Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 June 2022 doi:10.20944/preprints202111.0008.v4

Closed Form Solution of Plane-Parallel Turbulent Flow Along an Unbounded Plane Surface

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A century-old scientific conundrum is solved in this paper. The Prandtl mixing length modelled plane boundary turbulent flow is described by: $\frac{du^+}{dy^+}+\kappa^2 {y^+}^2\left(\frac{du^+}{dy^+}\right)^2=1$, together with boundary condition $y^+=0:\ u^+=0$. Only an approximate solution to this nonlinear ordinary differential equation (ODE) has been sought so far, however, the exact solution to this ODE has not been obtained. By introducing a transformation, $2\kappa y^+=\sinh\xi$, I successfully find the exact solution of the ODE as follows: $u^+=\frac{1}{\kappa}\ln(2\kappa y^+ +\sqrt{1+4\kappa^2 {y^+}^2})-\frac{2y^+}{1+\sqrt{1+4\kappa^2 {y^+}^2}}$.

Keywords: Exact solution, plane turbulent boundary flow, Prandtl mixing length modelling, Reynolds number

INTRODUCTION

The concept of the boundary layer was proposed by Prandtl in 1904 [1], initially he noticed that the fluid in the boundary layer is a laminar flow [2], but then Prandtl [3, 4] realized that the flow inside the boundary layer can be either laminar and turbulent, or just turbulent. The boundary layer is a revolutionary concept in fluid mechanics [6, 7], which has greatly promoted the development and application of fluid dynamics theory. Due to the uncertainty of the turbulent Reynolds stress, the proposed turbulent boundary layer equation still cannot be solved. To solve this uncertain problem, Prandtl [5] creatively proposed a mixing length model, giving an expression that uses mean flow velocity to represent turbulent Reynolds stresses. Prandtl used his proposed mixing length model to derive the differential equation of plane boundary layer. Since the derived ordinary differential equation (ODE) is nonlinear, it is a great pity that the exact solution of the ODE has not yet been obtained in the 100 years since it been derived. In order to be able to use this equation to study the urgently needed problems in engineering, Prandtl had to retreat to the second place, to find an approximate solution to this equation. Since Prandtl proposed the mixing length model in 1925, the approximate solution has been used to this day and has been adopted by all famous fluid mechanics textbooks [8, 9, 12]. Our question is, after 100 years of development, is it possible that the ODE derived by Prandtl can be solved precisely? If an exact solution to this equation can be obtained, then whether this exact solution can characterize boundary layer turbulence, and how does it differ from the DNS solution? If this exact solution does not describe the turbulent boundary layer very well, how to obtain results close to those of DNS by improving the Prandtl model.

For an arbitrary Reynolds number, finding a uniform analytical description of the non-uniform distribution of mean velocity across the turbulence boundary layer domain is a century-old puzzle. In this paper, we will solve the puzzle and find the exact solution within the framework of Prandtl mixing length modelling.

THE PRANDTL MIXING LENGTH THEORY OF THE PLANE BOUNDARY TURBULENT FLOW AND PRANDTL'S ASYMPTOTIC SOLUTIONS

For narrative convenience, let's briefly highlight the Prandtl mixing length model, as well as the Prandtl approximate solution.

Considering here a plane-parallel turbulent flow along an unbounded smooth plane surface (wall), and taking the direction of the flow as the x axis and the plane of the surface as the xz plane, so y is the direction orthogonal to the surface. Assuming that the turbulent flow is steady with constant pressure along the x axis, the y and z components of the mean velocity are zero, and all of the quantities depend only on y. In the following, we only need to study the upper flow due to the symmetry of the problem.

According to Reynolds [8–11], the Reynolds-averaged Navier-Stokes equation of the plane turbulent boundary flow is reduced to $\mu\frac{d^2\bar{u}}{dy^2}+\frac{d\tau'_{xy}}{dy}=0$, where μ is the dynamical viscosity, $\bar{u}$ is the mean velocity, and $\tau'_{xy}=-\rho\overline{u'v'}$ is the Reynolds stress [10, 11].

