*Article*

# Ordered Weighted Averaging (OWA), Decision Making Under Uncertainty, and Deep Learning: How Is This All Related?

**Vladik Kreinovich**

University of Texas at El Paso; vladik@utep.edu
* Correspondence: vladik@utep.edu
† Current address: Department of Computer Science, University of Texas at El Paso, 500 W. University, El Paso, Texas 79968, USA

**Abstract:** Among many research areas to which Ron Yager contributed are decision making under uncertainty (in particular, under interval and fuzzy uncertainty) and aggregation – where he proposed, analyzed, and utilized the use of Ordered Weighted Averaging (OWA). The OWA algorithm itself provides only a specific type of data aggregation. However, it turns out that if we allows several OWA stages one after another, we get a scheme with a universal approximation property – moreover, a scheme which is perfectly equivalent to deep neural networks. In this sense, Ron Yager can be viewed as a (grand)father of deep learning. We also show that the existing schemes for decision making under uncertainty are also naturally interpretable in OWA terms.

**Keywords:** Ordered Weighted Averaging (OWA); decision making under uncertainty; deep learning

## 1. Introduction: How to Process Data

### *1.1. Need for stages of data processing*

In order to start analyzing the relation between OWA, deep learning, and decision making under uncertainty – several computer-topics – let us first recall why computers are needed in the first place. The main objective of computers is to process data. Data processing is usually performed in several stages. Which processing algorithms should we select for each stage?

At each stage of a deterministic data processing, the result $y$ is uniquely determined by the inputs $x_1, \ldots, x_n$ – this is what "deterministic" means. In mathematical terms, this means that at each stage we are computing a function of the inputs $y = f(x_1, \ldots, x_n)$. Which functions should we select?

### *1.2. Linear stages*

The simplest possible functions are linear functions

$$f(x_1, \ldots, x_n) = a_0 + a_1 \cdot x_1 + \ldots + a_n \cdot x_n.$$

They are ubiquitous in our description of the physical world, and there is a natural explanation for this ubiquity: most real-world dependencies are smooth, and in a reasonable neighborhood of each point $x^{(0)} = \left(x_1^{(0)}, \ldots, x_n^{(0)}\right)$, a smooth function can be well-approximated by a linear one:

$$f(x_1, \ldots, x_n) \approx \left(x_1^{(0)}, \ldots, x_n^{(0)}\right) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \left(x_i - x_i^{(0)}\right);$$

see, e.g., [1,9].

It is therefore reasonable to allow linear stages, where a linear functions are computed.



**Citation:** Kreinovich, V. Ordered Weighted Averaging (OWA), Decision Making Under Uncertainty, and Deep Learning: How Is This All Related?. *Preprints* **2022**, *1*, 0. https://doi.org/

Received:
Accepted:
Published:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 January 2022 doi:10.20944/preprints202201.0028.v1



### 1.3. Need for nonlinear stages

We cannot only use linear stages, since in this case, all we would be able to compute are compositions of linear functions, and it is known that a composition of linear functions is always linear – while in real life, many processes are nonlinear; see, e.g., [1,9].

Which nonlinear stages should we consider?

### 1.4. Importance of symmetries and invariance

To decide which nonlinear stages we should consider, let us recall how we acquire knowledge about the laws of the physical world in the first place.

For example, how did we learn that any object left in mid-air will fall down with the acceleration of 9.81 m/sec$^2$? Someone (actually Galileo) performed this experiment several times, and got this result. Then this person moved to a different location, and got the same result, repeated this experiment at a later time – and got the same result. In other words, this person performed shifts in space, shifts in time, rotations, etc. – and the results did not change under these transformation. So, Galileo made a natural conclusion – that this phenomenon is invariant with respect to all such transformations, so if we repeat this experiment in another country, in another century, we will still observe the same behavior.

