*Article*

# Wavelet time scale modelling of brand sales and prices

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**Abstract:** Marketing is the manner of how to make our sales the best in the market, our prices the most accessible, our clients satisfactory, and thus our brand is the largest distributed. This needs sophisticated and advanced understanding of the total network related. Indeed, marketing data may be seen in different forms such as qualitative and quantitative. However, in the literature, it is easily noticed that large bibliography may be collected about qualitative studies, against few studies on the quantitative point of view. This is a major drawback that makes the marketing science still focusing on the design, although the market is strongly depending on quantities such as money and time. Indeed, marketing data may be a time series such as brand sales per specified periods, brand related-prices over specified periods, market shares, ..., etc. The purpose of the present work is to investigate some marketing models based on time series due to brands. We will precisely study the effect of the time scale on the persistence of brands sales in the market and on the forecasting of such a persistence according to the characteristics of the brand and the related market competition or competitors. Our study is acted on a sample of Saudi brands during the period November 22, 2017 to December 30, 2021.

**Keywords:** Wavelets; Multi scale; Mathematical models; Brand sales.

## 1. Introduction

The aim of this work is developing a mathematical approach for the so-called brands sales and forecasting using wavelet theory. Recall that such a theory has been proved to be a powerful tool in the economic and financial field. Financial indicators are subject to volatility and high fluctuations, which makes their study and their understanding using classical methods insufficient. One of the main power characteristic of wavelets is their ability to detect and/or localize fluctuations and volatility.

Marketing is defined as "the activity, set of institutions, and processes for crating, communicating, delivering and exchanging offerings that have value for customers, clients, partners and society at large ([4]). While marketing consists various activities, the principles of marketing are rely on 4ps which are product, price, promotion and place (distribution). Each one of the elements depends on side of qualitative data and quantitative data. The latter side shows the power of marketing when it comes to collect, analyse, optimise and execute based on relevant numerical data. Products choices and portfolio, prices hierarchies, number of outlets and stores and importantly promotion budget, especially branding, brand management and brand equity. Although brand or activity of branding has huge attrition in literature, brand equity refer to added value to products ([40]). Firms can benefit from strong brand in various ways: improve product value, provide a chance to extend brand on new products, expand into new market, get a high level of customer loyalty and tolerance and a well know brand gets a price value ([28]). Thus, brand sales and equity related data such as choice data, the brand design, the brand idea, time of purchases, the number of stores, the choice of the price, the competitors in the market, the state of the

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market itself effects strongly and simultaneously on the persistence and the success of the brand investment. Without sophisticated mathematical models that take into account all the factors and that permit a good understanding of the time and space evolution of the brand distribution and evolution in the market, the investment in the brand may fail.

Mathematical model in marketing may include many factors such number of sales in a specified period such as weeks, months, years, market shares per a specified period, price evolution, the history of the market and its situation, the competitor brands, ..., etc. The main characteristic of a mathematical model that it permits to understand many factors simultaneously according to time and space. It gathers the data into a time series form thus permit some order over time. This has a major important consequence that it yields and/or informs about the future state of the brand. The model will permit for a time series to extract, deduce and/or describe accurately the properties of the empirical data as best descriptor of the data.

In the present paper, we aim to investigate some multidimensional modeling of time series issued from brands sales and evolution. The model will be a general form including many variables such as sales, prices and advertising simultaneously. Our principal idea is based on the effect of the time factor on the brand evolution. In [50] and [51], the authors mentioned that one of the limitations of conventional approaches in econo-financial studies and models is the lack of dependence of the models on the time scale, and distinction between the time domain and the frequency ones which is a crucial task in both econometric and economic rationale.

Wavelets offer efficient algorithms for practical problems where classical techniques have shown limitations. Moreover, they provide an attractive mathematical formalism in the reformulation of several problems and in different scientific fields, especially in time series analysis. Wavelets form a mathematical tool that transforms time domain data into different frequency horizons. They represent the advantage of being localized in both the time and frequency domains. They allow to observe and to analyse data at different time scales, which in turns makes it possible to overcome the inadequacies of the classical analysis. For more backgrounds on wavelets and their applications the authors may refer to [3,8–11,57,60,61].

From an empirical point of view, we aim to apply the wavelet technique to estimate mathematical models for brands' marketing. The validity of models should also be a real foundation for effective financial decisions. Indeed, the effectiveness of a financial decision depends to a large extent on the accuracy of the valuation of the securities as well as the most precise knowledge possible of their subsequent evolution and their risks.

## 2. Literature review

Generally, the marketing researchers and/or modelers search models taking into account one or more variables as functions of time solely. Next, a step of analysis of the model and its efficiency to interpolate (approximate) and to extrapolate (predict) these variables in time.

By comparing to other fields such as financial and economic applications, the use of mathematical models and especially wavelet-based ones is still not well developed in marketing (See for instance [24,45]).

In [24], for example, the authors declared the difficulty of training in time series methods for traditional research models, the lack of adapted software to marketing modeling, the lack and sometimes the absence of good-quality time-series data, and finally, the absence of a substantive marketing area where time series modeling may be adopted as primary research tool.

The problem of time scale in marketing has been raised for several studies. In [53], it is mentioned that marketing phenomena depends on frequencies in time relatively to decisions and interactions in the market. In the classical investigations, the most common used time intervals are always weekly, monthly, quarterly and annual cycles. Each period has its special and/or specific characteristics. The weekly for example are for reductions

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of prices, the quarterly periods are in general related to to regular price adjustments and competitions.

