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Article

The systematic risk at the crisis - A multifractal non-uniform wavelet systematic risk estimation

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Abstract: The Capital Asset Pricing Model is a widely applied model to describe risky markets and to deduce their systematic risk. Its estimation therefore remains an important task in econo-financial studies. Empirically, it focuses on the impact of return interval on the betas. Existing studies somehow turn around the same idea of measuring the value of the beta according to the uniform intervals of time during a fixed period. However, it is noticed easily, and especially the last decade that many factors such as socio-political, and econo-environmental ones have led to a perturbation in the time-line of the worldwide development, and especially in countries and regions having political changes. This led us to introduce a new idea of risk estimation taking into account the non-uniform changes in markets by introducing a non-uniform wavelet analysis. We aim to explain the econo-political situation of Arab spring countries, and the effect of the revolutions on the market beta. The main novelty is firstly the construction of a dynamic backward-forward model for missing data, and next the application of random non-uniform wavelets. The proposed procedure will be acted empirically on a sample corresponding to TUNINDEX stock as a representative index of the Tunisian market actively traded over the period January 14, 2016 to January 13, 2021. The chosen 5-years period is important as it constitutes the first 5-years-after the revolution and depends strongly on the socio-econo-political stability in the revolutionary countries.

Keywords: Wavelets; Non-uniform wavelets; CAPM; Wavelets CAPM; systematic (market) risk; scaling; Arab spring revolutions.

JEL Classification: G11; C02; C22.

1. Introduction and motivations

It is well known that economies are based to a large extent on the corresponding financial markets, local as well as external. Local markets explain or reflect the internal policies of governments and the internal laws that govern the movement of money locally such as local prices, internal laws of economics and finance, production, development, etc. The interaction with foreign markets explains and reflects first of all the effectiveness of foreign affairs policy, import-export movements, the quality of the export, ..., etc.

To control the situation of the market, controllers such as governments apply usually qualitative and quantitative measurements such as purses and or indices. This permits to econo-financial agents such as Financial markets to reconcile the antagonistic objectives of their clientele such as profitability, security and liquidity.

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To guarantee a non risky or at least less risky (secure) portfolios, with high level of profitability, investors has to make into consideration may factors, such as time, securities, risk, profitability, policies, ... etc, (See [1–8]).

Since these studies, the CAPM has became a basic model in financial studies, and permitted in many cases the description of the link some factors raised above. However, its empirical validity has been investigated after by works of [9]. Next, in the 1990's decade, the empirical studies have been growing up, raising in some cases the problem of time intervals in the use of the CAPM. Indeed, Fama-French stated in [10] that the classical way of annual time intervals may not induce good results, and thus raised the necessity of using adapted time intervals for the estimation of systematic risk beta, such as monthly and weekly data due to many causes such as the seasonal effect of returns, the non-synchronization of actions, etc (See [1,2,11–27]). However, many studies already taking into account the time factor in the CAPM validation have concluded that the estimations may lead to biased beta when using even monthly, weekly and more precisely short (uniform) periods (see [28–34]). Recall that short uniform periods may induce a great problem of frequency estimation and/or volatility/fluctuation control in the market. In this context, the authors in [35] and [36] stated that the lack of distinction between the time domain and the frequency may be, indeed, causes of the the limitations of conventional approaches. See also [37].

Due to these studies, a great interest to the time factor has been considered by researchers on the validity of the CAPM and the relationship between the systematic risk and the return of securities. This issue is still relevant in financial studies.

One of the recent tools involved in the econo-financial studies to understand and to include the time factor in the models is the so-called wavelet approach. Indeed, Wavelets form a mathematical tool that transforms time domain data into different frequency horizons. They represent the advantage of being localized in both the time and frequency domains. Wavelets since their appearance and especially their involvement in financial time series and economic models understanding showed their efficiency to offer sophisticated algorithms. They constituted attractive mathematical tools in the modeling of several complicated situations such as in economics and finance. Financial indices are usually volatility and highly fluctuated, which imposes the need to advanced mathematical tools for their study, such as wavelets which have shown ability to detect and/or localize fluctuations and volatility.

Related to the risk in markets, Marfatia developed a wavelet approach in [35] for the impact of risks in international stock markets by involving wavelet techniques into time-varying conditional volatility. The wavelet modeling permitted to conclude that co-movement of risks between the US market and European markets are more strong explained at lower frequencies, contrarily to regional markets versus the country, where the co-movement is much stronger at higher frequencies, See also [36].

Related to the CAPM, wavelet variants have been developed leading to the so-called wavelet CAPM (WCAPM) by analysing data at different time scales, making it possible to overcome the inadequacies of the classical analysis of the CAPM. For more backgrounds on wavelets and their applications the authors may refer to [38–45].

The present study lies in the whole scope of the integration and/or the study and evaluation of risks in stock markets according to time changes. Focus are made on the involvement of wavelet theory in estimating the systematic risk beta relative to crisis and precisely sharp socio-political movements.

A main aim in our present paper is to understand the nature of the relationship linking systematic risk and the return on equities for the Tunisian stock markets as a representative of the so-called Arab spring revolutions' countries. Recall that, compared to other countries in such a set, Tunisia is the most stabilized country that could pass the revolutionary period without a great instability us in some other countries where after-revolution wars have been broken out between the regimes and opposite parts. Such situations have now influenced quasi all the world by inducing great movement of illegal

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immigration, refugees movements, appearance and spread of terrorist organizations, ..., etc.

The irregular movements have induced a perturbation in the time-line of the world-wide development, economy, finance, and especially in countries and regions having the socio-political changes. We aim precisely in the present work to introduce a new idea of risk estimation taking into account the non-uniform changes in markets by introducing a non-uniform wavelet analysis. We aim to explain the econo-political situation of Arab spring countries, and the effect of the revolutions on the market beta. The main novelty is firstly the construction of a dynamic backward-forward model for missing data, and next the application of random non-uniform wavelets. The proposed procedure will be acted empirically on a sample corresponding to TUNINDEX stock as a representative index of the Tunisian market actively traded over the period January 14, 2016 to January 13, 2021. The chosen 5-years period is important as it constitutes the first 5-years-after the revolution and depends strongly on the socio-econo-political stability in the revolutionary countries. The period of study is also characterized by the appearance of the corona pandemic spread in all the world. It is well known the influence of such pandemic on the economic situations on worldwide countries, especially the Arab spring ones. The crisis such as COVID-19 pandemic have also shown the weakness in worldwide countries relationships and supports to each other during severe moments. Until now, many countries classified as poor suffer from the severe situation of the pandemic without real and high support in view of vaccination availability. From an empirical point of view, the approach may also be a real foundation for effective financial decisions, which depend usually on the accuracy of the valuation of the securities, their evolution and their risks.

