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DERIVING VON KOCH'S INEQUALITY WITHOUT USING THE RIEMANN HYPOTHESIS.

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ABSTRACT. In this paper, we derive Von Koch's inequality without using the Riemann Hypothesis. As well known, Von Koch's inequality is equivalent to the Riemann Hypothesis. Namely, we show that the Riemann Hypothesis is true. Furthermore, we show that the upper limit of its can be lowered.

1. DERIVING VON KOCH'S INEQUALITY WITHOUT USING THE RIEMANN HYPOTHESIS.

1.1. **Introduction.**

The purpose of this paper is to derive that Von Koch's inequality without using the Riemann Hypothesis. As well known, Von Koch's inequality is equivalent to the Riemann Hypothesis. Namely, we show that the Riemann Hypothesis is true. The proof steps are as follows :

1.1.1. First, we first define for the constant $\alpha > 0$ and the variable $x > 1$ the following equation $R_\alpha^+(x)$:

$$R_\alpha^+(x) := \frac{\sqrt{2\pi}\alpha}{ex+1}, \tag{1.1}$$

where the constant α and the variable x are positive real numbers.

Using the function $R_\alpha^+(x)$, we prove the followings inequalities :

$$\mathrm{li}(x) \geq xR_\alpha^+(\log(x)), \tag{1.2}$$

and there exist a constant $C > 1(\in \mathbb{R})$ such that for all sufficiently large $x > 1$ the following inequality is satisfied :

$$|\pi(x) - \mathrm{li}(x)| \leq C(\mathrm{li}(x) - \frac{x}{\log(x)}). \tag{1.3}$$

1.1.2. Second, we define that for the constant $\alpha > 0$ and the variable x the distribution function :

$$\Pi_\alpha^+(x) := x(\exp(\frac{1}{\log(x)R_\alpha^+(\log(x))})). \tag{1.4}$$

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Thereby, we derive that for the constant α and sufficiently large $x > 1$, the following inequalities are satisfied :

$$\frac{x}{\log(x)} \leq \Pi^{+}_{\alpha}(x), \tag{1.5}$$

$$x\left(\frac{1}{\log(x)}\right)' \leq x\left(\frac{\Pi^{+}_{\alpha}(x)}{x}\right)'. \tag{1.6}$$

1.1.3. Third, using these results above, we derive that for the constant $\alpha > 0$ there exist a constant $C > 1(\in \mathbb{R})$ such that for all sufficiently large real number $x \geq 2(\in \mathbb{R})$, the following inequality is satisfied :

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{\sqrt{2\pi}\alpha}{48})^{\frac{1}{4}} \exp(\frac{e}{\sqrt{2\pi}\alpha})x^{\frac{1}{\sqrt{2\pi}\alpha}}(\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}). \tag{1.7}$$

Therefore, applying the variable α to $2/\sqrt{2\pi}$, that is,

$$\alpha := \frac{2}{\sqrt{2\pi}}, \tag{1.8}$$

we examine that the derivation of Von Koch's inequality as follows :

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{1/2}\log(x)). \tag{1.9}$$

Namely, we show that the Riemann Hypothesis is true.

1.1.4. Fourth, additionally, applying the variable α to $e/\sqrt{2\pi}$, that is,

$$\alpha := \frac{e}{\sqrt{2\pi}}, \tag{1.10}$$

we obtain the following result :

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{1/e}\log(x)). \tag{1.11}$$

We further raise the following problem :

$$e = \sup\{d > 1 \mid |\pi(x) - \mathrm{li}(x)| \leq O(x^{1/d}\log(x))\}. \tag{1.12}$$

We give details of the discussions as follows.

1.2. **Definition of functions and symbols.**
We first give some definitions and symbols used in this paper as follows.

Definition 1.1. *Let $x > 1$ be a positive real number ($x \in \mathbb{R}$).*

$$\pi(x) := \sum_{\substack{p \leq x \\ p:prime}} 1 \tag{1.13}$$

: *the number of prime numbers less than or equal to x.*

$$\log(x) := \log_e(x) \tag{1.14}$$

: $\log$ *is as natural logarithm, e is Napier number.*

$$\mathrm{li}(x) := \lim_{\epsilon \to 0}\left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^{x}\right)\frac{dt}{\log(t)} \tag{1.15}$$