Moreover, many phenomena such as pandemics (COVID-19), wars, climate changes, financial crisis, socio-political movements may induce severe perturbations on the usual considerations especially time periods, leading to a diversification on the use of the time scale..

This diversification of how to chose the time scale, and according to what factor, what character, and what aim, has led researchers in marketing field to think about including strongly the time scale in the models. In [45], the authors applied certain type of short, medium and long time horizons to investigate the effect of the time scale. However, their choice is not adapted to wavelet method.

In marketing, according to [53,54], the time scale is mostly applied at discrete, equally spaced intervals with a dominance of the Weekly periods, although this hypothesis is already and always criticized. In [46] for example, it is mentioned that marketing models still present aggregations even using weekly intervals. This leads researchers to conclude that more sophisticated tools of time and frequency scales should be applied.

Many methods have been tackled in this context such as Fourier analysis, spectral analysis. However, these tools have raised many problems such as the non stationary behavior of the marketing data, the size of samples, etc. See also [12,20,26,41,42,56].

Wavelet analysis has been introduced by the next in marketing models to overcome the limitations such as the non localization aspect of spectral analysis, and its concentration in the frequency domain. This is done by providing providing frequency decomposition of the statistical time series into components that are well localized in both time and frequency.

Michis mentioned also a very important characteristics of wavelet theory in [53–55]. It permits indeed to understand the causal relations between marketing time series over different cycles. The modelers will differentiate the marketing drivers influences in long duration cycles of sales. Wavelets are also good tools in estimation and prediction accuracy, and the handling of non-stationary time series. The application of wavelets in marketing resides in covering/uncovering the frequency activity characteristics of marketing models. Wavelet crystals generated from the inverse of the wavelet transform are applied to localized the variation in economic variables over different horizons. In [54], the author applied a wavelet based method for forecasting brand sales. The problem of multi-co-linearity has been investigated and resulting in correlated vectors of coefficients, which has been applied next to provide the most accurate forecasting and a best dimension reduction. Recall that brands' sales forecasting is important for investors, consumers and also for modelers because of its strong relationship with the budget planning and allocation of financial resources. Therefore, a successful sales forecasts will yield a good guide for sufficient production planning, timely distribution of the products in times of increased demand and directions for the planning of appropriate marketing activities to support product performance in the market place ([53–55]).

## 3. Wavelets toolkit for time series

Wavelet analysis allows the representation of time series into species relative to the time and frequency information known as time-frequency decomposition. It consists in decomposing a series in different frequency components with a scale adapted resolution and thus permits to observe and to analyze data at different scales. Wavelet analysis starts from one source function $\psi$ known as the mother wavelet and next composes dilation-translation copies to get a complete system for finite energy time series. Each wavelet basis element is defined for $j,k \in \mathbb{Z}$ as a copy of $\psi$ at the scale $j$ and the position $k$ by $\psi_{j,k}(t) = 2^{-j/2}\psi(2^j t - k)$. The quantity $2^j$ corresponds to the frequency of the series while the index $k$ localizes volatility or fluctuations. Let for $j \in \mathbb{Z}$ fixed, $W_j = span(\psi_{j,k}, k)$ known as the $j$-level detail space. A time series $X(t)$ is projected onto $W_j$ yielding a component $DX_j(t)$ given by

$$DX_j(t) = \sum_k d_{j,k}\psi_{j,k}(t) \tag{1}$$

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The $d_{j,k}$ are the detail coefficients of the series $X(t)$ expressed by means of the ordinary inner product in the functional space $L^2(\mathbb{R})$ as

$$d_{j,k} =< X, \psi_{j,k} >= \int_{\mathbb{R}} X(t)\psi_{j,k}(t)dt \tag{2}$$

The spaces $W_j$'s form an orthogonal decomposition covering the space of finite energy series $L^2(\mathbb{R})$. This means that the series $X(t)$ can be completely reconstructed as a sum of its projections on the detail spaces and that these projections are mutually uncorrelated. In wavelet theory, the mother wavelet yields a second function called father wavelet or scaling function denoted here by $\varphi$. (See [18]). Similarly to $\psi$, the function $\varphi$ yields dilation-translation copies $\varphi_{j,k}(t) = 2^{-j/2}\varphi(2^j t - k)$ generating subspaces $V_j$. The sequence $(V_j)_j$ is called a multi-resolution analysis (multi-scale analysis) on $\mathbb{R}$ and $V_j$ is called the $j$-level approximation space. It is well known in wavelet theory that $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$, which means that the approximation of the time series at the level $j$ and $j+1$ can be viewed from each other and so from any horizon $p \geq j+1$. In physics-mathematics this is called the zooming rule. It holds also that for all $j \in \mathbb{Z}$, $f(t) \in V_j$ iff $f(2t) \in V_{j+1}$, which reflects the fact that, not only the signal $f$ from horizon $j$ can be seen in the horizon $j+1$ but also his contracted or dilated copies. As for the detail subspaces, the approximation subspaces $V_j$'s satisfy also a completeness relation meaning that no information is lost when considering all approximations and a second property meaning that all the information is lost at finer scales. Finally the $Vj$'s satisfy a shift-invariance property in the sense that $f(t) \in V_j$ iff $f(t-k) \in V_j$, $j,k \in \mathbb{Z}$, which means that the multi-resolution analysis permits to detect the properties of the signal along the whole time support. Combining all the properties above we deduce that the approximation space is decomposed into a low-level approximation part supplemented with a detail one. Under these properties, the following decomposition is proved for $j \in \mathbb{Z}$,

$$X(t) = \sum_j DX_j(t) = \sum_{j\leq J} DX_j(t) + \sum_{j\geq J+1} DX_j(t) \tag{3}$$