We recall in particular, that Tunisia has a strategic geographic position in the mid of the south Mediterranean sea, which allows it to be a central point in the inter-changing between the north and the south rivers, and thus plays a great role in the economic, social and political movements in this region of the world. Relatively to the present work, and especially to the period chosen for study, Tunisia has established a long-standing co-operation with the European council since 2011, just after the revolution, to develop many sectors such as human rights, democracy, combating violence, improving the functioning of justice, the fight against corruption, which were immediately reflected in the partnership with Tunisia especially in the period 2015–2021. These priorities have and/or will have surely a great effect on the stability of the country, and thus permits the investment and the development to be pursued. As a consequence, the different after-revolution governments have established many social, political, economic and financial projects that have a direct impact on the national market as well as interacted ones. This makes it of interest to study such market and understand its complexities.

Now, from a practical point of view, in the present case of study, the data collected, based on yahoo finance and the web site investing.com, unfortunately has many missing values, in different time positions such as the beginning, the mid, and the end of the period of investigation. Consequently, our work becomes a twofold study. In a first step, we developed a wavelet-based method to reconstruct missing data. It consists of backward-forward method, in which we applied for indices starting adequately a forward prediction of the missing values, and for indices with missing data at the beginning we have applied a backward prediction to reconstruct the first values of the data. The missing mid values are estimated as overages of the application of the two previous procedures. The method looks like the lifting from a multiresolution horizon to the previous and vice versa. level The complete set is by the next applied for CAPM and thus the comprehension of the market.

The rest of the present work is organized as follows. In section 3, a literature review on forecasting systematic risk is developed. Section 4 is devoted to the development of our methodology. We especially re-develop the wavelet analysis of time series briefly to apply it by the next for missing data reconstruction which in turns will be applied

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for completing the data basis used later. The mathematical formulation of the CAPM is provided by the next. Finally, this section is achieved by the development of the wavelet CAPM. Section 5 is subject of our empirical results and their discussions on the Tunisian TUNINDEX stock market. Section 6 is a conclusion. The findings in the present work may be good basis for understanding current and future situation and may be thus a basis for investors' decisions in such markets. We intend that our method will be adapted to non-uniform, multifractal, random and stochastic aspects in the market, and to explain the market situation at the crisis.

2. The CAPM and the time factor review

This section is concerned to the literature review of the original construction of the CAPM. This come back to the early 1950's decade in the works of Markowitz [6] on portfolio selection. Markovitz thought that the CAPM is mainly applied for solving the problem of portfolio's structure. This needs to an estimation of the demand function of assets to incorporate the quantified treatment of risk, which in turns leads to the investigation of the market's equilibrium. However, Markovitz framework did not take into account the time factor, which causes a strong limitation. Besides, Markovitz model [6] stated that the investor optimal choices are based on the expected return of investments and the risk of the portfolio. The model suggests to select stocks relative to statistical criteria such as the profitability.

The original version of the CAPM has been a governing measure or index to explain and interpret the behaviour of the market for a long period such as the estimation of the capital cost for companies, the evaluation of the performance of managed portfolios despite the problems confronted in its empirical evaluation, and which are related to many causes in the market such as the availability of data.

The CAPM in its original and/or modified versions is based indeed on many assumptions that should be taken into consideration, but which may not be satisfied simultaneously in all situations. These may be resumed into several points.

- mono-periodicity,
- market perfection aspect without taxes nor transaction costs,
- homogeneity in anticipations,
- unlimited short selling,
- loans and borrowing at the risk-free and limitless rate,
- strictly increasing and strictly concave Von-Neumann-Morgenstern utility functions,
- mean-variance preferences based on restrictions relating to the return or the utility function,
- investor aversion to risk,
- competition and market efficiency.

Next, in quietly a decade, the mathematical formulation has been put out by Sharpe who continued to exploit such a formalism and its efficiency in explaining the market movement and extended the CAPM to overcome some cases of limitations due to the last assumptions by proposing the so-called CAPM with non-homogeneous anticipations ([46–48]).

In a parallel time or direction, Lintner investigated the risk assets and the selection of stock portfolios, and the CAPM with taxes in [3,4]. Other extensions have been also proposed by [33,49,50] such the CAPM with transaction costs and the CAPM in continuous time. In [51] however, the authors criticized the profitability of a stock as a criterion to confirm about the opportunity of the investment, and stated instead that it is not sufficient to do. The authors asserts that a good measure of the total risk of a portfolio may be the variance of its profitability.

Mathematically speaking the CAPM is a linear relationship between the expected return on the share or portfolio and the market premium where the linearity coefficient is often denoted β and is known as systematic risk. It tests the mean-variance efficiency and involves the ranking of assets and portfolios against systematic risks by practitioners. It thus analyses the return and risk of an investment. In [4] and [8], the main idea of the CAPM turns around the estimation of the prices of transferable securities allowing supply and demand to be balanced and allowing a general equilibrium of the market. Sharpe proved effectively that for each asset i, the coefficient β_i is the quotient of the covariance between the rate of return of asset i and the rate of return of the portfolio by the square of market risk. Other factor may affect the CAPM such that diversification which is indeed strongly related to the behaviour of the expectation-variance of the portfolio. The contribution of each security to the expected profitability of the portfolio is in fact proportional to its expected profitability.

In the literature, despite the governance of the original versions of the CAPM for a long period in financial studies, many criticisms related to the portfolio and/or the market have been addressed for such versions. In [9] for example, an empirical criticism has been pointed out relative to the representative portfolio of the market. See also [52–54].

The main drawback of these versions may be resumed in the fact that the estimated betas are assumed to be stable relatively to the time scale and thus relative to the whole period of study. However, in the real aspect of the financial market, the time factor is always present although being hidden in the model. For example, the standard deviation of the market is always variable according the time.

For this reason, many researchers have focused on the inclusion of the time factor, at least in the empirical extensions of the CAPM relatively to the time variations of the beta or the risk premium, or sometimes both of them. Marfatia and collaborators, for example, showed in a series of works ([35,36,55–57]) that different estimators of the risk may be obtained according to the difference in the time interval of returns (see also [28,32]). Handa et al rejected in [29] the monthly periods for the estimation of the CAPM beta, and accepted instead the annual returns.

