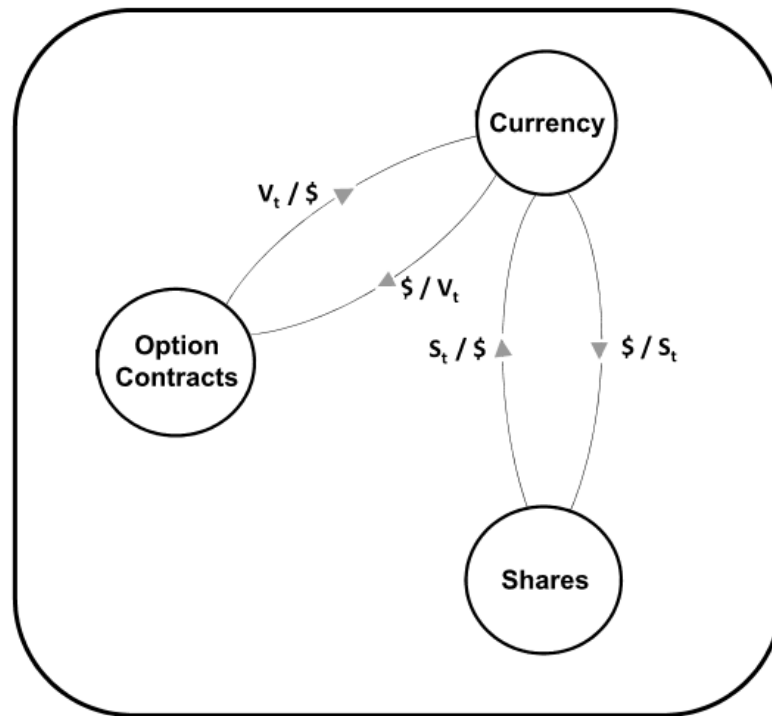


Figure 2. BSM market network connections between assets and implicit exchange rates.

yield higher returns than the risk-free rate. While arbitrage profits would clearly be desirable, they would ultimately only be a means to the end of asset acquisition. As a result, it makes sense to consider seriously the perspective of the asset allocator who views at least one primary function of the option contract as simply providing a mechanism to assume positions in the underlying security.

4. Black-Scholes-Merton: Relaxing the Implicit Assumption

4.1. Formalizing the Hedge Portfolio

The new risk factor that needs to be hedged must be measured in the shares basis. This reflects the reality that many portfolio managers ultimately regard holdings in the currency basis as a means to the end of acquiring positions in assets which, properly balanced, will conceivably lead to superior returns with less risk than a given benchmark.

With this perspective, risk-free profits in one numeraire alone are insufficient to attract trader interest. Opportunity cost across assets must also be considered. If a trading strategy facilitates the acquisition of a fixed amount of currency, while asset prices move such that fewer assets may be obtained after the strategy is implemented, the opportunity cost of the strategy will make it unattractive to such managers. To accommodate this, we apply the BSM analysis in a way that allows for the consideration of the terms of exchange between the non-currency assets, rather than a single fixed numeraire.

To formalize the statement of the portfolio and the exchange conversion factor, let $\phi_1 = \frac{s_t}{v_t}$ and $\phi_2 = \phi_1^{-1}$. By definition, ϕ_1 is the price of the underlying stock in units of the option, while ϕ_2 is the price of the option in units of shares of the underlying stock. As such, ϕ_2 assumes the role of V_t in the portfolio representation, yielding the number of shares for which the option may be implicitly exchanged. Both ϕ_1 and ϕ_2 vary stochastically. This risk affects the terms faced by the option holders should they exercise the implicit embedded option. At the same time, the portfolio manager may choose to hold an amount Δ_c of the currency numeraire. For the sake of simplicity, we assume units are a single U.S. dollar and so the numeraire may be exchanged for $\frac{1}{s_t}$ shares of the stock.

For hedging this conversion risk factor, the portfolio should be expressed in terms of a consistent numeraire, specifically shares of the stock itself. Accordingly, the modified representation of the delta-hedged portfolio employed by BSM, the equivalent of (9), mapped into the shares basis, becomes

$$(14) \quad \Pi_{S\text{-shares}} = V_t^{S\text{-shares}} - \Delta_C C(S_t),$$

where $C(S_t)$ is the per share purchasing power of the currency numeraire, $V_t^{S\text{-shares}} = \phi_2$ is the per share purchasing power of the option, and Δ_C is the amount of the cash numeraire held. $C(S_t)$ is given by

$$(15) \quad C(S_t) = \frac{1}{S_t}.$$

4.2. In Search of the Optimal Financial Strategy

Since (14) is measured in the shares basis, the risk presented by the fluctuating terms of exchange provided by the implicit embedded option and the reciprocal of the stock price must be represented by a continuous-time random process measured in the same basis. In order to provide a perfect hedge in both the currency and shares basis, fluctuations in both the stock price and the new terms of exchange need to be accounted for. For simplicity, we model risk in the shares basis rather than the currency basis.