The component $AX_J(t) = \sum_{j\leq J} DX_j(t)$ is called the approximation of $X(t)$ at the level $J$ and it reflects the trend or the global shape of $X(t)$. It also belongs to the space $V_J$. Thus, using the definition of the $V_J$'s, the component $AX_J(t)$ may be expressed using the basis $(\varphi_{J,k})_k$ as

$$AX_J(t) = \sum_k a_{J,k}\varphi_{J,k}(t) \tag{4}$$

where the $a_{J,k}$ are the approximation coefficients of the series $X(t)$ expressed by $a_{J,k} =< X, \varphi_{J,k} >$. As a result, we obtain the following relation known as the wavelet decomposition of $X(t)$

$$X(t) = AX_J(t) + \sum_{j\geq J+1} DX_j(t) \tag{5}$$

It is composed of one part reflecting the global behavior of the series and a second part reflecting the higher frequency oscillations or the fine scale deviations of the series near its trend. In practice we cannot obviously compute the complete set of coefficients. We thus fix a maximal level of decomposition $J$ and consider the decomposition for any $J_0 < J$,

$$X_J(t) = AX_{J_0}(t) + \sum_{J_0<j\leq J} DX_j(t). \tag{6}$$

There is no theoretical method for the exact choice of the parameters $J_0$ and $J$. However, the minimal parameter $J_0$ does not have an important effect on the total decomposition and usually chosen to be 0. But, the choice of $J$ is always critical. One selects $J$ related to the error estimates.

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In marketing, the wavelet theory and generally mathematical advanced models are still not widely developed. However, in finance, economics, management and generally actuarial sciences, wavelet modeling and analysis are rapidly growing as sophisticated tools recently discovered in these fields, which may be an encouraging task to extend wavelet and generally multi-scale models to marketing, business, because of their natural link with actuarial sciences. See [3,8–11,15–17,27,29,31–33,38,39,57,61,64,65].

## 4. Development of the mathematical model

In literature there are few models in marketing. One of them is due to Hansenes et al [36] adopted for both brands' sales and the marketing activities that surround them like price changes and promotions, and which states that

$$S_t = \beta_0 + \beta_1 P_t + \beta_2 CP_t + \beta_3 PR_t + \beta_4 D_t + \epsilon_t, \tag{7}$$

where

- $S_t$ stands for Sales value at the time $t$,
- $P_t$ for the Price at the time $t$, and which refers exactly to the relative average price over the range of stock keeping units, variants and pack sizes under the specific brand.
- $CP_t$ represents similarly defined relative average prices of the main competitive brands in the market.
- $PR_t$ is the PROMO variable at time $t$, and refers to promotional activities relevant to the brand.
- $D_t$ refers to the brand's numeric handling distribution. This is the percentage of stores in the province to which any of the brand's stock keeping units are distributed.
- Finally, $\epsilon_t$ is an error term.

Using wavelets, we will speak about multi-resolution or multi-horizons or levels model, where the variables have to be replaced by their projections on the multi-resolution and/or detail spaces as in (6). Therefore, for any level $j$ we get a $j$-level model as

$$S_{t,j} = \beta_{0,j} + \beta_{1,j} P_{t,j} + \beta_{2,j} CP_{t,j} + \beta_{3,j} PR_{t,j} + \beta_{4,j} D_{t,j} + \epsilon_{t,j}, \tag{8}$$

where

- $S_{t,j}$ reflects the Sales value at the time $t$, and the level or horizon $j$,
- $P_{t,j}$ is the Price at the time $t$, and the level $j$,
- $CP_{t,j}$ represents similarly defined relative average prices of the main competitive brands in the market, already at the horizon $j$,
- $PR_{t,j}$ is the PROMO variable at time $t$, and horizon $j$,
- $D_{t,j}$ refers to the brand's numeric handling distribution at the time $t$ and the level $j$,
- Finally, $\epsilon_t$ is an error term.

Considering this wavelet decomposition which leads to horizon-wise model will permit to explore more efficiently the behaviour of brand sales according to the time scale.

## 5. Results and discussion

To show the utility of the involvement of wavelets into the mathematical model, we applied it on a typical case consisting of the top 10 brands in Saudi Arabia. The choice of Saudi Arabia is justified by many reasons. ..... Such brands are resumed in Table 1 below. The chosen Saudi Brands to be applied is in turn justified by many reasons, mainly, these are the most Worldwide spread brands, which have been now known in many other countries, making thus an extension of their local origins and having thus a worldwide reputation, and constitutes therefore a success for the kingdom industry. Recall that the Saudi Arabia kingdom is mostly known as a consumer community, which relies on foreign imported consummation products in front of petroleum exportation. Encouraging national industry, and generally national labor is one of the major goals of the 2030-vision plan of the kingdom.