The concept of time becomes thus a crucial and indispensable factor for the validity of the CAPM, and still constitutes an essential and relevant issue. One of the challenging ideas that have been recently involved in the CAPM, and in the investigation of financial markets is due to wavelet theory. which permits essentially to show the presence of the time-scale division to overcome the shortcomings of the classical CAPM analysis. Our aim in the present work is to continue exploiting the validity of the CAPM by means of wavelet theory. The focuses are made on the Tunisian TUNINDEX stock market in a critical period known as the after-revolution state. We aim firstly to analyze the systematic risk and to point out its limitations taking into account the time factor. The second is to identify the relationship between equity returns and their systematic risks using the wavelet approach taking already the time factor into account.

In our knowledge, the application of wavelets for revolutions' transition periods, before and after needs more developments. The wavelet method however, has been applied for stable markets such as SP500 (USA), CAC40 (France), ISE (Istanbul) and GCC markets. [58] applied wavelets for the stock markets of the US, UK and Germany to estimate the best time scale for measuring systemic risk. The authors concluded that the relationship between risk and return is a multi-scale phenomenon. [59] analyzed the Santiago stock market in Chile using time-scaling methodology. [60,61] studied the French CAC40 index as market portfolio and the daily EURIBOR as the risk-free rate. The predictions of the CAPM are claimed to be more relevant in the short term than in the long term, which makes the French market different from those of the US, UK and Germany. [62] applied the wavelet multi-scaling method for the Istanbul Stock Exchange during the period from January 2003 to October 2007. It is shown that a positive relationship between risk and returns is most significant at the medium levels, concluding that the effect of market returns on an asset is stronger in this time horizon. In [39], the author developed a wavelet study for the largest GCC market Saudi Tadawul in order to understand such market on a critic period which post follows many important

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movements infusing directly and strongly on the market. In the next section, we will address the details of our mathematical methodology in exploiting wavelet theory differently from existing works in the analysis of the CAPM.

For more details, backgrounds, and applications of wavelet theory in finance, economics, management and generally actuarial sciences, the readers may refer for instance to [38–45,58], [59], [63–76].

3. Wavelets for financial series processing

3.1. The wavelet processing classical view

A wavelet may be defined simply as a short wave function oscillating as the cosine and sine waves, but with high frequency and small support. Such a behaviour is known as the localization in time-frequency, and/or time-space. As in Fourier analysis, to analyse signals such as financial time series, wavelet analysis also is based on a type of transform defined by means of a convolution of the analyzed signal with wavelets obtained from the same source function ψ called the mother wavelet, a square-integrable function with enough vanishing moments (oscillating) with necessary zero mean. To analyse financial time series, the mother wavelet should satisfy the hypothesis

$$\mathcal{A}_\psi = \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty, \tag{1}$$

known as the admissibility condition (See [77–81]). These wavelets are obtained by translation, and dilation of the mother wavelet by means of a dilation factor $a > 0$ (known as the scale), and a translation factor $b \in \mathbb{R}$ (known as the position) as

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi(\frac{x-b}{a}). \tag{2}$$

Therefore, the continuous wavelet transform (CWT) of an analyzed function $F \in L^2(\mathbb{R})$ is

$$C_F(a,b) = \int_{-\infty}^{+\infty} F(t)\psi_{a,b}(t)dt, \tag{3}$$

which depends as its form shows on the scale and the position parameters. This explains its nomination as the CWT of F at the scale a and the position b.

In the case of discrete signals such as the financial time series, this is of course evaluated by means of the discrete integrals or the discrete convolution. Indeed, it is proved in wavelet theory that a restrictive version of the CWT (called the discrete wavelet transform (DWT)) suffices to overcomes many cases such as the financial series. It is obtained by restricting the scale and the position factors to discrete grids. The most commonly known restriction is the dyadic grid $a = 2^{-j}$, and $b = k2^{-j}$, $j,k \in \mathbb{Z}$. The wavelet $\psi_{a,b}$ is therefore written as

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k). \tag{4}$$

The discrete wavelet transform (DWT) (called also the wavelet coefficient or detail coefficient at the level j, and the position k) is defined by

$$d_{j,k}(S) = C_F(2^{-j}, k2^{-j}). \tag{5}$$

In wavelet theory we know that the set $(\psi_{j,k})_{j,k\in\mathbb{Z}}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$, and consequently any element F may be decomposed in a series

$$F = \sum_{j,k} d_{j,k}(S)\psi_{j,k} \tag{6}$$

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known as the wavelet series of F, and guarentees a reconstruction formula of the original analyzed function F.

In the case of a discrete function S such as the financial time series this formulation may be easily adapted, by means of a convolution product permitting to write the wavelet detail coefficients $d_{j,k}(S)$ as a cpnvolution product

$$d_{j,k}(S) = \sum_{n} F(n)\psi_{j,k}(n). \tag{7}$$

This decomposition of the series S into an orthogonal-wise components series may be explained also and equivalently by means of a multiresolution framework by splitting it into two orthogonal parts,

$$S = \sum_{j\leq J,k} d_{j,k}(S)\psi_{j,k} + \sum_{j>J,k} d_{j,k}(S)\psi_{j,k}, \tag{8}$$

relative to a fixed integer $J \in \mathbb{Z}$. If we denote for $j \in \mathbb{Z}$, $W_j = spann(\psi_{j,k}\ ;\ k \in \mathbb{Z})$ (known as the detail spaces), and $V_j = \overset{\perp}{\bigoplus\limits_{l\leq j}} W_l$ (known as the approximation spaces), we get in one hand an orthogonal decomposition $V_j = V_{j-1} \overset{\perp}{\bigoplus} W_{j-1}$. On the other hand, the part

$$A_J(S) = \sum_{j\leq J,k} d_{j,k}(S)\psi_{j,k} \tag{9}$$

(called the approximation of F at the level J_0, which is also the projection of F on V_{J_0}) belongs to V_J, and the part

$$D_j(S) = \sum_{k} d_{j,k}(S)\psi_{j,k} \tag{10}$$

(called the detail component of F at the level j, which is also the projection of F on W_j) belongs to the space W_j. In other words, the series S is written as

$$S = A_J(S) + D_{J+1}(S) + D_{J+2}(S) + ... \tag{11}$$

It holds that the component $A_J(S)$ describes the global behavior, the trend, or the shape of S, and a second part reflects the higher frequency oscillations or the fine scale deviations of the series near its trend.