: *logarithmic integral function.*

□

Definition 1.2. *Let the constant $\alpha > 0$ be a positive real number ($\in \mathbb{R}$). For all positive real number $x > 1(\in \mathbb{R})$, the functions $R^{+}_{\alpha}(x)$ and $R^{-}_{\alpha}(x)$ are defined as follows:*

$$R^{+}_{\alpha}(x) := \frac{\sqrt{2\pi}\alpha}{ex+1}, \tag{1.16}$$

$$R^{-}_{\alpha}(x) := \frac{\sqrt{2\pi}\alpha}{ex-1}. \tag{1.17}$$

The function $R^{\pm}_{\alpha}(x)$ are combined $R^{+}_{\alpha}(x)$ and $R^{-}_{\alpha}(x)$ as follows:

$$R^{\pm}_{\alpha}(x) := \frac{\sqrt{2\pi}\alpha}{ex \pm 1}. \tag{1.18}$$

Therefore, the following equation is satisfied: (Refer to Feynman[2])

$$\frac{1}{xR^{\pm}_{\alpha}(x)} = \frac{e}{\sqrt{2\pi}\alpha}(1 \pm \frac{1}{ex}). \tag{1.19}$$

□

Lemma 1.3. *Let the constant $\alpha > 0$ be a positive real number ($\in \mathbb{R}$). For all real number $x > 1(\in \mathbb{R})$, the following condition is satisfied :*

$$\mathrm{li}(x) \geq x\left(\frac{\sqrt{2\pi}\alpha}{e\log(x)}\right) > x\left(\frac{\sqrt{2\pi}\alpha}{e\log(x)+1}\right) = xR^{+}_{\alpha}(\log(x)), \tag{1.20}$$

where the constant α is satisfied as follows :

$$\alpha \leq \frac{e}{\sqrt{2\pi}}, \quad (\alpha, x \in \mathbb{R}). \tag{1.21}$$

Proof. Using Definition(1.2) and apply integration by parts of the logarithmic integral function $\mathrm{li}(x)$. There is a positive integer $n \geq 1$ such that for all real number $x > 1$, the following inequalities are satisfied :

$$\begin{aligned}
\mathrm{li}(x) &= \lim_{\epsilon \to 0}\left(\int_{0}^{1-\epsilon} + \int_{1+\epsilon}^{x}\right)\frac{dt}{\log(t)} \\
&= \frac{0!x}{\log(x)} + \frac{1!x}{(\log(x))^2} + \cdots + \frac{(n-1)!x}{(\log(x))^n} + \lim_{\epsilon \to 0}\left(\int_{0}^{1-\epsilon} + \int_{1+\epsilon}^{x}\right)\frac{ndt}{(\log(t))^{n+1}} \\
&\geq x\left(\frac{1}{\log(x)} + \frac{1}{(\log(x))^2} + \frac{2}{(\log(x))^3} + \frac{6}{(\log(x))^4} + \cdots + \frac{(n-1)!}{(\log(x))^n}\right) \\
&\geq x\left(\frac{\sqrt{2\pi}\alpha}{e\log(x)}\right) \\
&\geq x\left(\frac{\sqrt{2\pi}\alpha}{e\log(x)+1}\right) = xR^{+}_{\alpha}(\log(x)).
\end{aligned} \tag{1.22}$$

Therefore, the following conditions are satisfied :

$$\mathrm{li}(x) \geq x\left(\frac{\sqrt{2\pi}\alpha}{e\log(x)}\right) > x\left(\frac{\sqrt{2\pi}\alpha}{e\log(x)+1}\right) = xR^{+}_{\alpha}(\log(x)). \tag{1.23}$$

□

1.3. The upper and the lower bound of $\pi(x)$.

We know as the result that the upper and the lower bound of $\pi(x)$ exists. (Lemma1.4) Using this result, we show that the upper bound of $|\pi(x)-\mathrm{li}(x)|$ exists. (Lemma1.5)

Lemma 1.4. *There exists positive real numbers $C_1 > 1$ and $1 > C_2 > 0$ such that for all positive real number $x \geq 2$, the following conditions are satisfied :*

$$C_2 \frac{x}{\log(x)} < \pi(x) < C_1 \frac{x}{\log(x)}, \quad (C_1, C_2, x \in \mathbb{R}). \tag{1.24}$$

(Refer to Wladyslaw [1]*)*

□

Lemma 1.5. *There exist a real number $C > 1 (\in \mathbb{R})$ such that for all sufficiently large real number $x > 2 (\in \mathbb{R})$, the following inequality is satisfied :*

$$|\pi(x) - \mathrm{li}(x)| \leq C\left(\mathrm{li}(x) - \frac{x}{\log(x)}\right). \tag{1.25}$$

Proof. The proof is described in the following subsection 2.1.

□

1.4. Definition of distribution functions.

On this subsection, we define distribution functions and prove some lemma for preparing to prove the main theorem(1.9).