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| Nomination | Brand | Description (Sector) |
|---|---|---|
| $B_1$ | Jarir Bookstore | Books and electronics |
| $B_2$ | Almarai | Dairy and poultry |
| $B_3$ | STC | Telecommunications |
| $B_4$ | AlAbdullatif | Household Durables |
| $B_5$ | EIC | The Electrical Industries Company |
| $B_6$ | AlAseel | Textiles, Apparel & Luxury Goods |

**Table 1.** Some top Saudi brands

Jarir Bookstore is founded at July 1974 by Abdulrahman Nasser Al-Agil. It is one of the largest retailers for books and electronics in the Kingdom, expanded now to many countries especially in GCC such Kuwait, Qatar and UAE. It is also one of the major components of the Saudi TADAWUL index.

Almarai company originally a partnership between Irish Alastair McGuckian, Paddy McGuckian and Prince Sultan bin Mohammed bin Saud Al Kabeer, is now one of the biggest dairy companies in KSA, and also in the whole Middle East region. It is founded 40 years ago, and now specialized in dairy, yogurt, juices, bakeries, poultry and also infant formula products.

The Saudi Telecommunications Company abbreviated STC is starting 19 years ago. It is offering basically telecommunications services and products. It is also now expanded in GCC countries and other continents such as India, Turkey, South Africa and Malaysia.

Al Abdullatif Industrial Investment Company is a national Saudi Arabia-based company specialized in both distribution and manufacture of weaving products such as blankets, rugs, carpets and intermediates such as nylon, polyester, ... etc.

The Electrical Industries Company abbreviated EIC started at 2005, and it constitutes since than a leading manufacturer of electrical products to satisfy growing demand of electrical equipment in the Kingdom.

Thob Al Aseel Company develops, imports, exports, wholesales, and retails fabrics and readymade clothes. It operates through Thobs and Fabrics segments. The company offers thobes, pants, underwear, pajamas, and sleeping robes, as well as miscellaneous products, such as T-shirts, ehram products, and cotton socks. It also sells women's, men's, and children's fabrics and clothing products, as well as sewing supplements. The company operates through a chain of retail outlets under the Al-Jedaie brand. Thob Al Aseel Co. was founded in 1970 and is headquartered in Riyadh, Saudi Arabia.

Table 2 below shows the descriptive statistics corresponding to prices and sales for the six brands.

| Brand | Mean | Median | Min | Max | Std | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| | | | The brands' prices | | | | |
| $B_1$ | 161.42 | 159.2 | 105.44 | 225 | 26.48 | 0.21 | 2.37 |
| $B_2$ | 52.91 | 53.3 | 36.95 | 63.7 | 4.07 | -0.08 | 3.15 |
| $B_3$ | 100.25 | 99.75 | 67.10 | 139.2 | 16.35 | 0.20 | 2.52 |
| $B_4$ | 15.88 | 12.66 | 8.15 | 40.35 | 8.12 | 1.76 | 4.64 |
| $B_5$ | 20 | 20.34 | 14.46 | 29.4 | 3.55 | 0.39 | 2.35 |
| $B_6$ | 35.52 | 31.35 | 14.88 | 69.75 | 16.44 | 0.65 | 2.01 |
| | | | The brands sles | | | | |
| $B_1$ | 153.63 | 109.31 | 7.56 | 4020 | 258.03 | 10.70 | 140.71 |
| $B_2$ | 564.28 | 418.03 | 34.70 | 12140 | 689.47 | 9.25 | 129.83 |
| $B_3$ | 890.12 | 512.49 | 33.30 | 1.21310 | 3967.75 | 27.64 | 831.52 |
| $B_4$ | 809.01 | 223.43 | 8.24 | 22200 | 1935.15 | 6.08 | 51.00 |
| $B_5$ | 2577.69 | 1720 | 111.39 | 37490 | 3161.13 | 4.89 | 40.35 |
| $B_6$ | 243.89 | 64.04 | 0.01 | 5250 | 570.35 | 4.70 | 29.97 |

**Table 2.** Descriptive statistics for brands $B_i$, $1 \leq i \leq 6$, prices and sales.

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The first deductions from Table 2 may be due to the Min-Max values, which are for quasi all the brands' prices and sales widely far apart, reflecting the existence of aberrant values or anomalies in the market. Besides, the mean and median values are also different, even widely far apart especially for sales. This big range in the prices and sales is more adequately detected and explained by means of time scale modeling, as apparently there are no logical causes for it.

Notice easily from Table 2 that the skewness is non-vanishing for all the brands prices and sales, which rejects the asymmetry hypothesis. All the brands' prices and sales are spread to the right of their mean values, except $B_2$-prices which has a small skewness, and a kurtosis close to 3, meaning that a quasi-normal distribution behaviour is hidden for this series of prices, with a negative skewness meaning a left-spreading tail for $B_1$-prices as it is shown in fact in Figure 1, $A_6$.

The kurtosis shows a non-normal value and/or behavior (being different from 3) for all brands' prices and sales, meaning heavier tails than the normal distribution. This fact is more visible by means of the detail components. See Figures 1–12.

The flatness and distortion features of all stocks' returns are different from each other. Moreover, for all the variables $B_1$, $B_3$, $B_4$, $B_5$ and $B_6$, the Jarque-Bera test leads to a returned value of $h = 1$, and returned $p$-value at the order of $e - 03$ at the 5% significance level, which indicates a rejection of the null hypothesis.

We may further notice that the standard deviation measure is somehow large, which reflects a sparse aspect for both prices and sales around their mean values. For some brands' prices and sales, the data is widely sparse. This fact may be explained by the use of big difference on prices as well as sales and non uniform view of the future by both consumers and managers in this market. Consumers demands on special types of products under the same brand leads to an increase of both volume sales and prices without taking into account the equilibrium with other products under the same brand title. Again, the JB test residues in the fact that the market is not developing a normal distribution way.