Under boundary conditions, i.e., $y=0:\ \bar{u}=0,\ u'=0,\ v'=0$ and $\mu\frac{d\bar{u}}{dy}=\tau_w$, the above equation can be integrated to

$$\mu\frac{d\bar{u}}{dy}-\rho\overline{u'v'}=\tau_w, \tag{1}$$

where τ_w is the wall friction force on a unit area of the surface. This force is clearly in the x direction. The quantity τ_w is the constant flux of the x component of momentum transmitted by the fluid to the surface per unit time. The first term on the left-hand side of Eq. 1 represents the effect of viscosity on the mean flow, whereas the second term is the Reynolds stress. In turbulent flow located some distance away from a wall, the Reynolds stress is of considerably greater magnitude than the viscous stress; however, the role of viscous stress increases as the distance to a smooth wall decreases until finally, at the wall, viscosity predominates.

According to the Prandtl mixing length theory [5], the Reynolds stress is proposed to be $\tau'_{xy} = \rho\ell^2|\frac{d\bar{u}}{dy}|\frac{d\bar{u}}{dy}$, where the mass density is ρ, the mixing length is constructed by dimensional arguments as $\ell = \kappa y$, and κ is a numerical constant, namely, the von Kármán constant. Hence, Eq. 1 becomes $\mu\frac{d\bar{u}}{dy}+\rho(\kappa y)^2|\frac{d\bar{u}}{dy}|\frac{d\bar{u}}{dy} = \tau_w$, namely,

$$\frac{d\bar{u}}{dy}<0: \quad \mu\frac{d\bar{u}}{dy} - \rho(\kappa y)^2\left(\frac{d\bar{u}}{dy}\right)^2 = \tau_w, \tag{2}$$

$$\frac{d\bar{u}}{dy}>0: \quad \mu\frac{d\bar{u}}{dy} + \rho(\kappa y)^2\left(\frac{d\bar{u}}{dy}\right)^2 = \tau_w, \tag{3}$$

and the boundary condition is

$$y = 0: \ \bar{u} = 0. \tag{4}$$

So far, no complete solutions have been obtained for either Eq. 2 or Eq. 3 due to their nonlinearity. Instead, asymptotic solutions have been constructed in two different regions (or sub-layers) [8] .

In order to obtain a solution of Eq. 2 or Eq. 3, Prandtl made a bold simplification according to the effects of viscosity in different regions. Dividing the boundary layer into two regions, the region near the wall surface is called the laminar flow region, in this region the viscosity dominates, and the region away from the wall surface is called inertial sub-layer, where the effect of viscosity can be ignored.

In the inertial sub-layer, the first terms of Eq. 2 and Eq. 3 are neglected, leading to $\rho(\kappa y)^2\left(\frac{d\bar{u}}{dy}\right)^2 = \tau_w$, the solution of which is a well-known Prandtl logarithmic law,

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa}\log(\frac{y\bar{u}}{\nu}) + \alpha - \frac{1}{\kappa}\log\alpha, \tag{5}$$

where $u_\tau = \sqrt{\tau_w/\rho}$ and $\nu = \mu/\rho$.

According to Nikuradse's famous experiments [14, 15], data fitting gives $\kappa = 0.4$ and $\alpha = 11.5$; hence, the Prandtl log-law is

$$\frac{\bar{u}}{u_\tau} = 5.75\log(\frac{y\bar{u}}{\nu}) + 5.5 \tag{6}$$

This expression becomes infinite at the boundary $y = 0$ and is inapplicable at very small distances y from the surface, since the effect of viscosity near the surface becomes non-negligible [9] .