Since the dynamics of ϕ_2 are derivable from a specification for $d\phi_1$, we begin with $d\phi_1$. Let dW_t be a continuous-time stochastic contribution. Following BSM, let

$$(16) \quad d\phi_1 = \alpha_0 \phi_1 dt + \alpha_1 \phi_1 dW_t.$$

which determines the functional form with which hedging will occur in the shares basis, and it will be the path by which option and stock prices may be directly recovered.

The specification in (16) has several advantages. It is parsimonious, it mirrors BSM, and it seems to be a rather minimal baseline requirement for hedging any more complex specification. If $\alpha_0 = \alpha_0(\phi_1)$, the deterministic trend component could be heuristically nudged away from zero and infinity. At the same time, at least approaching limits of zero and infinity are potential consequences of extremes of the exchange ratios. In the currency numeraire basis, should the price of the option fall below the quantization provided by units of the numeraire, it would appear to be zero. This would result in a divergence of ϕ_1 toward infinity, which this model accommodates.

This specification for exchange ratio dynamics differs from what would be derived had one started with (4), describing stock price changes in the currency numeraire, and (8), describing the resulting option price changes in the currency numeraire and cross-effects. The process of invoking (4) to produce (8) presumes the existence of $V \equiv V(S, t)$. If the new risk dimension is unhedgeable, this deterministic definition would not hold.

The term $d\phi_2$ may be derived from Itô's Lemma, which initially yields

$$(17) \quad d\phi_2 = \frac{\partial \phi_2}{\partial \phi_1} d\phi_1 + \frac{1}{2} \frac{\partial^2 \phi_2}{\partial \phi_1^2} (d\phi_1)^2.$$

Calculating the relevant partial derivatives produces $\frac{\partial \phi_2}{\partial \phi_1} = -\frac{1}{\phi_1^2}$ and $\frac{\partial^2 \phi_2}{\partial \phi_1^2} = 2\frac{1}{\phi_1^3}$. Ignoring terms smaller than dt , $(d\phi_1)^2 = \alpha_1^2 \phi_1^2 dt$, mirroring its counterpart in BSM. Inserting these and the specification for $d\phi_1$ into the above yields

$$(18) \quad d\phi_2 = -\frac{1}{\phi_1^2}(\alpha_0 \phi_1 dt + \alpha_1 \phi_1 dW_t) + \frac{1}{\phi_1^3} \alpha_1^2 \phi_1^2 dt.$$

Or,

$$(19) \quad d\phi_2 = -\frac{1}{\phi_1}(\alpha_0 - \alpha_1^2)dt - \frac{1}{\phi_1} \alpha_1 dW_t.$$

It is important to observe that (16) and (19) both describe geometric Brownian motions. This means that we could have specified either $d\phi_1$ or $d\phi_2$ first, and the functional form of the resulting dynamic process would be unaffected.

In like manner, we may derive the stochastic dynamics of $C(S_t) = \frac{1}{S_t}$ from (4). Noting that $\frac{1}{\phi_1} = \frac{V_t}{S_t}$, $\frac{\partial C}{\partial S} = -\frac{1}{S_t^2}$ and $\frac{\partial^2 C}{\partial S^2} = \frac{2}{S_t^3}$, by Itô's Lemma,

$$(20) \quad dC_t(S) = \left(-\frac{1}{S_t} \mu + \frac{2}{S_t} \sigma^2\right) dt - \frac{1}{S_t} \sigma dW_t,$$

which provides the continuous-time fluctuation in the per share purchasing power of one unit of the currency numeraire. Substituting $\phi_1 = \frac{S_t}{V_t}$ into (19), it is possible to express terms in the potentially hedged portfolio in the shares basis in terms of option and stock prices, as in BSM. This yields

$$(21) \quad d\phi_2 = -\frac{V_t}{S_t}(\alpha_0 - \alpha_1^2)dt - \frac{V_t}{S_t} \alpha_1 dW_t.$$