Our analysis acts by projecting equation (7) relatively to time scales to test the effect of time scale on the brands' movements. This will be conducted by splitting the brands' prices or values into crystals or horizons relative to different time scales instead of using the classical periods such as weeks, moths, years.

The coefficients of the linear regressions will be estimated by the usual ordinary least square (OLS) estimates of the variable $S_{t,j}$ on the variables $P_{t,j}$, $CP_{t,j}$ $PR_{t,j}$, $D_{t,j}$ in the model (8).

Next, we provide the wavelet multiscale analysis of each variable at the level 6, using herewith the well-known Daubechies wavelet Db8 (See [18]). Table 5 here-after shows the wavelet coefficients for each brand prices and sales.

| Brand | Level 1 | Level 2 | Level 3 | Level 4 | Level 5 | Level 6 |
|---|---|---|---|---|---|---|
| | | | The brands' prices | | | |
| $B_1$ | -0.0002 | 0.0007 | 0.0016 | 0.0147 | 0.0147 | 0.0912 |
| $B_2$ | 0.0000 | 0.0000 | -0.0001 | -0.0054 | -0.0054 | -0.0041 |
| $B_3$ | -0.0000 | -0.0004 | 0.0001 | 0.0042 | 0.0042 | 0.0656 |
| $B_4$ | -0.0000 | -0.0001 | -0.0008 | 0.0001 | 0.0001 | -0.0106 |
| $B_5$ | -0.0000 | 0.0002 | -0.0007 | -0.0045 | -0.0045 | -0.0061 |
| $B_6$ | 0.0000 | 0.0002 | -0.0008 | 0.0054 | 0.0054 | -0.0395 |
| | | | The brands' sales | | | |
| $B_1$ | -0.0073 | 0.0196 | -0.0206 | 0.0079 | 0.0079 | 0.0034 |
| $B_2$ | -0.0645 | -0.0049 | 0.0065 | 0.2211 | 0.2211 | 0.7129 |
| $B_3$ | 0.0012 | -0.4976 | -0.9052 | 36.9157 | 36.9157 | 20.2465 |
| $B_4$ | -0.2039 | 0.2151 | -0.0401 | 0.3270 | 0.3270 | 1.0747 |
| $B_5$ | -0.2954 | 0.2021 | -1.5023 | 4.1040 | 4.1040 | 21.1156 |
| $B_6$ | -0.0060 | 0.0060 | -0.0210 | 0.4643 | 0.4643 | 0.5981 |

**Table 3.** Mean wavelet coefficients for brands' prices and sales at different levels.

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Table 3 presents a global overview of the movement of brands (prices and sales) according to their mean wavelet coefficients estimated at different time horizons. We notice an overall increase for example for the prices of the brand $B_1$ accompanied by a decrease in sales, such a behavior is quietly happening for the brands $B_3$, $B_4$, $B_5$ and $B_6$, contrarily to $B_2$ where the global behavior is a little stable, provided with some extreme values, which appear mainly in the high levels for all the brands. This permits to conclude that effectively the wavelet application on these marketing time series starts to point out the influence of the time scale behavior hidden in these series, and that global description, although with wavelets, is not sufficient to extract the real and the hidden structure of the series. Therefore, a deep time scale study is necessary.

To show easily the time scale variations (fluctuations, increase, decrease) of these variables and for further understanding their behavior according to the time scale, we reproduced the prices and sales for the brands $B_i$, $1 \leq i \leq 6$, graphically. This will yield in turn and among other interpretations a good and easy reading and description of these variables according to the time scale. Figures 1, 3, 5, 7, 9, 11 illustrate the wavelet decomposition of the prices for each brand at the level of decomposition $J = 6$. Besides, we provided in Figures 2, 4, 6, 8, 10, 12 the wavelet decomposition of the brands' sales at the level $J = 6$. These graphs illustrate the variables with their trends and dynamics or fluctuations. The strong fitting between each variable and its approximation is clearly noticed.

B1PricesWD.png

**Figure 1.** The wavelet decomposition of the Jarir brand $B_1$ prices at the level 6.

B1SalesWD.png

**Figure 2.** The wavelet decomposition of the Jarir brand $B_1$ sales at the level 6.

B2PricesWD.png

**Figure 3.** The wavelet decomposition of the Almarai brand $B_2$ prices at the level 6.

B2SalesWD.png

**Figure 4.** The wavelet decomposition of the Almarai brand $B_2$ sales at the level 6.

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B3PricesWD.png

**Figure 5.** The wavelet decomposition of the STC brand $B_3$ prices at the level 4.

B3SalesWD.png

**Figure 6.** The wavelet decomposition of the STC brand $B_3$ sales at the level 4.

B4PricesWD.png

**Figure 7.** The wavelet decomposition of the AlAbdullatif brand $B_4$ prices at the level 6.

B4SalesWD.png

**Figure 8.** The wavelet decomposition of the AlAbdullatif brand $B_4$ sales at the level 6.

B5PricesWD.png

**Figure 9.** The wavelet decomposition of the EIC brand $B_5$ prices at the level 6.

B5SalesWD.png

**Figure 10.** The wavelet decomposition of the EIC brand $B_5$ sales at the level 6.

B6PricesWD.png

**Figure 11.** The wavelet decomposition of the AlAseel brand $B_6$ prices at the level 6.

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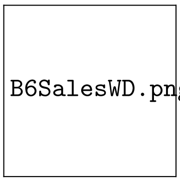


**Figure 12.** The wavelet decomposition of the AlAseel brand $B_6$ sales at the level 6.

Let us now investigate case-by-case these graphs and deduce therefore the effect of time scale on the corresponding brands' movements.