To fix the singularity problem, traditionally a viscous sub-layer is introduced in which the viscosity of the fluid begins to be important. The second terms of Eqs. 2 and 3 can be neglected, leading to $\mu\frac{d\bar{u}}{dy} = \tau_w$, the solution of which is the Prandtl linear law: $\bar{u} = \tau_w y/\mu$.

After obtaining these two sub-layer solutions, combining them to form a segmental solutions. Mathematically speaking, the above two segmental solutions, including linear and log-law solutions, are incomplete, because they are not whole domain solutions. Rather, they are a matching solution between the inertial sub-layer and viscous sub-layer of turbulence. Between the inertial region and viscous sub-layer, there is an intermediate region whose empirical solution has not been obtained [8, 9, 11, 16] .

That is to say, although Prandtl obtained an approximate solutions to Eqs. 2 and 3 through partition simplification, because the boundaries of its partition are blurred, an intermediate region is artificially created, however there is no corresponding differential equation in this intermediate region, so it is impossible to obtain its approximate solution at all. In this way, it is important to obtain exact solutions valid everywhere. It is the mathematical puzzle that has plagued the fluid mechanics community for a hundred years. Today let's take on the challenge to solve this puzzle.

EXACT SOLUTION

To find a singularity-free and consistent solution valid in the whole domain of y, we return to the complete governing equations, Eqs. 2 and 3, and work to find their solutions without any approximations, as in the preceding section. We find the exact solutions for both Eqs. 2 and 3, but the solution of Eq. 2 is not physically possible since two terms of the solution are infinite at $y = 0$.

Introducing the wall friction velocity $u^+ = \frac{\bar{u}}{u_\tau}$, and the characteristic wall coordinate $y^+ = \frac{yu_\tau}{\nu}$. The wall friction velocity is the characteristic velocity for turbulent flows at a given wall shear stress. The only physically possible equation is Eq. 3, which can be rewritten as

$$\frac{du^+}{dy^+} + \kappa^2 {y^+}^2\left(\frac{du^+}{dy^+}\right)^2 = 1, \tag{7}$$

together with the boundary condition

$$y^+ = 0: \ u^+ = 0. \tag{8}$$

From Eq.7, we can get $\frac{du^+}{dy^+} = \frac{-1\mp\sqrt{1+4\kappa^2{y^+}^2}}{2\kappa^2{y^+}^2}$, hence we have two solutions of Eq. 7, namely

$$u^+ = \begin{cases} \int \frac{-1-\sqrt{1+4\kappa^2{y^+}^2}}{2\kappa^2{y^+}^2}dy^+ = \frac{1}{2\kappa^2 y^+} - \int\frac{\sqrt{1+4\kappa^2{y^+}^2}}{2\kappa^2{y^+}^2}dy^+, \\ \\ \int \frac{-1+\sqrt{1+4\kappa^2{y^+}^2}}{2\kappa^2{y^+}^2}dy^+ = \frac{1}{2\kappa^2 y^+} + \int\frac{\sqrt{1+4\kappa^2{y^+}^2}}{2\kappa^2{y^+}^2}dy^+. \end{cases} \tag{9}$$

 doi:10.20944/preprints202111.0008.v4

The integral form solution in Eq. 9 can be found in the literature, such as Eq.(7.144) in Page 304 of Pope [12], however, no one complete the integration in analytically but in numeric integration.

Introducing a transformation

$$2\kappa y^+ = \sinh\xi = \frac{1}{2}(e^\xi - e^{-\xi}), \tag{10}$$

and derivative

$$dy^+ = \frac{\cosh\xi}{2\kappa}d\xi. \tag{11}$$

Since $\frac{d\sinh\xi}{d\xi} = \cosh\xi$, $\frac{d\cosh\xi}{d\xi} = \sinh\xi$, and $\cosh\xi = \sqrt{1+\sinh^2\xi} = \sqrt{1+4\kappa^2 y^{+2}}$ as well as $\coth\xi = \frac{\cosh\xi}{\sinh\xi} = \frac{\sqrt{1+4\kappa^2y^{+2}}}{2\kappa y^+}$, the integration in Eq.9 can be calculated as follows