To compute or to evaluate these parts, we have not to compute necessary all the coefficients appearing in the decomposition. We instead serve of some filters and algorithms permitting to reduce the cost of the computation. Indeed, it holds in wavelet theory that there exists a special function φ (known as the scaling function or the father wavelet) for which $V_j = spann(\varphi_{j,k}\ ;\ k \in \mathbb{Z})$, where the $\varphi_{j,k}$'s are defined similarly to the $\psi_{j,k}$. Such function is related to the mother wavelet by means of the so-called 2-scale relation

$$\varphi = \sum_{k\in\mathbb{Z}} h_k\varphi_{1,k}, \quad \text{and} \quad \psi = \sum_{k\in\mathbb{Z}} g_k\varphi_{1,k}, \tag{12}$$

where

$$h_k = \int_{-\infty}^{+\infty} \varphi(t)\varphi_{1,k}(t)dt, \quad \text{and} \quad g_k = (-1)^k h_{1-k}. \tag{13}$$

The approximation $A_J(S)$ will be therefore expressed by means of the basis of V_J as

$$A_J(S) = \sum_{k} a_{J,k}(S)\varphi_{J,k}, \tag{14}$$

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where the coefficients $a_{j,k}$ (known as the approximation or scaling coefficients of S) are evaluated as the $d_{j,k}$ by replacing the function ψ by φ. The relation (12) permits to compute the wavelet coefficients recxursively as

$$a_{j,k}(S) = \sum_{l\in\mathbb{Z}} h_l a_{j+1,l+2k}(S), \tag{15}$$

$$d_{j,k}(S) = \sum_{l\in\mathbb{Z}} g_l a_{j+1,l+2k}(S), \tag{16}$$

and

$$a_{j+1,k}(S) = \sum_{l} h_{l-2k} a_{j,l}(S) + \sum_{l} g_{l-2k} d_{j,l}(S). \tag{17}$$

This means that the wavelets permit to deduce the decomposition of the series S at different levels from each other by means of the filters $H = (h_k)_k$ (discrete wavelet low-pass filter), and $G = (g_k)_k$ (discrete wavelet high-pass filter).

In practice, obviously, a restriction a maximal level is needed. For this we fix two integers $J > J_0 \in \mathbb{Z}$ and consider the approximation

$$S_J = A_{J_0}(S) + \sum_{J_0<j\leq J} D_j(S). \tag{18}$$

The lower index J_0 is in fact more flexible, does not have an important effect on the total decomposition and usually chosen to be 0. The choice of J is always critical, and is related to the eventual error estimates relative to the error applied to get the closeness of S_J to the origin S. See [77–81] for more details.

3.2. The non-uniform wavelet processing

Non-uniform wavelets constitute an extension or improvement of wavelets by involving some modifications on the original way of use of wavelets. In the original wavelet analysis, we usually search for perfect representations of functions, time series, and generally analyzed objects by means of a set of uniformly-spaced samples. However, in nature, and practice, it may exists many cases of objects with non-uniformly-spaced samples such as structures sampled well-below the Nyquist rate. Indeed, the authors investigated in [82] a new approach called non-uniform wavelet sampling based on combining wavelet pre-processing with non-uniform sampling in order to alleviate the main issues of existing converter solutions, such as signal noise. A specialized variant of nonuniform wavelet band-pass sampling has been proposed by combining traditional band-pass sampling with nonuniform wavelet sampling. The method has been shown to be performant and efficient relative to the classical approaches.

In [83], a framework of non-uniform wavelets has been proposed for a special choice of Haar wavelet. Non-uniform Haar scaling and wavelet functions have been been introduced provided with a corresponding non-uniform multiresolution analysis and non-uniform wavelet transform and inverse transform. The approach has been tested for the case of bounded intervals, compression and reconstruction.

In [84], the dependence or the link between returns, betas and the return interval has been investigated. Based on the well-known Fama-french three factor model, the authors showed that both the mean returns and the betas are affected by the choice of time interval over which returns are measured. The authors proposed there the use of stochastic (thus, non-uniform) time intervals to describe more the situation of the market.

In the present work, we are concerned with this last approach, which will be applied as a suitable non-uniform wavelet sampling for the study of the CAPM. Other examples and methods to construct non-uniform wavelets and sampling may be found in [82]. To do this, we will recall here the construction of the non-uniform Haar wavelet due to [83].

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Haar wavelet system and multiresolution is based on the Haar scaling function $\varphi = \chi_{[0,1[}$, and the mother wavelet $\psi = \chi_{[0,1/2[} - \chi_{[1/2,1[}$. The idea to construct a non-uniform scaling function, nun-uniform wavelet and a non-uniform corresponding multiresolution analysis adapted to time series starts by subdividing the time interval of the series into suitable subdivision. Assume without loss of the generality that the series is defined on an interval $I = [0, T]$ and let $\{\Delta_m\}_{m=0}^{M}$ be a partition of I, with $M \in \mathbb{N}$ given such that $\Delta_m = \{x_k^{(m)}\}_{k=0}^{2^m}$, where

$$\begin{cases} x_0^{(m)} = 0 < \cdots < x_k^{(m)} < \cdots < x_{2^m}^{(m)} = T, \\ \text{and} \\ x_k^{(m-1)} = x_{2k}^{(m)}, \text{(this guarantees that } \Delta_m \text{ is a refinement of } \Delta_{m-1}), \end{cases}$$

for all $m \in \{1, 2, \ldots, M-1, M\}$, and $k \in \{0, \ldots, 2^{m-1}\}$. For each m, there will be a non-uniform scaling function $\varphi^{(m)}$ defined on each interval $x_k^{(m)} < \cdots < x_{k+1}^{(m)}$ by

$$\varphi^{(m)}(x) = \varphi_k^{(m)}(x) = \varphi\left(\frac{x - x_k^{(m)}}{x_{k+1}^{(m)} - x_k^{(m)}}\right) = \chi_{[x_k^{(m)}, x_{k+1}^{(m)}[}(x), \; x \in [x_k^{(m)}, x_{k+1}^{(m)}[.$$