Figure 1 illustrating the wavelet decomposition of $B_1$-prices shows a somehow increasing behaviour clearly illustrated by the 6-level trend $A_6$, reminiscent of some perturbation at medium to high scales. However, the trend shows no cycles (periodic behaviour) for the $B_1$-prices. The perturbation is confirmed in the detail components $D_1$ to $D_6$. According to $D_1, D_2$ a somehow pseudo-periodicity seems to take place with some differences in the maximum prices. This may be explained by the fact that at short time scales (like weeks in the classical treatments of the time factor) prices are influenced by competitors in one hand and the global behavior of the market (National as well as international). One essential cause to be realized is the COVID-19 pandemics which covered a great part of the period of study. In which, sales are surely decreased, so that, to compensate the quantities in stocks and thus to recover losses, the company has to the increase of prices as many other cases. This fact is easily shown in the $B_1$-sales graphs in Figure 2, where we observe easily an upward of sales whenever prices decrease and a downward of sales whenever prices increase. Besides both prices and sales are volatile at high time scales according to the detail components $D_3$ to $D_6$. We may thus conclude for this brad that short time scales are more comprehensive and suitable to understand for consumers, investors, as well as managers.

According to Figure 3, a slight variation around its mean (median) with no exact periodicity, but instead a slight pseudo-periodicity structure. This behavior is repeated with the sales illustrated in figure 4. However, some anomalies appear essentially in all time scales. This behavior can be naturally understood from the fact that Almarai is a completely national company based also on completely national products. Therefore, compared to $B_1$ for example, its prices and volumes are not strongly affected by the importation. In addition, its products are the oldest, the best in the market (to the majority of consumers) and, these are daily consumed. All these causes allow prices stability in the market despite even slight growth in the volume of production. Besides, prices and sales seem to reflect the same behavior at low and high time scales. These fluctuations are clearly illustrated by the detail components $D_1$ to $D_6$s, nevertheless, the fluctuations remain slightly around the mean at all the time scales.

Figures 5 and 6 illustrate the behavior of brand $B_3$ prices and sales relative to the wavelet decomposition at the level 6. Here also, we notice a global increase in both series, which is clearly show in the trend $A_6$. This increase is sometimes interspersed by little fluctuations. This brand is one of the major destinations for communications needs in the entire Kingdom. During the COVID-19 quarantine, sales of STC have been growing up (clearly in $A_6$) due to the increase of the remote and/or distance translations in quasi all domains. Nevertheless, we notice some pseudo-periodicity at low time sales.

The investigation of Figure 7 and 8 shows a quasi-stability at short and medium time scales, followed by a global increase at higher horizons. This increase may be also due to the big demand of consumers as an alternative during the quarantine period due to COVID-19. The pseudo-periodicity and stability appearing at short and medium horizons is effectively broken down at the high time scales due to these reasons. Moreover, with the lack of importation of similar products, the consumers turned to national brands, which contributed to the increase in sales and consequently the prices. On another side, a great part of this type of sales are in fact imported (in raw, or elementary commodities). Therefore, with the economic recession afflicting the market for the long COVID-19 period, merchants have to increase the prices to compensate a part of their losses. Besides, other costumers

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and companies use many products of this brand, such as polyester and nylon. The only way in critical periods such as COVID-19 is to use the national stocks. This induces an increase in both sales and prices.

The brand $B_5$ illustrated by Figures 9 and 10 for prices and sales respectively is characterized by a small skewness, kurtosis and a quite small Std. This indicates some stability in both prices and sales reminiscent of some small fluctuations but no important extreme values. As its trend $A_6$ indicates, prices started to be decreasing in short time scales, and turned to increase in high horizons, with a slow grow-up at medium levels. Sales seems to be stable globally at short horizons, decreasing at medium levels and next increasing at higher time scales. Recall that EIC is a national manufacture of eclectic instrument. Such instruments need many raw materials in a large part imported from outside of the Kingdom. Therefore, with the COVID-19 situation and the orientation towards encouraging the national and local industry and production, one of the main goals of the 2030-vision program, these companies have known a growing- up and development in both the volumes of sales as well as prices. Nevertheless, the graphical representations of the detail components do not reflect any cycles as expected in the classical studies.

Figures 11 and 12 illustrate the time scale behavior of brand $B_6$ prices and sales according to the wavelet decomposition at the level $J-6$. It is easily noticed a global increasing of both prices and sales especially at higher scales. By investigating the detail components $D_1$ to $D_6$ we may exclude easily the idea of cycles for the concerned series especially at low/medium horizons. At higher time scales, the series become more and more volatile as for the preceding brands. The increase at the higher horizons are due to the orientation of consumers to local/national products, as there is no importation from outside. Moreover, this brand is specialized majorly in national wearing, which is a fashion that is not widely known outside the GCC countries, except some companies in China designated to GCC market. This outside companies stopped their exportation at COVID-19 period, which induced therefore the increasing activation for national wearing products. Besides, due to the 2030-vision program, these local/national brands and the producing companies are forced to implement the government directions and guidelines to develop and activate national production.