$$\begin{aligned}
&\int \frac{\sqrt{1+4\kappa^2 y^{+2}}}{2\kappa^2 y^{+2}} dy^+ \\
&= \int \frac{\cosh\xi}{\frac{1}{2}\sinh^2\xi}\frac{\cosh\xi}{2\kappa}d\xi \\
&= \frac{1}{\kappa}\int \frac{\cosh^2\xi}{\sinh^2\xi}d\xi \\
&= \frac{1}{\kappa}\int \cosh\xi d\left(\frac{-1}{\sinh\xi}\right) \\
&= -\frac{1}{\kappa}\left(\frac{\cosh\xi}{\sinh\xi} - \int\frac{d\cosh\xi}{\sinh\xi}\right) \\
&= -\frac{1}{\kappa}\left(\frac{\cosh\xi}{\sinh\xi} - \int\frac{\sinh\xi}{\sinh\xi}d\xi\right) \\
&= -\frac{1}{\kappa}\left(\frac{\cosh\xi}{\sinh\xi} - \int d\xi\right) \\
&= \frac{1}{\kappa}(\xi - \coth\xi).
\end{aligned} \tag{12}$$

From the transformation $2\kappa y^+ = \frac{1}{2}(e^\xi - e^{-\xi})$, and noting $e^\xi > 0$, then we have $e^\xi = 2\kappa y^+ + \sqrt{1+4\kappa^2y^{+2}}$, and the inversion $\xi = \ln(2\kappa y^+ + \sqrt{1+4\kappa^2y^{+2}})$. With those relations, the integration can be completed as follows

$$\begin{aligned}
&\int \frac{\sqrt{1+4\kappa^2 y^{+2}}}{2\kappa^2 y^{+2}} dy^+ = \frac{1}{\kappa}(\xi - \coth\xi) \\
&= \frac{1}{\kappa}\ln(2\kappa y^+ + \sqrt{1+4\kappa^2y^{+2}}) - \frac{\sqrt{1+4\kappa^2y^{+2}}}{2\kappa^2 y^+}.
\end{aligned} \tag{13}$$

Substitute the integration in Eq. 12 into the first of Eq.9, we have two solutions

$$u^+ = \begin{cases} -\frac{1}{\kappa}\ln(2\kappa y^+ + \sqrt{1+4\kappa^2y^{+2}}) \\ \quad + \frac{2y^+}{-1+\sqrt{1+4\kappa^2y^{+2}}} + C, \\ \\ \frac{1}{\kappa}\ln(2\kappa y^+ + \sqrt{1+4\kappa^2y^{+2}}) \\ \quad - \frac{2y^+}{1+\sqrt{1+4\kappa^2y^{+2}}} + C, \end{cases} \tag{14}$$

The first solution in Eq.14 is singular solution, because of its singularity $\frac{2y^+}{-1+\sqrt{1+4\kappa^2y^{+2}}}$ at $y^+ \to 0$, we can not use it as a solution.

After carefully checking, the 2nd solution Eq.14 is singularity-free, hence the exact closed form and singularity-free solution of Eq.7 is obtained as

$$\begin{aligned}
u^+ = &\frac{1}{\kappa}\ln(2\kappa y^+ + \sqrt{1+4\kappa^2y^{+2}}) \\
&- \frac{2y^+}{1+\sqrt{1+4\kappa^2y^{+2}}} + C,
\end{aligned} \tag{15}$$

where C is integration constant that can be determined by the boundary condition in Eq. 8.

The solution in Eq.15 is singularity free because of the cancellation of singularity of the term $\lim_{y^+\to 0}\left(\frac{2y^+}{1+\sqrt{1+4\kappa^2y^{+2}}}\right) = 0$, which leads to the limit $\lim_{y^+\to 0} u^+ = C$.