Similarly, the non-uniform wavelet $\psi^{(m)}$ will be defined as

$$\begin{aligned} \psi^{(m)}(x) &= \psi_k^{(m)}(x) = \psi_{\alpha_k^{(m)}}^{(m)}\left(\frac{x - x_k^{(m)}}{x_{k+1}^{(m)} - x_k^{(m)}}\right) \\ &= \alpha_k^{(m)} \chi_{[x_{2k+1}^{(m+1)}, x_{2k+2}^{(m+1)}[}(x) - (1 - \alpha_k^{(m)}) \chi_{[x_{2k}^{(m+1)}, x_{2k+1}^{(m+1)}[}(x), \end{aligned}$$

where

$$\alpha_k^{(m)} = \frac{x_{2k+1}^{(m+1)} - x_k^{(m)}}{x_{k+1}^{(m)} - x_k^{(m)}}.$$

The non-uniform multiresolution analysis will be the sequence of spaces

$$V_m = spann\{\varphi_k^{(m)};\; k = 0, \ldots, 2^m - 1\},$$

as approximation spaces, and

$$W_m = spann\{\psi_k^{(m)};\; k = 0, \ldots, 2^m - 1\},$$

as detail spaces. In [83], it is proved that this construction leads as in the classical cases, to a 2-scale relation and some construction/reconstruction algorithms. More precisely, the following result is proved.

Proposition 1. *([83])*
The functions $\varphi_k^{(m)}$ and $\psi_k^{(m)}$ satisfy the following relations.

$$\varphi_k^{(m-1)} = \varphi_{2k}^{(m)} + \varphi_{2k+1}^{(m)},$$

$$\psi_k^{(m-1)} = -(1 - \alpha_k^{(m-1)})\varphi_{2k}^{(m)} + \alpha_k^{(m)} \varphi_{2k+1}^{(m)},$$

$$\varphi_{2k}^{(m)} = \alpha_k^{(m-1)} \varphi_k^{(m-1)} - \psi_k^{(m-1)},$$

$$\varphi_{2k+1}^{(m)} = (1 - \alpha_k^{(m-1)})\varphi_k^{(m-1)} + \psi_k^{(m-1)}.$$

Let $f \in V_m$ be written as

$$f = \sum_{k\in\mathbb{Z}} a_k^{(m)} \varphi_k^{(m)} = \sum_{k\in\mathbb{Z}} a_k^{(m-1)} \varphi_k^{(m-1)} + \sum_{k\in\mathbb{Z}} c_k^{(m-1)} \psi_k^{(m-1)}.$$

We have the following filter rules for the reconstruction,

$$a_k^{(m-1)} = (1-\alpha_k^{(m-1)})a_{2k+1}^{(m)} + \alpha_k^{(m-1)} a_{2k}^{(m)},$$
$$c_k^{(m-1)} = a_{2k+1}^{(m)} - a_{2k}^{(m)},$$

and the following for the decomposition,

$$a_{2k}^{(m)} = a_k^{(m-1)} - (1-\alpha_k^{(m-1)})c_k^{(m-1)},$$
$$a_{2k+1}^{(m)} = a_k^{(m-1)} + \alpha_k^{(m-1)} c_k^{(m-1)}.$$

4. Methodology

Our aim as raised above is the development of a general version of the CAPM that takes into account two main factors such as the missing data and the time factor by applying wavelet time-frequency analysis. Our new idea consists firstly in applying a backward-forward method to reconstruct missing data, and next a non-uniform time scaling in the evaluation of the wavelet CAPM by means of a set of non uniformly-timed samples corresponding to the so-called non-uniform wavelets instead of classical wavelets.

4.1. The original mathematical formulation of the CAPM

The CAPM in its initial form is mathematically expressed by means of the equation

$$R_{i,t} = \alpha_i + \beta_i R_{m,t} + u_{i,t}, \tag{19}$$

based on the return rate of an action i at the period time t denoted $R_{i,t}$, the return of the market measured by means of a general index at the same period of time t and denoted $R_{m,t}$, the beta coefficient or the systematic risk which is a specific factor to each action i that indicates the relation between the fluctuations of the return rate of the action i and the fluctuations of the general index of the market. This is denoted here β_i. The factor $u_{i,t}$ is a random factor relative to the hidden facts of the return $R_{i,t}$ that are not explained by the market. It is generally an error term. Finally, the parameter α_i is a calibration factor for a null expectation of the error $u_{i,t}$.

According to the model (19), the CAPM splits the total variability of an action into two main parts. One part is relative to the market, and corresponds to the systematic risk. The second is related to the specific characteristics of the action, and reflects the variations of the specific prices of such action.

In the literature, due to the movements in the markets, the appearance of new crisis, many socio-polytical and economic movements in the world, the CAPM has been modified many times to be adapted to the situation studied. For example, in [8] the risk beta is evaluated as the ratio of the covariance of the assets and the market by the variance of the market during the whole period of study,

$$\beta_i = \frac{Cov(R_i, R_m)}{Var(R_m)} = \frac{\sigma_{im}}{\sigma_m^2}, \tag{20}$$

leading to an expectation based formulation

$$E(R_i) = r_f + \beta_i (E(R_m) - r_f). \tag{21}$$

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where $E(R_i)$ is the expected return of the action i, $E(R_m)$ is the expected return of the market, and r_f is the so-called remuneration of the risk-free.

Another form of the CAPM depends on the number of investors in the market. Denote K the total number of such investors, and assume that each investor $k = 1, 2, \ldots, K$ have to allocate a wealth T_k in a number M_k of actions. The risk beta is evaluated as the ratio the covariance σ_{ik} between the return of action i and the portfolio by the variance σ_k^2 of the optimal portfolio (relative the the k-th investor), $\beta_{ik} = \frac{\sigma_{ik}}{\sigma_k^2}$. The CAPM is written on the form

$$E_i = r_f + \beta_{ik}(E_k - r_f), \tag{22}$$

where E_i is the expectation of the return of action i, and E_k is the expectation of the return of the optimal portfolio detained by the k-th investor. See [1,2,50].

Some extending variants of the CAPM may be also found in the literature such as the inflation-based CAPM, and taxation-based CAPM ([85]), CAPM depending on heterogeneity of anticipations ([62]).

In our present work, the model of the CAPM expressed by means equation (21), where we recall that $E(R_i)$ is the expected return due to the asset i, r_f is the risk-free rate, $E(R_m)$ is the market portfolio expected return, and β_i is the measure of risk for asset i evaluated by means of equation (20).