Overall, the investigation of all these graphs, especially their detail components, reduces the idea of cycles and stationarity for these marketing series, and show instead a volatile behavior, which increases with the time scale. This indicates that the market concerned is emergent, and economic laws have to be applied carefully for investors and managers. These facts may be related to many reasons. One is the COVID-19 crisis which have been positively used for some cases, especially telecommunications supplies, and national products including foods (Almarai) and wearing (Al assel), and bad situation for many brands relying on imported raw materials such as EIC. Besides, the labor resettlement, which induced an important number of non-highly qualified labor in many sectors. This fact, although it constitutes a principal goal in 2030-vision programs, it may reduce the productivity.

We estimate that more investigation of this volatile movement has to be applied by improving more the tools, such as the implementation of stochastic factors in the model and/or the experimentation non-uniform time scale models as in [60].

In the remaining we will focus on the validation of the mathematical model (8). For this aim we consider different cases of competitors for the brands applied as illustrated in Table 4 below. We recall that all these brands are distributed on the whole kingdom to all the provinces, and have besides online purchase and delivery service throughout all the Kingdom. This gives somehow the same opportunities for the distribution of their sales or products in all times and places. This is confirmed numerically as the numerical experimentation induced that $7.28e-12 \leq \beta_3 \leq 1.2e-09$ and $\beta_4 = 0$.

The investigation of Table 5 shows an important influence of the prices for both the original wavelet-less model (7), and the wavelet time scale model (8, reflected in the values of the coefficient $\beta_1$. For the brand $B_1$ for example, this coefficient is globally important

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| Brands | Top Competitor(s) |
|---|---|
| $B_1$ - Jarir Bookstore | Alobeikan ($CPB_1$) |
| $B_2$ - Almarai | Nadec ($CPB_2$) |
| $B_3$ - STC | Zain ($CPB_3$) |

**Table 4.** Some main competitors of brands $B_1$, $B_2$ and $B_3$.

| Coefficients | Model (7) | $A_6$ | L1 | L2 | L3 | L4 | L5 | L6 |
|---|---|---|---|---|---|---|---|---|
| | | | | The brand $B_1$ | | | | |
| $\beta_0$ | $-95.21$ | $-167.57$ | $-0.01$ | 0.01 | $-0.01$ | $-0.02$ | $-0.09$ | $-0.24$ |
| $\beta_1$ | 1.93 | 2.45 | 1.27 | 12.75 | 4.55 | 2.85 | $-7.13$ | $-4.82$ |
| $\beta_2$ | $-6.05$ | $-7.21$ | $-10.78$ | $-8.27$ | 15.78 | $-24.60$ | $-3.54$ | $-12.48$ |
| | | | | The brand $B_2$ | | | | |
| $\beta_0$ | $-42.76$ | $-131.92$ | $-0.06$ | 0.02 | $-0.002$ | $-0.42$ | 0.21 | 1.10 |
| $\beta_1$ | 10.31 | 12.98 | 227.21 | 166.02 | $-10.61$ | 45.20 | $-6.22$ | $-34.84$ |
| $\beta_2$ | 2.17 | 0.31 | 53.16 | $-72.86$ | $-14.65$ | 60.14 | $-6.97$ | 23.62 |
| | | | | The brand $B_3$ | | | | |
| $\beta_0$ | 139.14 | 72.38 | $-0.006$ | $-0.45$ | $-0.82$ | $-14.90$ | 35.88 | 21.80 |
| $\beta_1$ | $-5.91$ | $-7.47$ | $-590.90$ | 80.18 | $-314.70$ | 41.34 | 01.99 | 16.75 |
| $\beta_2$ | 122.77 | 139.51 | 2436.79 | $-396.59$ | 351.10 | $-394.74$ | $-980$ | $-167.01$ |

**Table 5.** Model (8) coefficients estimations for brands $B_i$, $1 \leq i \leq 6$, at different levels.

at the approximation component $A_6$, which describes the general behavior (trend) of the time series. By going in the microscopic scale, we notice that this coefficient reflects an important influence on sales over the short horizons $L1$, and becomes higher at medium to higher horizons $L2, L3, L4, L5, L6$, even being negative at the very high time scales. The competitor price however is negatively influencing the model, estimated by a negative coefficient $\beta_2$, except at the medium level $L3$ where a pick-up or an extreme value appears. An even higher effect exists at the higher horizons.

The same interpretations are also noticed for the brands $B_2$ and $B_3$, where both the prices and the competitive prices reflect important influences expressed by means of high coefficients $\beta_1$ and $\beta_2$. For the brand $B_2$ we notice higher positive influences at the short time scales for $\beta_1$, followed by a pick-down at medium level $L3$, and next becoming higher bu negative for the long term horizons. Nevertheless, the prices are globally positively influencing.

Besides, null values for $\beta_4$ suggest that no influence of the variable $D_t$. This somehow natural for the Saudi market, especially for national brands. Indeed, all the chosen brands and their associated sales executing companies own collection centers, warehouses and distribution points throughout the Kingdom. In addition, there is a feature, e-sale service, and delivery service available in all the emirates as well as all the governorates in the Kingdom, even small cities and villages. This makes the distributive property not the subject of great competition between the firms. During the quarantine period of COVID-19, the state was keen to provide and support companies to distribute their products throughout the country. Besides, in relation to the 2030-vision programs, the state plans to provide the same opportunities for the national companies to encourage the production as well as the investment in national brands and the competition. This has an important effect on the economy as well as the society as it will provide jobs for national labor, and thus effects the quality of life in the society, especially from the economic point of view.