Such an invariance is behind all physical laws. When Ohm discovered his famous law – that voltage is proportional to current – he first found out that this relation holds in his lab. But then Ohm and others saw that the same relation happens in all locations, at all moments of time – so it is indeed a universal physical law.

Physicists understand that invariances are the basis of all physical laws – see, e.g., [1,9], to the extend that in modern physics, new theories are usually proposed not in terms of differential equations – as in Newton's times – but in terms of the corresponding symmetries. Moreover, it turned out that many original physical theories, proposed first in terms of the differential equations, can actually be equivalently reformulated in terms of appropriate symmetries: this is true for Maxwell's equations that describe electromagnetism, Einstein's General Relativity equations that describe gravity, Schroedinger's equation that describes quantum phenomena, etc.; see, e.g., [2,3,6].

It is therefore reasonable to select nonlinear data processing stages which are invariant – and ideally, invariant under as many naturaol transformations as possible.

### 1.5. Which transformation are natural?

What are the natural transformations?

Some natural transformations come from the fact that we can use different measuring units to describe the same value of the physical quantity. For example, the height of 1.7 m is the same as the height of 170 cm. It is therefore reasonable to require that the result of data processing not change if we change the value of the measuring unit – i.e., if we apply the corresponding transformation $x \mapsto \lambda \cdot x$ for an appropriate value $\lambda$. It is thus reasonable to add these transformations to our list of natural transformations.

The numerical values of many quantities like temperature or time also depend on the choice of a starting point. If we take, as a starting point for measuring time, a moment $x_0$ years before the current Year 0, then all the numerical values will change from their original values $x$ to the new values $x + x_0$. So, we should also add transformations $x \mapsto x + x_0$ to our list of natural transformations.

In addition to such linear transformations, we often have nonlinear ones. For example, we can measure the strength of an earthquake by the amount of released energy or by its logarithm – known as the Richter scale. We can measure the intensity of a sound by its energy or in a logarithmic scale – in decibels. There are many other natural nonlinear transformations, the only thing they all have in common is that they are all continuous and strictly increasing: what is larger in the original scale remaind larger after the transformation. It is therefore reasonable to consider all continuous increasing transformations as natural.

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Yet another class of natural transformations corresponds to the case when several inputs $x_i$ corresponds to the values of the same type of quantity – e.g., they are all moments of time, or they are all energy values. In this case, a reasonable idea is that we can permute them and get the exact same result. For example, if we bring together several objects of weights $x_1, \ldots, x_n$, the overall weight $f(x_1, \ldots, x_n)$ should not depend on the order in which we brought these bodies together. Such permutations should therefore also be added to the list of natural transformations.

*1.6. Which functions are invariant with respect to all natural transformations?*

Now that we decided which transformations to look into, let us analyze which computational stages – i.e., which functions $y = f(x_1, \ldots, x_n)$ – are invariant with respect to all these transformations.

If all the values $x_i$ and $y$ represent different quantities – which can be independently transformed – then invariance would mean that for all continuous strictly increasing transformations $t_i(x_i)$ and $t(y)$, if we have $y = f(x_1, \ldots, x_n)$, then we should have $t(y) = f(t_1(x_1), \ldots, t_n(x_n))$. Such invariance is not possible: indeed, if we take $t_i(x_i) = x_i$ and $t(y) = y + 1$, then this would mean that $y = f(x_1, \ldots, x_n)$ implies $y + 1 = f(x_1, \ldots, x_n)$ and that $y = y + 1$ and, thus, $0 = 1$ – which is clearly impossible.

It is therefore reasonable to consider the case when all the quantities $x_i$ and $y$ represent the same quantity. In this case, invariance means that for any continuous strictly increasing function $t(x)$, if we have $y = f(x_1, \ldots, x_n)$, then we should have $t(y) = f(t(x_1), \ldots, t(x_n))$. Since all the values $x_1, \ldots, x_n$ represent the same quantity, invariance with respect to all natural transformations means also that the function $f(x_1, \ldots, x_n)$ should be invariance with respect to all permutations.