Now, empirically, as in the literature, the risk or the coefficient β_i will be evaluated by applying the usual ordinary least square method via a linear regression on the form

$$R_{it} - r_{ft} = \alpha_{it} + \beta_i(R_{mt} - r_{ft}) + u_{it} \tag{23}$$

where the index t is designated for the time, u_{it} is the error term while α_{it} is a calibration or regulation constant. By means of the variance measure, this means that the variance of the asset i is splitted into two parts such as

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2. \tag{24}$$

The first part $\beta_i^2 \sigma_m^2$ corresponds to the firm's systematic risk. while the second part $\sigma_{\varepsilon_i}^2$ corresponds to the firm's unsystematic risk.

4.2. Backward-forward wavelet reconstruction of missing data

The present section is concerned to the development of a wavelet approach for the well-known problem of missing and uncertain data confronted usually in financial time series. We aim to develop a prior procedure to complete the gaps of data used in the study. Indeed, in the present case of study, we noticed that the origin data bases such as web sites, although they are widely used, there is always and already a big lack of data. Usually the simple method to fill the gaps may be done by searching for different suppliers. This in turns may lead again to uncertain data, costs, and also loss of time. We thus thought about sophisticated methods to fill the gaps of data instead of applying non scientific methods.

Our procedure to be applied here is in fact a modified or a compilation of the one developed in [39,41] to reconstruct missing data on short time intervals, differently from classical methods where in the majority of cases, the prediction is just applied to construct future values based usually on a long-time training interval from past values. In the present study, some cases of data has missing values in the beginning (the past) that have to be filled. Moreover, in some situations such as the target period applied here, some characteristics such as seasonality, periodicity and/or auto-regressive aspect could not be applied. This is obvious for many causes, such as the main characteristic of the period which is known as the spring Arab revolution period, and which in turns was followed by many important political, social, financial and economic movements not present in the past, and which the governments work hardly to overcome them.

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Let $S(t)$, $t \in [0, T]$ be the financial time series. To be adequate with the notations of statistical series, we will write (t_i, S_i), $i = 1, 2, \ldots, N$, the financial series, and consider its wavelet decomposition at a level J, relative to a (father,mother) wavelet system (φ, ψ),

$$S^J = \sum_k a_{J_0,k}\varphi_{J_0,k} + \sum_{j=J_0+1}^{J} \sum_k d_{j,k}\psi_{j,k}. \tag{25}$$

The missing values are classified into two classes.

- For a missing value S_k, $k = 1, 2, \ldots, N$, we compare the lengths of the known segments after and before this values. We next chose the one with greater length. Denote for example L_k such greater length, and $I_k = \{i_1^k, i_2^k, ..., i_{L_k}^k\}$ the interval corresponding.
- Consider next the truncated time series $\widetilde{S}_i$ corresponding to (t_i, S_i), $i \in I_k$, and its wavelet decomposition as in (25).
- If I_k is an after-interval to the missing value, we estimate the value $S(i_1^k - 1)$ by

$$\widehat{S_{i_1^k-1}} = \frac{\sum_{m=1}^{L_k} S_{i_m^k} K\left(\frac{t_{i_m^k} + t_{i_1^k-1}}{h}\right)}{\sum_{m=1}^{L_k} K\left(\frac{t_{i_m^k} + t_{i_1^k-1}}{h}\right)}. \tag{26}$$

- If I_k is a before-interval to the missing value, we estimate the value $S(i_{L_k}^k + 1)$ by

$$\widehat{S_{i_{L_k}^k+1}} = \frac{\sum_{m=1}^{L_k} S_{i_m^k} K\left(\frac{t_{i_m^k} - t_{i_{L_k}^k+1}}{h}\right)}{\sum_{m=1}^{L_k} K\left(\frac{t_{i_m^k} - t_{i_{L_k}^k+1}}{h}\right)}. \tag{27}$$

  where K is a suitable prediction kernel, and h is a suitable prediction window.
- Whenever the after and before intervals have the same length we take the mean value of the two predicted ones.
- It remains the case where many successive values in the series are missing. We thus consider the extremities of the missing segments as starting points to be predicted.
- Finally, each predicted value is added to the series, and the new series is re-considered for the next step.

In the present work, a recursive procedure is applied as in [86], by acting the previous rules (26)-(27) on the approximation and the detail coefficients by considering

$$\widehat{a_{j,L_k+1}} = \frac{\sum_{m=1}^{L_k} a_{j,m} K\left(\frac{m-(L_k+1)}{h}\right)}{\sum_{m=1}^{L_k} K\left(\frac{m-(L_k+1)}{h}\right)}, \quad \text{and} \quad \widehat{d_{j,L_k+1}} = \frac{\sum_{m=1}^{L_k} d_{j,m} K\left(\frac{m-(L_k+1)}{h}\right)}{\sum_{m=1}^{L_k} K\left(\frac{m-(L_k+1)}{h}\right)},$$

and similarly

$$\widehat{a}_{j,i_1^k-1} = \frac{\sum_{m=1}^{L_k} a_{j,m} K\left(\frac{m+(i_1^k-1)}{h}\right)}{\sum_{m=1}^{L_k} K\left(\frac{m+(i_1^k-1)}{h}\right)}, \quad \text{and} \quad \widehat{d}_{j,i_1^k-1} = \frac{\sum_{m=1}^{L_k} d_{j,m} K\left(\frac{m+(i_1^k-1)}{h}\right)}{\sum_{m=1}^{L_k} K\left(\frac{m+(i_1^k-1)}{h}\right)}.$$

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This means that to forecast the value of the series, the procedure is transformed to the prediction of its wavelet/scaling components. We thus obtained a dynamic recursive scheme consisting in applying firstly a partial estimator at short horizons to all the observations to yield firstly the predicted values. These values are included as new observations to predict the next. We then follow the same steps until reaching the desired horizon. For backgrounds, more details, and other applications of analog methods to the present one, the readers may refer to [39,74,86,87].

4.3. The wavelet CAPM processing

The idea of the wavelet estimation of the CAPM goes back essentially to the works of [44,67], and [59]. In the first references, the authors introduced the general framework of wavelet analysis of time series, and the eventual applications in finance and economics. Next, Fernandez [59] has focused essentially on the CAPM and its improvement by means of wavelets by adopting the time scale factor in the empirical study. The investigations and applications of the wavelet methods in the CAPM have been next growing up (see [23,34,35,39,44,45,57–63,67,68,70,75,76]) .