Applying the boundary condition in Eq. 8, $u^+(0) = 0$, gives $C = 0$. Finally, the exact closed form solution of Eq. 7 is found as follows

$$\begin{aligned}
u^+ = &\frac{1}{\kappa}\ln\left(2\kappa y^+ + \sqrt{1+4\kappa^2y^{+2}}\right) \\
&- \frac{2y^+}{1+\sqrt{1+4\kappa^2y^{+2}}}.
\end{aligned} \tag{16}$$

Notice the identity of hyperbolic arcsine $\sinh^{-1}x = \ln(x+\sqrt{1+x^2})$, $x \in (-\infty,\infty)$, the closed form solution can also be expressed as follows

$$u^+ = \frac{1}{\kappa}\sinh^{-1}(2\kappa y^+) - \frac{2y^+}{1+\sqrt{1+4\kappa^2y^{+2}}}. \tag{17}$$

And or in a more compact form

$$\begin{aligned}
u^+ = &\frac{1}{\kappa}\ln\left(2\ell^+ + \sqrt{1+4(\ell^+)^2}\right) \\
&- \frac{1}{\kappa}\frac{2\ell^+}{1+\sqrt{1+4(\ell^+)^2}}.
\end{aligned} \tag{18}$$

where $\ell^+ = \kappa y^+$. The singularity-free solution in Eq.16 is valid for the whole domain of $y^+ \in [0,\infty]$. The only unknown in the solution is the von Kármán constant. This constant was introduced to build the Prandtl mixing length model, and the Prandtl model itself cannot determine its value, but can only be determined by other methods, such as experiments [14, 15].

 doi:10.20944/preprints202111.0008.v4

VALIDATION AND DISCUSSIONS

To visualize the contribution of each term in Eqs. 18, the comparisons are depicted in Fig.1.

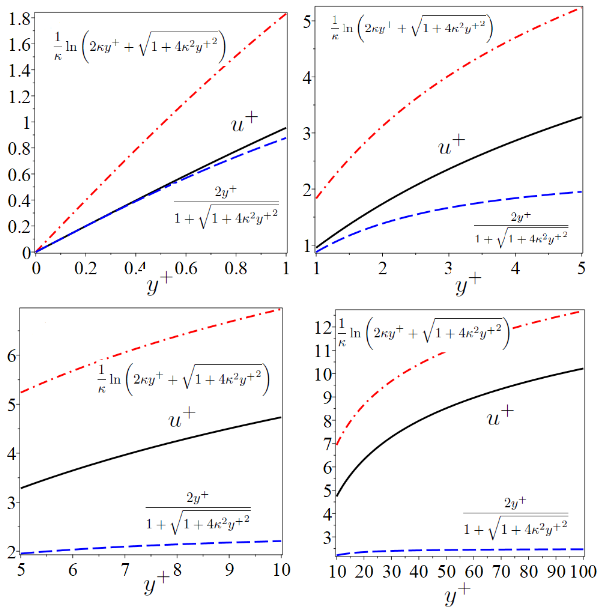

FIG. 1: The $u^+ - y^+$ curve. The exact solution u^+ in Eq. 18 is represented by black line, and $\frac{1}{\kappa}\ln\left(2\ell^+ + \sqrt{1+4(\ell^+)^2}\right)$ in Eq. 18 is represented by red line. $\frac{1}{\kappa}\frac{2\ell^+}{1+\sqrt{1+4(\ell^+)^2}}$ is represented in blue line. The plot indicates that the mean velocity of the Prandtl solution is dominated for very larger y^+.

Strictly speaking, for the Prandtl mixing length modelling, the solution in Eq. 16 reveals that the velocity profile of plane turbulent boundary flow is not logarithmic as stated in literature such as in [16], owing to the exist of the 2nd term " $-\frac{2y^+}{1+\sqrt{1+4\kappa^2 y^{+2}}}$". However, roughly speaking, for very large y^+, the logarithmic law of the turbulent velocity profile can still be approximately maintained. Similarly, close to the boundary surface, the velocity profile is not linear of distance to the wall as well, because $\frac{2y^+}{1+\sqrt{1+4\kappa^2 y^{+2}}} \neq 2\kappa y^+$.