The promotional coefficient $\beta_3$ reflects also a small value (quietly null) which suggests that the promotion factor is not highly influencing the movement of prices and sales. This finding is also reasonable in KSA, as along the whole year all these top brands apply promotional prices. Every week, promotions in different products under the same brand are announced. This is also one important point that leads to a growing-up in selling their products, and thus influencing and inducing the increase in sales especially.

In Figures 13, 14 and 15, comparisons between the brands' prices and sales for $B_1$, $B_2$ and $B_3$ with their competitive brands' $CPB_1$, $CPB_2$, $CPB_3$ respectively are illustrated graphically at the maximum level $J = 6$ of wavelet decomposition by means of the filtered prices.

B1CPB1A6Prices.png B1CPB1A6Sales.png

**Figure 13.** The wavelet approximation $A_6$ for $B_1$ and $CPB_1$ prices and sales at the level 6.

B2CPB2A6Prices.png B2CPB2A6Sales.png

**Figure 14.** The wavelet approximations $A_6$ for $B_2$ and $CPB_2$ prices and sales at the level 6.

B3CPB3A6Prices.png B3CPB3A6Sales.png

**Figure 15.** The wavelet approximation $A_6$ for $B_3$ and $CPB_3$ prices and sales at the level 6.

These last figures are in complete coherence with the results previously discussed, especially in Table 3. Indeed, we notice from Figure 13 the variation of the $B_1$ prices and sales in front of those of its competitor $CPB_1$. Prices and sales for both of them are globally increasing, with a superiority of $B_1$ prices. However, the sales present besides some oscillations, and a superiority of the competitor sales at the higher time scales. This fact may be explained by the fact that $B_1$ sales are in fact imported products from outside the kingdom, except for little types. However, the competitor Alobeikan is in a major part focusing on national printing press concerned with printing books internally, or at least imported and distributed in GCC countries. This makes its sales growing up during the COVID-19 period which represents here the long-term horizons.

The brand $B_2$ and its competitor $CPB_2$ are both concerned with national foods, and thus are not widely influenced by the COVID-19, as they use already internal products. Nevertheless the dominance of the brand $B_1$ sales and prices is clearly illustrated in Figure 14.This is also compatible with Table 3 where the corresponding coefficient $\beta_2$ is globally positive, even low for $A_6$, and perturbed (oscillating between positive and negative values) for some medium to high time scales.

Figure 15 reflects the variation of sales/prices for the brand $B_3$ and its competitor $CPB_3$. We notice easily the dominance (superiority) of the prices of the brand $B_3$ compared to its competitor prices. However, this dominance may be itself the cause to the reversed order in view of sales, where the competitor shows a superiority in the market. This may be due to the demand for the products of the second brand as its prices are within the reach of the middle class, regardless of the quality of the product. We recall that a large segment of the population is made up of foreign workers whose wages are low compared to the citizens, which makes this dominant group tend to more cheap prices for daily consumption. It should also be noted that the first label focuses on the traditional national dress of citizens more than other clothes.

## 6. Conclusion

In many cases of time series such as marketing ones, classical studies suggest non-stationarity and short time scales. Nevertheless, studies confirm the existence of time-frequency aspect. Many mathematical tools and models have been applied by researchers to explain the time scale behavior of marketing series. Wavelets are the last and the most powerful tools in such a direction because of their ability to both explain the time and frequency dependencies in the time series.

In the present work, we proposed essentially to test the impact of the time scale by applying wavelets and their supports instead of the classical methods based on classical periods applied in the field of marketing, economy, finance, .... etc. These periods are always expressed as weeks, months, and years.

Wavelets are therefore exploited to improve marketing models by projecting such models on different horizons due to the wavelet processing. These horizons reflect the time scales, and permitted a microscopic look to the situation of brands prices and sales in one of the largest economies in the world, the Saudi Arabia. The finding showed that effectively, the wavelet time scale models permit a good understanding of the brands movements especially during crisis such as the last pandemics, and in view of econo-political plans such as the 2030-vision of the kingdom.

Nevertheless, the present study showed also that even the wavelet time scale models may be improved, and many hidden structures may be discovered more by involving more tools. We suggest for the moment to include random and/or non uniform wavelets and time scales for a future extension of the existing models.

Sophisticated models will face the marketing managers, such as investors and analysts, with the necessary and useful information to fix their decisions and analyses.

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Conceptualization, T. M. Alanazi and A. Ben Mabrouk; Methodology, T. M. Alanazi and A. Ben Mabrouk; Software, T. M. Alanazi and A. Ben Mabrouk; Validation, T. M. Alanazi and A. Ben Mabrouk; Formal analysis, T. M. Alanazi and A. Ben Mabrouk; Investigation, T. M. Alanazi and A. Ben Mabrouk; Resources, T. M. Alanazi and A. Ben Mabrouk; Data curation, T. M. Alanazi and A. Ben Mabrouk; Writing—original draft preparation, T. M. Alanazi and A. Ben Mabrouk; Writing—review and editing, T. M. Alanazi and A. Ben Mabrouk; Visualization, T. M. Alanazi and A. Ben Mabrouk; Supervision, T. M. Alanazi and A. Ben Mabrouk; Project administration, T. M. Alanazi and A. Ben Mabrouk; Funding acquisition, T. M. Alanazi and A. Ben Mabrouk. All authors have read and agreed to the published version of the manuscript.

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**Conflicts of Interest:**

The authors declare no conflict of interest.

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