Now, we are ready to describe the class of all invariant functions.

**Definition 1.** *A continuous real-valued function $f(x_1, \ldots, x_n)$ of $n$ real variables is called* invariant *if it satisfies the following two conditions:*

- *for every continuous increasing function $t(x)$, if we have $y = f(x_1, \ldots, x_n)$, then we have $t(y) = f(t(x_1), \ldots, t(x_n))$, and*
- *for every permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and for all possible values $x_i$, we have*

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = f(x_1, \ldots, x_n).$$

There are known examples of invariant functions: namely, for each $i$ from 1 to $n$, we can take $f(x_1, \ldots, x_n) = x_{(i)}$, where $x_{(i)}$ is the $i$-th of the values $x_j$ when we sort them in the increasing order:

- $x_{(1)}$ is the smallest of $n$ values $x_1, \ldots, x_n$;
- $x_{(2)}$ is the second smallest of the values $x_1, \ldots, x_n$,
- $\ldots$, and
- $x_{(n)}$ is the largest of the values $x_1, \ldots, x_n$.

Interestingly, there are the only invariant functions:

**Proposition 1.** *A functions $f(x_1, \ldots, x_n)$ is invariant if and only it coincides with one of the functions $x_{(1)}, \ldots, x_{(n)}$.*

**Proof.** It is easy to check that all the functions $x_{(i)}$ are invariant in the sense of the above Definition. So, to complete our proof, it is sufficient to show that every invariant function is equal to one of the functions $x_{(i)}$.

Indeed, let $f(x_1, \ldots, x_n)$ be an invariant function. Let us first find what is the value $f(1, \ldots, n)$. Our claim is that this value must be equal to one of the values $1, \ldots, n$. Indeed, if the value $v \stackrel{\text{def}}{=} f(1, \ldots, n)$ was different from these $n$ numbers, then we could construct a continuous strictly increasing function $t(x)$ for which $t(1) = 1$, $\ldots$, $t(n) = n$, and $t(v) = v + \varepsilon$ for some small $\varepsilon > 0$. For example, we can form this function by using linear

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interpolation between the corresponding points. In this case, the first part of the definition of an invariant function would mean that $t(v) = v + \varepsilon = f(1, \ldots, n)$. However, since $v = f(1, \ldots, n)$, we get $v + \varepsilon = v$ and $\varepsilon = 0$ – while we know that $\varepsilon > 0$.

Thus, the value $f(1, \ldots, n)$ is indeed equal to one of the values $1, \ldots, n$. Let us denote this value by $i$, then we get $f(1, \ldots, n) = i$.

Now let us consider the case when $x_1 < x_2 < \ldots < x_i < \ldots < x_n$. In this case, we can use linear interpolation to build a continuous strictly increasing piece-wise linear function $t(x)$ that maps 1 into $x_1$, 2 into $x_2$, ..., and $n$ into $x_n$. For this transformation $t(x)$, invariance and the fact that $f(1, \ldots, n) = i$ imply that $f(x_1, \ldots, x_n) = x_i$. In this case, $x_i = x_{(i)}$, so we have $f(x_1, \ldots, x_n) = x_{(i)}$.

If we have $n$ values $x_1, \ldots, x_n$ which are all different, then we can perform a permutation $\pi$ that places these values in an increasing order $x_{(1)} < \ldots < x_{(n)}$. For this permutation, the second part of the definition of invariance implies that $f(x_1, \ldots, x_n) = f\left(x_{(1)}, \ldots, x_{(n)}\right)$. For the sorted values, we already know that $f\left(x_{(1)}, \ldots, x_{(n)}\right) = x_{(i)}$, so we conclude that

$$f(x_1, \ldots, x_n) = x_{(i)}.$$

We have almost proved the proposition, the only thing that remains is to take care of the case when some of the values $x_1, \ldots, x_n$ are equal to each other. In this case, we can change the values a little bit – e.g., permute them so that they are in non-decreasing order and then take $X_i = x_i + i \cdot \delta$ for some small $\delta > 0$. For these changed values, we have $X_1 < X_2 < \ldots < X_n$, so $f(X_1, \ldots, X_n) = X_{(i)}$. Since the function $f(x_1, \ldots, x_n)$ is continuous, in the limit $\delta \to 0$, we get the desired equality $f(x_1, \ldots, x_n) = x_{(i)}$.