Definition 1.6. *Let the constant $\alpha > 0$ be a positive real number $(\in \mathbb{R})$. For all real number $x > 1 (\in \mathbb{R})$, the distribution functions $\Pi^+_\alpha(x)$ and $\Pi^-_\alpha(x)$ are defined as follows:*

$$\Pi^+_\alpha(x) := x \exp\left(\frac{1}{\log(x) R^+_\alpha(\log(x))}\right), \tag{1.26}$$

$$\Pi^-_\alpha(x) := x \exp\left(\frac{1}{\log(x) R^-_\alpha(\log(x))}\right). \tag{1.27}$$

The distribution function $\Pi^\pm_\alpha(x)$ are combined and described by $\Pi^+_\alpha(x)$ and $\Pi^-_\alpha(x)$ as follows:

$$\Pi^\pm_\alpha(x) := x \exp\left(\frac{1}{\log(x) R^\pm_\alpha(\log(x))}\right). \tag{1.28}$$

□

Lemma 1.7. *Let the constant $\alpha > 0$ be a positive real number $(\in \mathbb{R})$. For all sufficiently large real number $x > 1 (\in \mathbb{R})$, the following condition is satisfied :*

$$\frac{x}{\log(x)} \leq \Pi^\pm_\alpha(x). \tag{1.29}$$

Proof. For all sufficiently large real number $x > 1$, the following conditions are satisfied :

$$\frac{1}{\log(x)} \leq \exp\left(\frac{e}{\sqrt{2\pi\alpha}}(1 \pm \frac{1}{e\log(x)})\right) = \exp\left(\frac{1}{\log(x) R^\pm_\alpha(\log(x))}\right). \tag{1.30}$$

Therefore, the following condition is satisfied :

$$\frac{x}{\log(x)} \leq x \exp\left(\frac{1}{\log(x) R^\pm_\alpha(\log(x))}\right). \tag{1.31}$$

□

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Lemma 1.8. *Let the constant* $\alpha > 0$ *be a positive real number* $(\in \mathbb{R})$*. For all sufficiently large real number* $x > 1(\in \mathbb{R})$*,*

$$x\left(\frac{1}{\log(x)}\right)' \leq x\left(\frac{\Pi^{+}_{\alpha}(x)}{x}\right)', \tag{1.32}$$

where the constant α *is satisfied as follows:*

$$1 \leq \frac{1}{\sqrt{2\pi\alpha}} \exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right), \quad (that\ is,\ \alpha \leq \frac{e}{\sqrt{2\pi}}). \tag{1.33}$$

Proof. For all sufficiently large $x > 1$, the following equations are satisfied :

$$x\left(\frac{1}{\log(x)}\right)' = x\left(\frac{1}{x(\log(x))^2}\right), \tag{1.34}$$

$$\begin{aligned} x\left(\frac{\Pi^{+}_{\alpha}(x)}{x}\right)' &= x\left(\exp\left(\frac{e}{\sqrt{2\pi\alpha}}(1+\frac{1}{e\log(x)})\right)\right)' \\ &= x\left(\frac{1}{\sqrt{2\pi\alpha}}\frac{1}{x(\log(x))^2}\exp\left(\frac{e}{\sqrt{2\pi\alpha}}(1+\frac{1}{e\log(x)})\right)\right). \end{aligned} \tag{1.35}$$

The right side of each equations(1.34) and (1.35) are divided by $1/(\log(x))^2$. Hence, the following inequality is satisfied :

$$1 \leq \frac{1}{\sqrt{2\pi\alpha}} \exp\left(\frac{e}{\sqrt{2\pi\alpha}}(1+\frac{1}{e\log(x)})\right). \tag{1.36}$$

Therefore, the following condition is satisfied :

$$x\left(\frac{1}{\log(x)}\right)' \leq x\left(\frac{\Pi^{+}_{\alpha}(x)}{x}\right)'. \tag{1.37}$$

□

1.5. Inequalities for evaluating the number of prime numbers.

We show that the following Theorem(1.9) is satisfied using the result of Lemma(1.5) is satisfied.

Theorem 1.9. *Inequalities for evaluating the number of prime numbers (1).*
Let the constant $\alpha > 0$ *be a positive real number* $(\in \mathbb{R})$*. There exist a positive real number* $C > 1(\in \mathbb{R})$ *such that for all sufficiently large real number* $x \geq 2(\in \mathbb{R})$*, the following conditions are satisfied :*

$$|\pi(x) - \mathrm{li}(x)| \leq C\left(\frac{\sqrt{2\pi\alpha}}{48}\right)^{\frac{1}{4}} \exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right) x^{\frac{1}{\sqrt{2\pi\alpha}}} \left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp\left(\frac{1}{\log(x)}\right), \tag{1.38}$$

where the constant $\alpha > 0$ *is satisfied as follows :*

$$1 \leq \frac{1}{\sqrt{2\pi\alpha}} \exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right), \quad (that\ is,\ \alpha \leq \frac{e}{\sqrt{2\pi}}), \tag{1.39}$$

$$\frac{1}{\sqrt{2\pi}} \leq \alpha \leq C\frac{e}{\sqrt{2\pi}}, \tag{1.40}$$

$$\exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right) = \lim_{x\to\infty} \exp\left(\frac{1}{xR^{\pm}_{\alpha}(x)}\right). \tag{1.41}$$

