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When the Riemann Hypothesis might be false

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Abstract Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^\gamma \times n \times \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Let $q_1 = 2, q_2 = 3, \dots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^m q_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer. If the Riemann Hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that Robin inequality does not hold and $n < (4.48311)^m \times N_m$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m .

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

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1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [4]. As usual $\sigma(n)$ is the sum-of-divisors function of n [2]:

$$\sum_{d|n} d$$

where $d | n$ means the integer d divides to n and $d \nmid n$ means the integer d does not divide to n . Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say Robins(n) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

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The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and \log is the natural logarithm. The importance of this property is:

Theorem 1.1 *If the Riemann Hypothesis is false, then there are infinitely many natural numbers $n > 5040$ such that $\text{Robins}(n)$ does not hold [4].*

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [2]. $\text{Robins}(n)$ holds for all natural numbers $n > 5040$ that are square free [2]. In addition, we show that $\text{Robins}(n)$ holds for some $n > 5040$ when $\frac{\pi^2}{6} \times \log \log n' \leq \log \log n$ such that n' is the square free kernel of the natural number n . Let $q_1 = 2, q_2 = 3, \dots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^m q_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [2]. Based on the theorem 1.1, we know this result:

Theorem 1.2 *If the Riemann Hypothesis is false, then there are infinitely many natural numbers $n > 5040$ which are an Hardy-Ramanujan integer and $\text{Robins}(n)$ does not hold [2].*

We prove if the Riemann Hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\text{Robins}(n)$ does not hold and $n < (4.48311)^m \times N_m$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m .

2 A Central Lemma

These are known results:

Lemma 2.1 [2]. For $n > 1$:

$$f(n) < \prod_{q|n} \frac{q}{q-1}. \quad (2.1)$$

Lemma 2.2 [3].

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \quad (2.2)$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all natural numbers n . The bound is too weak to prove $\text{Robins}(n)$ directly, but is critical because it holds for all natural numbers n . Further the bound only uses the primes that divide n and not how many times they divide n .

Lemma 2.3 *Let $n > 1$ and let all its prime divisors be $q_1 < \dots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

Proof We use that lemma 2.1:

$$f(n) < \prod_{i=1}^m \frac{q_i}{q_i - 1}.$$

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Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\begin{aligned} \frac{1}{1 - \frac{1}{q^2}} \times \frac{q+1}{q} &= \frac{q^2}{q^2 - 1} \times \frac{q+1}{q} \\ &= \frac{q}{q-1}. \end{aligned}$$

Then by lemma 2.2,

$$\prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^m \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^m \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}. \end{aligned}$$

3 A Particular Case

We can easily prove that Robins(n) is true for certain kind of numbers:

Lemma 3.1 Robins(n) holds for $n > 5040$ when $q \leq 5$, where q is the largest prime divisor of n .

Proof Let $n > 5040$ and let all its prime divisors be $q_1 < \dots < q_m \leq 5$, then we need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq e^\gamma \times \log \log n$$

according to the lemma 2.1. For $q_1 < \dots < q_m \leq 5$,

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^\gamma \times \log \log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^\gamma \times \log \log(5040) < e^\gamma \times \log \log n$$

and therefore, the proof is complete when $q_1 < \dots < q_m \leq 5$.

4 Helpful Lemmas

For every prime number $p_n > 2$, we define the sequence $Y_n = \frac{e^{\frac{1}{2 \times \log(p_n)}}}{(1 - \frac{1}{\log(p_n)})}$.

Lemma 4.1 For every prime number $p_n > 2$, the sequence Y_n is strictly decreasing.

Proof For every real value $x \geq 3$, we state the function

$$f(x) = \frac{e^{\frac{1}{2 \times \log(x)}}}{(1 - \frac{1}{\log(x)})}$$

which is equivalent to

$$f(x) = g(x) \times h(u)$$

where $g(x) = e^{\frac{1}{2 \times \log(x)}}$ and $h(u) = \frac{u}{u-1}$ for $u = \log(x)$. We know that $g(x)$ decreases as $x \geq 3$ increases, Moreover, we note that $h(u)$ decreases as $u > 1$ increases where $u = \log(x) > 1$ for $x \geq 3$. In conclusion, we can see that the function $f(x)$ is monotonically decreasing for every real value $x \geq 3$ and therefore, the sequence Y_n is monotonically decreasing as well. In addition, Y_n is essentially a strictly decreasing sequence, since there is not any natural number $n > 1$ such that $Y_n = Y_{n+1}$.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x .