Nevertheless, we can still divide the mean flow velocity into two parts according to the scale of the wall coordinate y^+, when y^+ is less than 1, the average flow rate near the wall surface obeys the quasi-linear law, and when y^+ is greater than 1, the mean flow velocity away from the wall obeys the logarithmic law. This understanding can be summarized as follows

$$u^+ \sim \begin{cases} \frac{2y^+}{1+\sqrt{1+4\kappa^2 y^{+2}}}, & (y^+ < 1), \\ \frac{1}{\kappa}\ln\left(2\kappa y^+ + \sqrt{1+4\kappa^2 y^{+2}}\right) + 4.2, & (y^+ > 1). \end{cases} \tag{19}$$

For further validation, a comparison between the closed form solution in Eq. 16 and direct numerical simulation (DNS) solution [17, 18] is depicted in Fig. 2. Although we successfully obtained the exact solution of plane turbulent boundary flow based on the Prandtl mixing length model $\ell^+ = \kappa y^+$, where $\ell^+ = \ell u_\tau/\nu$, unfortunately, Fig. 2 shows that the exact solution does not perfectly agree with the DNS solution [17, 18] for a smooth wall.

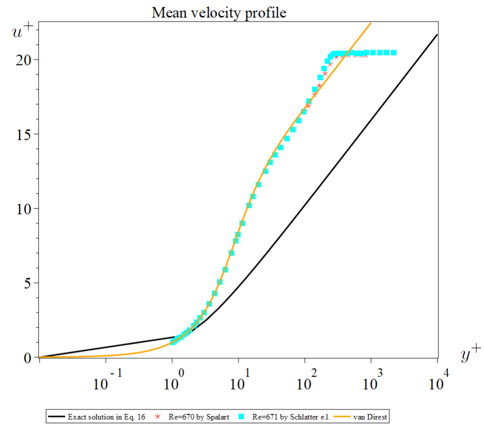

FIG. 2: Comparisons: the exact solution, van Driest mixing length theory [19], and the direct numerical simulation (DNS) solution [17, 18].

As can be seen from Fig.2, there is a difference between the exact solution and the results of the DNS. The difference between an exact solution and a numerical solution (DNS) is not an indication of something wrong with the solution process, but rather reveals the flaws of the Prandtl model itself. The solution we obtained is indeed an exact solution. If we want to get more accurate predictions, we must improve this model. For this, a modification scheme had been introduced by Driest [19], who proposed the mixing length should be modified to $\ell^+ = \kappa y^+[1-\exp(-\frac{y^+}{A})]$, where the damping parameter $A \approx 26$. The Driest's solution can be seen in Fig.2, which shows that Driest's solution [19] is more consistent with the results of DNS.

CONCLUSIONS AND PERSPECTIVES

To the best of the author's knowledge, within the frame of the Prandtl mixing modelling, the exact solution of the plane turbulent boundary flow is obtained for the first time [20]. Obtaining the exact solution allows us to soberly know the scope of the Prandtl mixing model. Obviously, the Prandtl model is inaccurate in the sense of the DNS accuracy, but the exact solution of the differential equation corresponding to the Prandtl model contains an essential information, i.e., the solution includes a logarithmic function, and later studies have shown that the mean velocity profile of the boundary layer flow can be roughly regarded as obeying the logarithmic law. This study may contribute to a better understanding of the turbulent boundary layer and even to the hypersonic boundary layer heating phenomena [21–24].

Acknowledgements

This work was supported by Xi'an University of Architecture and Technology (Grant No. 002/2040221134). The author wishes to express his appreciation his student, Mr. Zhe Liu, for extracting Fig. 2's DNS data from Ref.[17, 18]. .

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