Proof. Using Lemma(1.5), this theorem is proof. The proof is described in the following subsection 2.2. □

Corollary 1.10. *Inequalities for evaluating the number of prime numbers (2). There exist a positive real number $C > 1(\in \mathbb{R})$ such that for all $\epsilon > 0(\in \mathbb{R})$ and for all sufficiently large $x \geq 2(\in \mathbb{R})$, the following conditions are satisfied:*

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{48})^{\frac{1}{4}} \exp{(e)} x (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}). \tag{1.42}$$

Proof. Using Theorem(1.9), put the constant $\alpha > 0$ as follows:

$$\alpha = \frac{1}{\sqrt{2\pi}}. \tag{1.43}$$

Therefore, the inequality(1.42) are satisfied. □

The result of Corollary(1.10) is similar to the following result : (Refer to Wladyslaw[1])

$$(\exists C > 0)|\pi(x) - \mathrm{li}(x)| \leq O(x \exp(-C\sqrt{log(x)})). \tag{1.44}$$

Comparing inequalities(1.42) and (1.44), the following condition is satisfied :

$$O\Big(x(\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)})\Big) \leq O(x \exp(-C\sqrt{log(x)})). \tag{1.45}$$

Namely, the asymptotic of (1.42) gives better than that of (1.44).

Therefore, put $\alpha = 2/\sqrt{2\pi}$. the theorem above are satisfied as follows :

Corollary 1.11. *Inequalities for evaluating the number of prime numbers (3). There exist a positive real number $C > 1(\in \mathbb{R})$ such that for all $\epsilon > 0(\in \mathbb{R})$ and for all sufficiently large $x \geq 2(\in \mathbb{R})$, the following condition is satisfied:*

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{24})^{\frac{1}{4}} \exp{(\frac{e}{2})} x^{\frac{1}{2}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}). \tag{1.46}$$

Proof. Using Theorem(1.9) and the following conditions are satisfied:

$$1 \leq \frac{1}{\sqrt{2\pi}\alpha} \exp{(\frac{e}{\sqrt{2\pi}\alpha})}. \tag{1.47}$$

Put the constant $\alpha > 0$ as follows:

$$\alpha = \frac{2}{\sqrt{2\pi}} \; (\geq \frac{1}{\sqrt{2\pi}}). \tag{1.48}$$

Hence, the following inequalities is satisfied :

$$\begin{aligned} 1 &\leq \frac{1}{\sqrt{2\pi}\alpha} \exp{(\frac{e}{\sqrt{2\pi}\alpha})} \\ &= \frac{1}{\sqrt{2\pi}\frac{2}{\sqrt{2\pi}}} \exp{(\frac{e}{\sqrt{2\pi}\frac{2}{\sqrt{2\pi}}})} \quad (\because \alpha = \frac{2}{\sqrt{2\pi}}) \\ &= \frac{1}{2} \exp{(\frac{e}{2})}, \quad (= 1.946424 \cdots). \end{aligned} \tag{1.49}$$

The constant $\alpha > 0$ is satisfied the above conditions of (1.47) and (1.48). Therefore, there exist a positive real number $C > 1$ such that for all sufficiently large $x \geq 2$, the following condition is satisfied :

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{24})^{\frac{1}{4}} \exp{(\frac{e}{2})} x^{\frac{1}{2}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}). \tag{1.50}$$

□

Furthermore, we obtain the following conclusion.

Corollary 1.12. *Von Koch's inequality*

$$(\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2)\ |\pi(x) - \mathrm{li}(x)| \leq C x^{\frac{1}{2}} \log(x), \tag{1.51}$$

where $C, \epsilon, x \in \mathbb{R}$. *Namely,*

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{2}} \log(x)). \tag{1.52}$$

Proof. Fixed $\epsilon > 0$. For all sufficient large $x \geq 2$, the following conditions are satisfied :

$$\left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp(\frac{1}{\log(x)}) < \log(x) < x^{\epsilon}. \tag{1.53}$$

Therefore,there exist a positive real number $C > 0$ such that for all sufficiently large $x \geq 2$, the following inequalities are satisfied :

$$\begin{aligned} &|\pi(x) - \mathrm{li}(x)| \\ &\leq C(\frac{1}{24})^{\frac{1}{4}} \exp{(\frac{e}{2})} x^{\frac{1}{2}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}) \\ &\leq C x^{\frac{1}{2}} \log(x). \end{aligned} \tag{1.54}$$

□

Furthermore, Let $\alpha = e/\sqrt{2\pi}$. the following inequality is satisfied :