Lemma 4.2 [5]. For $x \geq 41$:

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Besides, we know that

Lemma 4.3 [5]. For $x \geq 286$:

$$\prod_{q \leq x} \frac{q}{q-1} < e^\gamma \times (\log x + \frac{1}{2 \times \log(x)}).$$

We will prove another important inequality:

Lemma 4.4 Let q_1, q_2, \dots, q_m denote the first m consecutive primes such that $q_1 < q_2 < \dots < q_m$ and $q_m > 286$. Then

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \times \log(Y_m \times \theta(q_m)).$$

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Proof From the theorem 4.2, we know that

$$\theta(q_m) > \left(1 - \frac{1}{\log(q_m)}\right) \times q_m.$$

In this way, we can show that

$$\begin{aligned} \log(Y_m \times \theta(q_m)) &> \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right) \times q_m\right) \\ &= \log q_m + \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right)\right). \end{aligned}$$

We know that

$$\begin{aligned} \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right)\right) &= \log\left(\frac{e^{\frac{1}{2 \times \log(q_m)}}}{\left(1 - \frac{1}{\log(q_m)}\right)} \times \left(1 - \frac{1}{\log(q_m)}\right)\right) \\ &= \log\left(e^{\frac{1}{2 \times \log(q_m)}}\right) \\ &= \frac{1}{2 \times \log(q_m)}. \end{aligned}$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \times \left(1 - \frac{1}{\log(q_m)}\right)\right) \geq \left(\log q_m + \frac{1}{2 \times \log(q_m)}\right).$$

Due to the theorem 4.3, we prove that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \times \left(\log q_m + \frac{1}{2 \times \log(q_m)}\right) < e^\gamma \times \log(Y_m \times \theta(q_m))$$

when $q_m > 286$.

5 Proof of Main Theorems

The next theorem implies that Robins(n) holds for a wide range of natural numbers $n > 5040$.

Theorem 5.1 *Let $\frac{\pi^2}{6} \times \log \log n' \leq \log \log n$ for some $n > 5040$ such that n' is the square free kernel of the natural number n . Then Robins(n) holds.*

Proof Let n' be the square free kernel of the natural number n . Let n' be the product of the distinct primes q_1, \dots, q_m . By assumption we have that

$$\frac{\pi^2}{6} \times \log \log n' \leq \log \log n.$$

For all square free $n' \leq 5040$, Robins(n') holds if and only if $n' \notin \{2, 3, 5, 6, 10, 30\}$ [2]. However, Robins(n) holds for all natural numbers $n > 5040$ when $n' \in \{2, 3, 5, 6, 10, 15, 30\}$ due to the lemma 3.1. When $n' > 5040$, we know that Robins(n') holds and so

$$f(n') < e^\gamma \times \log \log n'.$$

By the previous lemma 2.3:

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

Suppose by way of contradiction that Robins(n) fails. Then

$$f(n) \geq e^\gamma \times \log \log n.$$

We claim that

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} > e^\gamma \times \log \log n.$$

Since otherwise we would have a contradiction. This shows that

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} > \frac{\pi^2}{6} \times e^\gamma \times \log \log n'.$$

Thus

$$\prod_{i=1}^m \frac{q_i + 1}{q_i} > e^\gamma \times \log \log n',$$

and

$$\prod_{i=1}^m \frac{q_i + 1}{q_i} > f(n'),$$

This is a contradiction since $f(n')$ is equal to

$$\frac{(q_1 + 1) \times \cdots \times (q_m + 1)}{q_1 \times \cdots \times q_m}.$$

Theorem 5.2 *If the Riemann Hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that Robins(n) does not hold and $n < (4.48311)^m \times N_m$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m .*

Proof Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of some natural number $n > 5040$ as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents a_1, \dots, a_m . The primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since the natural number $n > 5040$ could be an Hardy-Ramanujan integer. We assume that Robins(n) does not hold. Indeed, we know there are infinitely many Hardy-Ramanujan integers such as $n > 5040$ when the Riemann Hypothesis is false according to the theorem 1.2. From the lemma 4.4, we know that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \times \log(Y_m \times \theta(q_m)) = e^\gamma \times \log \log(N_m^{Y_m})$$

when $q_m > 286$. In this way, if Robins(n) does not hold, then $n < N_m^{Y_m}$ since by the lemma 2.1 we have that

$$f(n) < \prod_{i=1}^m \frac{q_i}{q_i - 1}.$$

That is the same as $n < N_m^{Y_m-1} \times N_m$. We can check that $q_m^{Y_m-1}$ is monotonically decreasing for all primes $q_m > 286$ due to the lemma 4.1. Certainly, the function

$$g(x) = x \left(\frac{\frac{1}{e^{2 \times \log(x)}}}{\left(1 - \frac{1}{\log(x)}\right)} - 1 \right)$$

complies that its derivative is lesser than zero for all real numbers $x > 286$. Indeed, a function $g(x)$ of a real variable x is monotonically decreasing in some interval if the derivative of $g(x)$ is lesser than zero and the function $g(x)$ is continuous over that interval [1]. We know that q_m could comply with $q_m \geq 1000000!$ for infinitely many Hardy-Ramanujan integers $n > 5040$ such that Robins(n) does not hold, where $(\dots)!$ is the factorial function. Certainly, if q_m would have an upper bound by some positive value, then there would not be infinitely many natural numbers $n > 5040$ which are an Hardy-Ramanujan integer and Robins(n) does not hold because of the theorem 5.1. Consequently, it is enough to show that

$$q_m^{Y_m-1} \leq g(1000000!) < 4.48311$$

for all primes $q_m \geq 1000000!$. Moreover, we would obtain that

$$q_m^{Y_m-1} > q_j^{Y_m-1}$$

for every integer $1 \leq j < m$. Finally, we can state that $n < (4.48311)^m \times N_m$ since $N_m^{Y_m-1} < (4.48311)^m$ when $n > 5040$ could be any of the infinitely many Hardy-Ramanujan integers such that Robins(n) does not hold and $q_m \geq 1000000!$.

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