Now, we specify the dynamics for $d\Pi_{S\text{-shares}}$ by inserting (21) and (20) into (14),

$$(22) \quad d\Pi_{S\text{-shares}} = -\frac{V_t}{S_t}(\alpha_0 - \alpha_1^2)dt - \frac{V_t}{S_t} \alpha_1 dW_t - \Delta_C \left(\left(-\frac{1}{S_t} \mu + \frac{2}{S_t} \sigma^2\right) dt - \frac{1}{S_t} \sigma dW_t \right).$$

which, may be simplified as

$$(23) \quad d\Pi_{S\text{-shares}} = \left(-\frac{V_t}{S_t}(\alpha_0 - \alpha_1^2) - \Delta_C \left(-\frac{1}{S_t} \mu + \frac{2}{S_t} \sigma^2 \right) \right) dt + \left(\Delta_C \frac{1}{S_t} \sigma - \frac{V_t}{S_t} \alpha_1 \right) dW_t.$$

4.3. Hedging Impossibility

It is possible now to address the issue of whether or not one can hedge the purchasing power of the portfolio (14) while at the same time keeping the prices of the stock and the

option strictly positive. These terms are necessary so that the dynamics in (8) don't encounter a sink at $S_t = 0$, which would make further analysis trivial.

Suppose that the purchasing power of the portfolio is hedgeable. In that case, we can eliminate dW_t by solving for Δ_c . This produces the quantity of the currency numeraire held to hedge option price fluctuations,

$$(24) \quad \Delta_c = \frac{V_t}{\sigma} \alpha_1,$$

which when inserted into (23) produces the analogue of (11).

$$(25) \quad d\Pi_{S\text{-shares}}^{\text{Hedged}} = \left(-\frac{V_t}{S_t} (\alpha_0 - \alpha_1^2) - \left[\frac{V_t}{\sigma} \alpha_1 \right] \left(-\frac{1}{S_t} \mu + \frac{2}{S_t} \sigma^2 \right) \right) dt.$$

If a risk-free interest rate, r , paid in quantity of shares existed, we could derive the equivalent of (13) from (25) for the shares basis. This results in the relation

$$(26) \quad \left(-\frac{V_t}{S_t} (\alpha_0 - \alpha_1^2) - \left[\frac{V_t}{\sigma} \alpha_1 \right] \left(-\frac{1}{S_t} \mu + \frac{2}{S_t} \sigma^2 \right) \right) dt = r \left(\frac{V_t}{S_t} - \left[\frac{V_t}{\sigma} \alpha_1 \right] \frac{1}{S_t} \right) dt.$$

Subtracting the right-hand side and eliminating dt provides the equivalent of (13) for the shares basis, with the implicit embedded option now structurally recognized

$$(27) \quad -\frac{V_t}{S_t} (\alpha_0 - \alpha_1^2) - \left[\frac{V_t}{\sigma} \alpha_1 \right] \left(-\frac{1}{S_t} \mu + \frac{2}{S_t} \sigma^2 \right) - r \frac{V_t}{S_t} + r \left[\frac{V_t}{\sigma} \alpha_1 \right] = 0.$$

Unlike (13), (27) is not a partial differential equation. This is a consequence of explicitly recognizing the structural relationships provided by the implicit embedded option, which makes partial derivatives in the shares basis readily computable. This also makes any proof by contradiction more accessible. Factoring out $\frac{V_t}{S_t}$ from each term, it may be seen that

$$(28) \quad \frac{V_t}{S_t} \left(-(\alpha_0 - \alpha_1^2) - \left[\frac{1}{\sigma} \alpha_1 \right] (-\mu + 2\sigma^2) - r + r \alpha_1 \right) = 0.$$

Of course, no risk-free interest rate, r , exists in the shares basis, although if and only if hedging is possible, r could serve as a proxy for a deterministic, hedged growth rate from a given dynamic strategy. Regardless, if (28) is able to yield nonzero deterministic values for the option in the currency or shares basis, we may reasonably assume that the multiplier on the right of $\frac{V_t}{S_t}$ in (28) is nonzero for any scenario of interest. If the multiplier were zero, then from (28), we would conclude that the conversion rate $\frac{V_t}{S_t}$ is an entirely free variable, and the risk in the portfolio cannot be dynamically hedged.