Corollary 1.13. *The reduction of the upper limit of the inequality that evaluate the number of prime number.*

$$(\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2)\ |\pi(x) - \mathrm{li}(x)| \leq C x^{\frac{1}{e}} \log(x), \tag{1.55}$$

where $C, \epsilon, x \in \mathbb{R}$. *Namely,*

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{1/e} \log(x)). \tag{1.56}$$

Proof. Using theorem(1.9), put the constant $\alpha > 0$ as follows:

$$\alpha = \frac{e}{\sqrt{2\pi}} (\geq \frac{e}{\sqrt{2\pi}}). \tag{1.57}$$

Thus, the following conditions are satisfied :

$$\begin{aligned} 1 &\leq \frac{1}{\sqrt{2\pi}\alpha} \exp{(\frac{e}{\sqrt{2\pi}\alpha})} \\ &= \frac{1}{\sqrt{2\pi}\frac{e}{\sqrt{2\pi}}} \exp{(\frac{e}{\sqrt{2\pi}\frac{e}{\sqrt{2\pi}}})} \quad (\because \alpha = \frac{e}{\sqrt{2\pi}}) \\ &= \frac{1}{e} \exp{(1)}, \quad (= 1). \end{aligned} \tag{1.58}$$

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Therefore the following condition is satisfied :

$$\begin{aligned}
&|\pi(x) - \mathrm{li}(x)| \\
&\leq C(\frac{e}{48})^{\frac{1}{4}} \exp(1) x^{\frac{1}{e}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}) \\
&\leq C x^{\frac{1}{e}} \log(x).
\end{aligned} \tag{1.59}$$

□

Hereby, we have an problem whether the upper limit of the real number d that satisfies the formula

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{1/d} \log(x)) \tag{1.60}$$

is the Napier number e or not. Namely, the following problem can be considered.

Problem 1.14. *The upper limit of the inequality that evaluate the number of prime number.*

$$e = \sup\{d > 1 \mid (\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2)\ |\pi(x) - \mathrm{li}(x)| \leq C x^{\frac{1}{d}} \log(x)\}, \tag{1.61}$$

where $C, \epsilon, x \in \mathbb{R}$. *Namely,*

$$e = \sup\{d > 1 \mid |\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{d}} \log(x))\}. \tag{1.62}$$

□

We expect the problem(1.14) to be correct. In future, we try to solve this problem.

2. The Proof of the discussion above.

On this section, we first give the proof of Lemma(1.5) and second Theorem(1.9) as follows.

2.1. Proof of Lemma(1.5).

Proof. For sufficiently large $x > 2$, the sizes of the number of prime numbers $\pi(x)$ and logarithmic integral function $\mathrm{li}(x)$ are switched infinitely. Namely, the positive and negative of the difference $\mathrm{li}(x) - \pi(x)$ are switched infinitely. (Refer to Wladyslaw [1]) Therefore, we consider the following two cases Case 1) and Case 2) separately :

$$\text{Case 1) : } \pi(x) \leq \mathrm{li}(x), \tag{2.1}$$

$$\text{Case 2) : } \mathrm{li}(x) < \pi(x). \tag{2.2}$$

We first discuss the Case 1) and second the Case 2).

Case 1) : When the inequality $\pi(x) \leq \mathrm{li}(x)$ is satisfied :
We show that there exist a real number $C > 0$ such that for all real number $x > 2$ the following condition is satisfied :

$$\mathrm{li}(x) - \pi(x) \leq C\left(\mathrm{li}(x) - \frac{x}{\log(x)}\right). \tag{2.3}$$

Because, using Lemma(1.4), there exist a real number $C_1 > 1$ such that the following conditions are satisfied :

$$\mathrm{li}(x) < C_1 \mathrm{li}(x), \tag{2.4}$$

$$\pi(x) < C_1 \frac{x}{\log(x)}. \tag{2.5}$$

Subtracting both side of inequality(2.5) from that of inequality(2.4), for all real number $x > 2$ the following conditions are satisfied :

$$\mathrm{li}(x) - \pi(x) \leq C_1 \left(\mathrm{li}(x) - \frac{x}{\log(x)} \right). \tag{2.6}$$

Therefore, put $C = C_1$, the following inequality is satisfied :

$$|\pi(x) - \mathrm{li}(x)| \leq C \left(\mathrm{li}(x) - \frac{x}{\log(x)} \right). \tag{2.7}$$

The end of the Case 1) proof. We next discuss the Case 2) as follows.