The wavelet CAPM consists in decomposing the variance of the financial series issued from the whole market index, and the actions in the market (or the portfolio) into sub-variances corresponding to the decomposition levels j (called the j-level variance or the variance at the scale j). More precisely, denote as in the previous sections $S(t)$, $t \in [0, T]$ a financial time series, and σ_S^2 its variance. Denote also $\sigma_S^2(j)$ the variance of the projection $A_j(S)$ on the j-level approximation space V_j due to the multiresolution. The variance σ_S^2 is splitted as

$$\sigma_S^2 = \sum_j \sigma_S^2(j). \tag{28}$$

This leads to the analysis of the sub-variances relative to the different horizons j instead of the total one, which facilitates the analysis and the interpretation. The j-level variance $\sigma_S^2(j)$ is evaluated as in [39,40,59,62] as

$$\widehat{\sigma}_X^2(j) = \frac{1}{2^j(N_j - L_j)} \sum_{k=L_j-1}^{N_j-1} d_{j,k}^2, \tag{29}$$

L is the length of the wavelet filter, $N_j = [N/2^j]$ is the number of wavelet coefficients involved in the j-level approximation of the series, and $L_j = [2^{-j}(L-2)(2^j-1)]$ stands for the number of boundary wavelet coefficients at the level j. The same principle will be also applied for the j-level covariance between a couple (X, Y) of financial series

$$\widehat{\sigma}_{XY}^2(j) = \frac{1}{2^j(N_j - L_j)} \sum_{k=L_j-1}^{N_j-1} d_{j,k}^X d_{j,k}^Y. \tag{30}$$

To apply our wavelet method to the estimation of the CAPM, the series considered will be replaced by the returns due to the market and those due to the actions. The coefficient beta estimated in (20) will be replaced by the so-called wavelet beta at the level j as

$$\widehat{\beta}_i(j) = \frac{\widehat{\sigma}_{R_iR_m}(j)}{\widehat{\sigma}_{R_m}^2(j)} \tag{31}$$

where $\widehat{\sigma}_{R_iR_m}(j)$ is the wavelet covariance of the component i of the portfolio with the market at the scale j estimated by (30), with X being replaced by the j-level approximation/details of the return R_i, and Y is replaced by the j-level approximation/details of the return R_m, and finally, $\widehat{\sigma}_{R_m}^2(j)$ is the j-level wavelet variance due to the j-level approximation/details of the return R_m of the market.

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The final step allowing to explain the explanatory power of the market returns on the portfolio returns, is concerned to the estimation of the determination coefficient $R_i^2(j)$ for each return time scale as follows

$$R_i^2(j) = \beta_i(j)^2 \frac{\sigma_{R_m}^2(j)}{\sigma_{R_i}^2(j)}. \tag{32}$$

The main procedure in the present work is to serve of the non-uniform wavelet analysis to estimate the risk beta, compare with existing methods (based on uniform estimations), and deduce eventually the power of the new process. We will thus denote for the rest of the paper β^c to designate the classical beta estimated without use of wavelets, β^u to designate the classical wavelet beta, and β^{nu} to designate the non-uniform wavelets introduced here. Similarly, we will adopt the same notations (upper-scripts) for the R^2 determination coefficient.

5. The multifractality of the market

Financial time series are always characterized by the stylized facts. The nature or the aspect of the distribution tail has always to be noticed. This may be done via the kurtosis measure which permits to confirm if the series is leptokurtic or not.

Financial time series have also high frequency components, explained always by the volatility clustering. This behavior is always explained by the presence of many hidden factors and/or aspects such as the random or stochastic behavior of markets.

To understand these problems in the market, the researchers have included the so-called scaling law invariance in the study of the volatility or the market in general. Many tools have been thus investigated for the aim of understanding aspects of non stationary, auto-regression, filtering, support vector machine models and prediction, neural networks models and predicting.

However, it seems that wavelets are the last tool involved, and are the most efficient compared to past tools. In financial series, indeed, there are underlying the fluctuations causal information cascades from increasing (large) to decreasing (small) time scales called self-similarity law or generally scaling law. Such facts may be easily visualized with the wavelet representation.

To confirm the presence of the fractal/multifractal behaviour of the market, such as the self similarity, a multifractal test should be done consisting in evaluating the multifractal spectrum of the index.

An original way to evaluate such a spectrum starts by evaluating the increments of the index. For a series S, these are defined as

$$\Delta S_i = S(t_{i+1}) - S(t_i),$$

for a suitable subdivision of the time interval. Next, we compute an associated partition function by means of the q-increments as

$$Z(q) = \sum_{i=0}^{N_\nu} |\Delta x_i|^q.$$

The scaling law $\tau(q)$ is obtained by regressing the logarithm $\log Z(q)$ of the partition function $Z(q)$ against the step time $\log \nu$ in the sense that

$$Z(q) \approx \nu^{\tau(q)}.$$

The scaling function $\tau(q)$ yields next the multifractal spectrum by means of the Legendre transform

$$d(\alpha) = \inf_q (\alpha q - \tau(q)).$$

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In general, for mono-fractal series, we get a linear scaling function. In the case where the series is volatile (highly volatile), the scaling function should be a concave curve. So, the multifractal test will consist in visualizing the scaling law function τ or equivalently the multifractal spectrum $d(\alpha)$.

Another way to explore the volatile behavior of the market is to conduct the so-called detrended fluctuation analysis (DFA) in its original form or its recent wavelet extension. The DFA of the series $S_i = X_{t_i}$, $i = 1, 2, \ldots, N$ statrs by computing its integrated form

$$IS_t = \sum_{i=1}^{t} (S_i - \overline{S}), \tag{33}$$

where $\overline{S}$ is the mean value of the series X_t. Let next $n \in \mathbb{N}$ be fixed and I_k, $k = 1, ..., n$ be a subdivision of the time interval I_N. For all k, the least squares fit (denmoted IS_t^k) of the series IS_t is associated IS_t^k. An overage fluctuation function is next introduced as

$$AF_{S_t}(n) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (IS_i - IS_i^n)^2}.$$

The DFA consists in evaluating the power-law dependence of the function $AF_{S_t}(n)$ on n. When $AF_{S_t}(n) \sim n^{\alpha}$, the series is self fluctuating with scaling exponent α.

The DFA is next extended or improved by the so-called multifractal detrended fluctuation analysis (MDFA), which is based essentially on two parameters,

- the Hurst-Hölder exponent which permits to estimated the local regularity of the series and consequently leads to the fluctuations inside it.
- the multifractal spectrum of the series which allows to conclude about its possible multifractality of it.