The proposition is proven. □

## 2. What Is Ordered Weighted Averaging and How It Is Related to Deep Learning

### *2.1. Reminder*

In the previous section, we have shown that a reasonable way to form a computational process is to compose if of two types of stages:

- linear stages, when we compute a linear combination of the inputs

$$a_0 + a_1 \cdot x_1 + \ldots + a_n \cdot x_n,$$

  and
- nonlinear stages, when we compute the $i$-th value $x_{(i)}$ in the increasing order.

If we allow only one stage, this is all we can compute: linear combinations and $x_{(i)}$. This is not enough, we can have more complex dependencies. So, we need more than one stage.

A natural question is: what if we use two stages?

### *2.2. What if we use two stages?*

If we use two stages, then it does not make sense to have two linear stages one after another: this way, we compute the composition of two linear functions, and since this composition is also linear, this means that we can compute it in a single stage. Similarly, it does not make sense to have two non-linear stages one after another: indeed, after the first nonlinear stage, we get sorted values, so an additional sorting of these results will not add any information.

So, we have two options:

- We can first perform a non-linear stage and compute the values $x_{(i)}$, and then perform a linear stage, i.e., compute a lineqr combination of these values

$$a_0 + a_1 \cdot x_{(1)} + \ldots + a_n \cdot x_{(n)}.$$

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This is exactly Yager's Ordered Weighted Averaing (OWA); see, e.g., [10]. To be more precise, OWA is a particular case of this formula when $a_0 = 0$, $a_i \geq 0$, and

$$a_1 + \ldots + a_n = 1.$$

- Alternatively, we can first perform a linear stage and then perform a nonlinear stage. In this case, we get $i$-th element in the list of some linear combinations of inputs. As a particular case, we get the main units of deep learning (see, e.g. [4]): the Rectified Linear Unit (ReLU) $\max(0, a_0 + a_1 \cdot x_1 + \ldots + a_n \cdot x_n)$, convolution (linear transformation), and max-pooling, when we compute $\max(x_1, \ldots, x_n)$.

If we consider only two stages, we can see the relation between OWA and deep learning, but the results are clearly different. In both cases, not all functions can be represented by these 2-stage computations.

To be able to compute any function with any given accuracy – i.e., to have a universal approximation property – we therefore need to have more than two stages. And here things become interesting.

### *2.3. What if we use many stages?*

As we have mentioned, a linear stage (we will denote it by $L$) must be followed by a nonlinear stage (we will denote it by $NL$), and vice versa. Depending on which stage we start with, we have two possible sequences:

$$L \to NL \to L \to NL \ldots$$

or

$$NL \to L \to NL \to L \ldots$$

For both sequences, we can attach each $NL$ stage to the previous $L$ stage – as in deep neural networks – and get a sequence

$$(L \to NL) \to (L \to NL) \ldots$$

or

$$NL \to (L \to NL) \to (L \to NL) \ldots$$

This way, we get a general neural network – which is known to have a universal approximation property.

Alternatively, we can attach each $L$ stage to the previous $NL$ stage – as in OWA – and get a sequence

$$L \to (NL \to L) \to (NL \to L) \ldots$$

or

$$(NL \to L) \to (NL \to L) \ldots$$

In other words, the same computations as performed by a deep neural network can be interpreted as several layers of OWA operations (and linear layers).