Case 2) : When the inequality $\mathrm{li}(x) < \pi(x)$ is satisfied :
Assume that the inequalitty (1.25) is not satisfied (reductio ad absurdum). Namely, assume that for all $C > 0$, there exist a sufficiently large real number $x > 2$, the following condition is satisfied :

$$\pi(x) - \mathrm{li}(x) > C \left(\mathrm{li}(x) - \frac{x}{\log(x)} \right). \tag{2.8}$$

Using Lemma(1.3) and Lemma(1.4), the following conditions are satisfied :

$$\pi(x) < C_1 \frac{x}{\log(x)} \quad (\because \text{Lemma}(1.4)), \tag{2.9}$$

$$x \left(\frac{\sqrt{2\pi}\alpha}{e \log(x)} \right) < \mathrm{li}(x) \quad (\because \text{Lemma}(1.3)), \tag{2.10}$$

where the constant α is satisfied as follows :

$$0 < \alpha \leq \frac{e}{\sqrt{2\pi}}, \quad (\alpha \in \mathbb{R}). \tag{2.11}$$

Adding the both side of inequalities(2.9) and (2.10), thus the following condition is satisfied :

$$\pi(x) + x \left(\frac{\sqrt{2\pi}\alpha}{e \log(x)} \right) < C_1 \frac{x}{\log(x)} + \mathrm{li}(x). \tag{2.12}$$

Transform the inequality (2.12) above, the following condition is satisfied :

$$\pi(x) - \mathrm{li}(x) < C_1 \frac{x}{\log(x)} - x \left(\frac{\sqrt{2\pi}\alpha}{e \log(x)} \right). \tag{2.13}$$

Hence, for sufficiently large $x (\geq 2)$, the following inequalities are satisfied :

$$\begin{aligned} \pi(x) - \mathrm{li}(x) &< C_1 \frac{x}{\log(x)} - x \left(\frac{\sqrt{2\pi}\alpha}{e \log(x)} \right) \\ &< \left(C_1 - \frac{\sqrt{2\pi}\alpha}{e} \right) \frac{x}{\log(x)}. \end{aligned} \tag{2.14}$$

Therefore, the following conditions are satisfied:

$$\pi(x) - \mathrm{li}(x) < \left(C_1 - \frac{\sqrt{2\pi}\alpha}{e}\right)\frac{x}{\log(x)}. \tag{2.15}$$

Because the inequality

$$(C_1 - \frac{\sqrt{2\pi}\alpha}{e}) > 0 \tag{2.16}$$

is satisfied, thus the following inequality is satisfied :

$$C_1 > \frac{\sqrt{2\pi}\alpha}{e}. \tag{2.17}$$

Using the assumption(2.8) and the inequality(2.15), the following inequalities are satisfied :

$$0 < C\left(\mathrm{li}(x) - \frac{x}{\log(x)}\right) < \pi(x) - \mathrm{li}(x), \tag{2.18}$$

$$\pi(x) - \mathrm{li}(x) < \left(C_1 - \frac{\sqrt{2\pi}\alpha}{e}\right)\frac{x}{\log(x)}. \tag{2.19}$$

Using two inequalities(2.18) and (2.19) above, the following inequality is satisfied :

$$C\left(\mathrm{li}(x) - \frac{x}{\log(x)}\right) < \left(C_1 - \frac{\sqrt{2\pi}\alpha}{e}\right)\frac{x}{\log(x)}. \tag{2.20}$$

According to the assumption(2.8), the positive real number $C > 1$ can be choose arbitrary. Hence, on inequality(2.20) above, choose C as follow :

$$C = \frac{\sqrt{2\pi}\alpha}{e}. \tag{2.21}$$

Thus, the following condition is satisfied :

$$\frac{\sqrt{2\pi}\alpha}{e}\left(\mathrm{li}(x) - \frac{x}{\log(x)}\right) < \left(C_1 - \frac{\sqrt{2\pi}\alpha}{e}\right)\frac{x}{\log(x)}. \tag{2.22}$$

Hence, the following inequality is satisfied :

$$\frac{\sqrt{2\pi}\alpha}{e}\mathrm{li}(x) < C_1\frac{x}{\log(x)}. \tag{2.23}$$

The both sides of inequality(2.23) is divided by $\sqrt{2\pi}\alpha/e$. Therefore, for the constant $\alpha > 0$ such that $C_1 > \sqrt{2\pi}\alpha/e$, the following inequality is satisfied :

$$\mathrm{li}(x) < C_1\left(\frac{e}{\sqrt{2\pi}\alpha}\frac{x}{\log(x)}\right). \tag{2.24}$$

Namely, using inequality(2.24), for the constant $\alpha > 0$ such that $C_1 > \sqrt{2\pi}\alpha/e$, there exists sufficiently large $x \geq 2$, the following condition is satisfied :

$$\mathrm{li}(x) \leq \frac{x}{\log(x)}, \quad (\because\ C_1 > \frac{\sqrt{2\pi}\alpha}{e}). \tag{2.25}$$

However, the inequality(2.25) contradicts the fact that for sufficiently large $x > 2$, the following inequality is satisfied :

$$\frac{x}{\log(x)} < \mathrm{li}(x). \tag{2.26}$$

Namely, the assumption(2.8) is inconsistent. Thus, there exist a real number $C > 1(\in \mathbb{R})$ such that for all sufficiently large real number $x > 2(\in \mathbb{R})$, the following inequality is satisfied:

$$\pi(x) - \mathrm{li}(x) \leq C\Big(\mathrm{li}(x) - \frac{x}{\log(x)}\Big). \tag{2.27}$$

The end of the Case 2) proof.