Secondarily, from a practitioner's perspective, every term in the multiplier other than r is continuous and must be estimated by some process. Any such set of estimators must

be associated with a continuous probability density function, in which the probability of a single fixed value is by definition equal to zero. This means that as a practical matter,

$$(29) \quad Pr\left(-(\hat{\alpha}_0 - \hat{\alpha}_1^2) - \left[\frac{1}{\sigma}\hat{\alpha}_1\right](-\hat{\mu} + 2\hat{\sigma}^2) - r + r\hat{\alpha}_1 = 0\right) = 0.$$

Dividing both sides of (28) by the multiplier yields

$$(30) \quad \frac{V_t}{S_t} = 0.$$

At this point, several paths lead to serious logical contradictions or to violations of the terms of hedgeability. For example, assuming $S_t \neq 0$, trivially produces

$$(31) \quad V_t = 0.$$

Were (31) to be satisfied, it would produce a collapse of the implicit exchange relationship between the stock and the option. Were (25) still valid and able to be derived from a dynamic specification, we would find that $\Delta_C = 0$, meaning that no holdings exist in the currency basis. We would also observe that $\phi_2 = \frac{V_t}{S_t} = 0$, rendering the portfolio described in (18) and (25) valueless. In other words, if the purchasing power of the cash and option portfolio is hedgeable, then the option is worthless.

As a result, it may be concluded that the assumption of hedgeability in the shares basis fails. Moreover, the implication of (31), that the value of the option equals zero, stands in stark contrast to the predictions of BSM. So long as the underlying has nonzero volatility, BSM always predicts that an option will have a positive arbitrage-free value.

5. Concluding Remarks

For almost half a century, option pricing theory has relied heavily on the seminal dynamic hedging and replication arguments set forth by the BSM approach. These arguments presume a restricted subset of the market network, a subset over which any resulting dynamics are restricted to unfold. This paper focuses on what may occur when the BSM approach is extended to a more practically relevant general market network. Specifically, we consider explicitly an implicit option embedded in all option contracts, one that arises from the nature of the market itself. This implicit option is exercised by selling the option to which it is attached and exchanging it for another asset according to relative price ratios. The effect of this is to add an early-exercise conversion provision that provides option holders an additional means to enter positions in underlying assets.

For the holder of a call option, the underlying may be acquired either by exercising the option according to the terms of the contract or by exchanging the option for shares in the underlying. This allows the holder of a put option to enter a long position in the underlying by exchanging the put option for a certain number of shares of the underlying, which reveals the put option to contain an inherent exotic call provision on the underlying. Alternatively, the holder of a put option may exercise the contract terms of the option in order to take a negative position in the underlying. For example, as the price of the underlying stock decreases, the price of a put option will typically rise, *ceteris paribus*. This means that the put option may be used to purchase more shares of the underlying asset than a given call option may have made possible, even though the put contract itself only expressly facilitates entering a negative position.

Adapting the problem structure to the terms of the embedded option, the portfolio hedged in the original BSM framework is unhedgeable within the resulting market network of assets and exchange rates. This leads to a breakdown in deterministic option pricing approaches, as well as in popular risk-management metrics for options that are widely employed by portfolio managers and regulatory institutions.

By integrating a complete market network structure into the BSM analysis of option hedging, several issues become apparent. The strategy provided by the implicit conversion property of any asset in a market network removes the ability of the BSM approaches to provide Δ -hedging, resulting in a stochastic portfolio. Without the ability to derive a deterministic partial differential equation for what once seemed to be hedged portfolio dynamics, contract terms themselves become irrelevant to the hedging problem. Moreover, the “Greeks” used in risk-management are derived from the terms $\theta = \frac{\partial V}{\partial t}$, $\Delta = \frac{\partial V}{\partial S}$, and $\Gamma = \frac{\partial^2 V}{\partial S^2}$, all of which require a deterministic partial differential equation.

At the same time, the exercise highlights the fundamental objective of the portfolio manager – to acquire positions in underlying assets, using whatever means seem most appropriate. In fact, this pursuit of returns in assets subject to risk is implied by the no-arbitrage assumption of BSM. The results demonstrate that under these assumptions, the goal of increasing share acquisition cannot be achieved without some assumption of risk. For investors with the proper risk tolerances, the resulting dynamics for the unhedgeable portfolio may provide a means of indirectly acquiring shares. Such investors would purchase the option expressly in hopes of converting it to stock through the implicit mechanism granted by the market itself. If the resulting exchange rate between the derivative and underlying trends strongly and stably in a favorable direction, the option may be a risky but strategically optimal shortcut to greater share acquisition.

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