So, whatever we can compute with deep neural networks – and we can compute many important things [4] – can also be interpreted as a multi-stage applications of OWA. In this sense, OWA can be viewed as a natural predecessor of deep neural neural networks – and Ron Yager can be viewed as a (grand)father of deep learning.

## 3. OWA and Decision Making Under Uncertainty

### *3.1. How to make a decision under interval uncertainty: reminder*

In the ideal case, when we make a decision, we know the exact benefit $u_i$ from each of the alternatives $i$. In this case, a reasonable idea is to select the alternative with the largest benefit: $u_i \to \max$.

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In practice, however, we usually do not know the exact values of the corresponding benefits. At best, for each alternative $i$, we know the bounds $\underline{u}_i$ and $\overline{u}_i$ on the actual (unknown) value of the future benefit $u_i$. In other words, all we know about the actual value $u_i$ is that this value belongs to the interval $[\underline{u}_i, \overline{u}_i]$. How can we make a decision in this situation?

Reasonable assumptions on the rational decision making imply that we should select an alternative for which the following combinations is the largest $\alpha \cdot \overline{u}_i + (1-\alpha) \cdot \underline{u}_i$, where the value $\alpha \in [0,1]$ depends on the decision maker; see, e.g., [5,7,8]. This formula was first proposed by an economist Leo Hurwicz – who later received a Nobel Prize for this – and is thus known as Hurwicz optimism-pessimism criterion. This name comes from the fact that:

- When $\alpha = 1$, then the Hurwicz combination is simply equal to $\overline{u}_i$. This means that when we make a decision, for each alternative, we only take into account the most optimistic value $\overline{u}_i$.
- When $\alpha = 0$, then the Hurwicz combination is simply equal to $\underline{u}_i$. This means that when we make a decision, for each alternative, we only take into account the most pessimistic value $\underline{u}_i$.
- When $\alpha$ is between 0 and 1, this means that take both optimistic and pessimistic values into account.

### *3.2. Hurwicz criterion is a particular case of OWA*

Let us show that the above-described Hurwicz criterion is actually a particular case of OWA.

Indeed, how do we find the bounds $\underline{u}_i$ and $\overline{u}_i$? We usually consider several possible scenarios $s = 1, 2, \ldots, S$, and for each alternative decision $i$ and for each scenario $s$, we compute the resulting gain $u_{is}$. Then:

- we compute $\overline{u}_i$ as $\max(u_{i1}, u_{i2}, \ldots, u_{iS})$, and
- we compute $\underline{u}_i$ as $\min(u_{i1}, u_{i2}, \ldots, u_{iS})$.

In these terms, Hurwicz optimism-pessimism criterion means selecting the alternative with the largest possible value of the quantity

$$\alpha \cdot \max(u_{i1}, u_{i2}, \ldots, u_{iS}) + (1-\alpha) \cdot \min(u_{i1}, u_{i2}, \ldots, u_{iS}).$$

With respect to the inputs $u_{i1}, u_{i2}, \ldots, u_{iS}$, this formula takes exactly the OWA form

$$\alpha \cdot u_{i(S)} + (1-\alpha) \cdot u_{i(1)}.$$

(By the way, in this case, as in the original OWA, the weights $\alpha$ and $1-\alpha$ are both non-negative, and their sum is 1.)

**Funding:** This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes), and by the AT&T Fellowship in Information Technology.

It was also supported by the program of the development of the Scientific Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

**Acknowledgments:** The author is greatly thankful to Prof. Dr. Luis Martínez López, Dr. Lesheng Jin, Dr. Zhen-Song Chen, and Prof. Dr. Humberto Bustince, editors of this special issue, for their invitation, encouragement, and help.

**Conflicts of Interest:** The author declares no conflict of interest.

 doi:10.20944/preprints202201.0028.v1



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 doi:10.20944/preprints202201.0028.v1