Consequently, using the discussion Case 1) and Case 2) above, there exist a positive real number $C > 1$ such that for all positive real number $x > 2$, the following condition is satisfied :

$$|\pi(x) - \mathrm{li}(x)| \leq C\Big(\mathrm{li}(x) - \frac{x}{\log(x)}\Big). \tag{2.28}$$

□

2.2. Proof of Theorem(1.9).

Proof. Using Lemma(1.5) and Lemma(1.8), for the constant $\alpha > 0$, there exist a positive real number $C > 1$ such that for all sufficiently large $x \geq 2$, the following formulas are satisfied :

$$\begin{aligned}
&|\pi(x) - \mathrm{li}(x)| \\
&\leq C(\mathrm{li}(x) - \frac{x}{\log(x)}) \quad (\because \text{Lemma}(1.5)) \\
&= C\Big(\lim_{\epsilon\to 0}\Big(\int_0^{1-\epsilon} + \int_{1+\epsilon}^{x}\Big)\frac{dt}{\log(t)} - \Big(\Big[\frac{t}{\log(t)}\Big]_0^{1-\epsilon} + \Big[\frac{t}{\log(t)}\Big]_{1+\epsilon}^{x}\Big)\Big) \\
&= C\Big|\lim_{\epsilon\to 0}\Big(\int_0^{1-\epsilon} + \int_{1+\epsilon}^{x}\Big)t\Big(\frac{1}{\log(t)}\Big)' dt\Big| \quad (\because \text{Partial Integral Formula}) \\
&\leq C\Big|\lim_{\epsilon\to 0}\Big(\int_0^{1-\epsilon} + \int_{1+\epsilon}^{x}\Big)t\Big(\frac{\Pi_\alpha^+(t)}{t}\Big)' dt\Big| \quad (\because \text{Lemma}(1.8)) \\
&= C\Big|\int_0^{x} t\Big(\frac{\Pi_\alpha^+(t)}{t}\Big)' dt\Big| \\
&= C\Big|\int_0^{x} t\Big(\exp\big(\frac{e}{\sqrt{2\pi}\alpha}(1 + \frac{1}{e\log(t)})\big)\Big)' dt\Big| \\
&= C\int_0^{x} t\Big(\frac{1}{\sqrt{2\pi}\alpha}\frac{1}{t(\log(t))^2}\exp\big(\frac{e}{\sqrt{2\pi}\alpha}(1 + \frac{1}{e\log(t)})\big)\Big)dt \\
&= C\frac{1}{\sqrt{2\pi}\alpha}\exp\big(\frac{e}{\sqrt{2\pi}\alpha}\big)\int_0^{x}\frac{1}{(\log(t))^2}\big(\exp\big(\frac{1}{\sqrt{2\pi}\alpha}\frac{1}{\log(t)}\big)\big)dt.
\end{aligned} \tag{2.29}$$

Put the positive real number $a > 0(\in \mathbb{R})$ as follows :

$$a := \frac{1}{\sqrt{2\pi}\alpha}. \tag{2.30}$$

Using the condition of inequality(1.40), that is,

$$\frac{1}{\sqrt{2\pi}} \leq \alpha, \tag{2.31}$$

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the positive real number $a > 0$ is satisfied as follows :

$$a \leq 1. \tag{2.32}$$

Put the variable v as follows :

$$v := \frac{a}{\log(t)}. \tag{2.33}$$

Hence, the following conditions are satisfied :

$$dv = \frac{-a dt}{t(\log(t))^2}, \tag{2.34}$$

$$t = \exp(\frac{a}{v}). \tag{2.35}$$

Apply change of variables to the formula(2.29) as follows :

$$\begin{aligned} &\int_0^x \frac{1}{(\log(t))^2} (\exp(\frac{1}{\sqrt{2\pi}\alpha}\frac{1}{\log(t)}))dt \\ &= \frac{1}{a}\int_{-\infty}^{\frac{a}{\log(x)}} \left| -\exp(\frac{a}{v})\exp(v)\right| dv \\ &= \frac{1}{a}\int_{-\infty}^{\frac{a}{\log(x)}} \exp(\frac{a}{v})\exp(v)dv \\ &\leq \frac{1}{a}\int_{-\infty}^{\frac{1}{\log(x)}} \exp(\frac{a}{v})\exp(v)dv \qquad (\because \frac{a}{\log(x)} \leq \frac{1}{\log(x)}) \\ &\leq \frac{1}{a}\left(\int_{-\infty}^{\frac{1}{\log(x)}} \exp(\frac{2a}{v})dv\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\frac{1}{\log(x)}} \exp(2v)dv\right)^{\frac{1}{2}}. \\ &(\because \text{Schwarz inequality}) \end{aligned} \tag{2.36}$$

Next, set the variable u as follows :

$$u := \frac{1}{v}. \tag{2.37}$$

Hence, the following equation is satisfied :

$$dv = \frac{-du}{u^2}. \tag{2.38}$$

Apply to change of variables the inside of first bracket on the formula(2.36) as follows :

$$\begin{aligned} &\int_{-\infty}^{\frac{1}{\log(x)}} \exp(\frac{2a}{v})dv \\ &= \int_0^{\log(x)} \left|\exp(2au)(-\frac{1}{u^2})\right| du \\ &\leq \left(\int_0^{\log(x)} \exp(4au)du\right)^{\frac{1}{2}} \left(\int_0^{\log(x)} \left|\frac{1}{u^4}\right| du\right)^{\frac{1}{2}} \\ &(\because \text{Schwarz inequality}) \\ &\leq (\frac{1}{4a}\exp(4a\log(x)))^{\frac{1}{2}} (\frac{1}{3(\log(x))^3})^{\frac{1}{2}} \\ &= \frac{1}{2}(\frac{1}{a})^{\frac{1}{2}}\exp(2a\log(x))(\frac{1}{3(\log(x))^3})^{\frac{1}{2}}. \end{aligned} \tag{2.39}$$

Furthermore, the inside of second bracket on the formula(2.36) is satisfied as follows:

$$\int_{-\infty}^{\frac{1}{\log(x)}} \exp(2v)dt \leq \frac{1}{2}\exp(\frac{2}{\log(x)}). \tag{2.40}$$

Using formulas(2.36), (2.39) and (2.40), the formula(2.29) is expressed as follows :

$$\begin{aligned}
&|\pi(x) - \mathrm{li}(x)| \\
&\leq \frac{C}{\sqrt{2\pi\alpha}} \exp\big(\frac{e}{\sqrt{2\pi\alpha}}\big)\frac{1}{a} \\
&\cdot \left(\frac{1}{2}(\frac{1}{a})^{\frac{1}{2}} \exp(2a\log(x))\big(\frac{1}{3(\log(x))^3}\big)^{\frac{1}{2}}\right)^{\frac{1}{2}} \left(\frac{1}{2}\exp(\frac{2}{\log(x)})\right)^{\frac{1}{2}} \\
&= \frac{C}{2}(\frac{1}{3a})^{\frac{1}{4}} \exp\big(\frac{e}{\sqrt{2\pi\alpha}}\big)\exp(a\log(x))\big(\frac{1}{\log(x)}\big)^{\frac{3}{4}} \exp\big(\frac{1}{\log(x)}\big) \\
&= \frac{C}{2}(\frac{1}{3a})^{\frac{1}{4}} \exp\big(\frac{e}{\sqrt{2\pi\alpha}}\big)x^a \left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp\big(\frac{1}{\log(x)}\big) \\
&(\because \exp(a\log(x)) = x^a) \\
&= C(\frac{\sqrt{2\pi\alpha}}{48})^{\frac{1}{4}} \exp\big(\frac{e}{\sqrt{2\pi\alpha}}\big)x^{\frac{1}{\sqrt{2\pi\alpha}}} \left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp\big(\frac{1}{\log(x)}\big). \\
&(\because a = \frac{1}{\sqrt{2\pi\alpha}})
\end{aligned} \tag{2.41}$$

Consequently, the inequality(1.38) of Theorem(1.9) is satisfied.

□

3. Conclisions.

We derived that Von Koch's inequality without using the Riemann Hypothesis in this paper. Using the fact that Von Koch's inequality is equivalent to the Riemann Hypothesis, we obtained that the Riemann Hypothesis is true.

Furthermore, we have shown that the upper limit of $|\pi(x) - \mathrm{li}(x)|$ can be lowered. Namely, we obtained a result that the inequality $|\pi(x) - \mathrm{li}(x)| \leq O(x^{1/e}\log(x))$, which has stricter conditions than the Riemann Hypothesis.

We started this paper to the introduction of the functions $R_\alpha^+(x)$. We obtained the functions $R_\alpha^+(x)$ using the concept of Entropy (Second law of thermodynamics). We think that Number theory and Entropy are closely related.

We would like to explain how to derive the distribution functions used in this paper if it gives us an opportunity,

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