The main idea improved by the MDFA is by considering general increments of the form

$$\Delta_{\nu}^{m} S(t) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} S(t + j\nu),$$

for $m \in \mathbb{N}$. Such an increment should be next estimated as

$$\Delta_{\nu}^{m} S(t) \sim \nu^{H_S(t)}.$$

The exponent $H_S(t)$ plays the role of the so-called Hurst-Holder exponent.

However, to avoid the difficult use of the functional spaces mathematical definition of such exponents, the MDFA has been re-developed using the wavelet tools. Indeed, in the wavelet MDFA (WMDFA), the partition function is defined as

$$Z_j(q) = 2^{-j} \sum_{k} |d_{j,k}|^p, \tag{34}$$

at a level of decomposition j. the $d_{j,k}$ are the well known wavelet coefficients (detail) or sometimes the wavelet leaders (applied in the last variant of WMDFA). To test the multifractal structure and compute next the multifractal spectrum we seek a function $\tau(q)$ that reflects the scaling law of $Z_j(q)$ as

$$Z_j(q) \sim 2^{-j\tau(q)}.$$

Backgrounds and details on multifractal analysis of financial (time) series, DFA, MDFA and WMDFA may be found in [38,62,77,78,88–97].

6. Empirical results and discussions

In this part, we will examine the influence of non-uniform time scale law on the understanding of the systematic risk beta relative to the Tunisian index TUNINDEX due to the Tunisian market as a representative of the so-called spring Arab countries. The influence of the political and social movements and instabilities on the economic situation and the future of the market will be investigated. In the literature on the CAPM, it is somehow confirmed that a linear dependence between action's returns and their systematic risks, and the risk of the market return exists. However, the dependence on the time scale is always hidden, and not explained explicitly in the mathematical model. Therefore, empirical investigations remain necessary to check such a dependence and influence. We thus focus in the present work on testing the hypotheses stating that

- The time scale law explains the stability/instability of the risk beta.

We propose in the empirical section to test a wavelet methodology of beta estimation on daily returns for TUNINDEX collected on the period of January 14, 2016 to January 13, 2021 resulting of a sample size $N = 1253$. We focused on a portfolio composed of 73 actions as listed in Table 1 with corresponding sectors. According to the Tunisian stock market web site, the first Tunisian market index was launched in September 30th 1990 under the name of BVMT index. TUNINDEX index and sector indices were created in December 31st 1997 and December 31, 2005 respectively. Theses indices aim to provide managers, analysts and savers with a measure of portfolios performance and stock market activities as a whole. Upon the decision of the expert committee, since January 02, 2009, BVMT index is no longer disseminated and the method of calculation of the TUNINDEX index and the sector indexes are changed. The TUNINDEX index and the sector indexes are no longer weighted by the total market capitalization but by the free float adjusted market capitalization. This method of calculation, already used for other major indices around the world, ensures greater coherence between the real nature of companies on the market and how it is expressed in the indices. (see http://www.bvmt.com.tn/en-gb/content/documentation). In fact during 2008-2013 a slightly different composition has been approved, but with no essential and great difference with the present one. It is based on the sector indices listed as Automobiles and parts, Banks, Construction, Consumer Goods, Industries, Raw Materials, Consumer Services, Financial Services, and Financial Companies. However, such a subdivision did not focus on the Energy sector for example, which is an important one in the econo-financial market in Tunisia as Oil and Phosphate for example are main energy exportation products. Therefore, an independent consideration of this sector is of great interest. Healthcare also is an active, and very important sector. Recall that, Tunisia was one of the rare countries that reached zero Corovavirus case at the first waves of the COVID-19. Tunisia has been also participated in fighting COVID-19 in many European countries such as Italy and Germany by medical groups. However, the last period, the country has confronted with a severe wave of COVID-19. The majority of analyzers explained this fact by the wide corruption in the last governments, especially the last one. Besides, the n on experienced governments after the revolution are strong cause of some failures. The sector of Technology did not appear also in the 2008-2013 composition, despite its importance. Recall here that a big number of industrial technological companies have already branches installed in Tunisia, due to the qualified, and also sheep labor as well as the short distance to Europe. The sector of Transportation needs also to be considered because of its basic role in the daily life of the people. It also serves to the transportation of import-export goods and materials such as Oil and Phosphate, and agricultural crops such as olive oil, dates, etc.

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Table 1: Tunindex Companies and Sectors.

Sector	Company Name	Abbreviation
Basic Materials	ALKIMIA chemical company	ALKM
	Carthage Cement	CC
	Les Industries Chimiques du Fluor	ICF
	One Tech Holding	OTH
	Papier et Carton	STPAP
	Societe Tunisienne de Verreries	STVR
Capital Goods	Les Ateliers Mecaniques du Sahel	AMS
	Manufacture De Panneaux Bois Du Sud	MPBS
	Les Ciments de Bizerte	SCB
	Servicom	SERVI
	Essoukna	SOKNA
	Société Moderne de Céramique	SOMOC
	Société Tunisienne d'Email	SOTEM
	Société Tunisie Profilé Aluminium	TPR
Consumer	Accumulateur Tunisien Assad SA designs	ASSAD
	Cerealis	CREAL
	Délice Holding	DH
	Euro-Cycles	ECYCL
	Societe Generale Industrielle de Filtration	GIF
	Hannibal Lease	HANL
	Land'Or	LNDOR
	Electrostar	LSTR
	New Body Line	NBL
	Office Plast	PLAST
	Poulina Group Holding	POULA
	Société d'Articles Hygiéniques	SAH
	Atelier du Meuble Interieurs	SAMAA
	Société Frigorifique et Brasserie de Tunis	SFBT
	Société de Production Avicole de Teboulba	SOPAT
	Universal Auto Distributors Holding	UADH
Energy	Air liquid	AL
	Transport des Hydrocarbures par Pipelines	STPIL
Healthcare	ADWYA company	ADWYA
	Pharmaceutiques de Tunisie	SIPHA
	nité de Fabrication des Médicaments	UMED
Technology	Advanced e-technologies	AETEC
	Appareillage et Matériels Electriques	SIAM
	Entreprises de Télécommunications	SOTE
	Telnet Holding	TLNET
Transportation	Société Tunisienne de l'Air	TAIR

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Financial	Amen Bank	AB
	Arab Tunisian bank	ATB
	Banque De L'Habitat	BH
	BH Assurance	BHASS
	Banque Internationale Arabe Tunisie	BIAT
	Banque Nationale Agricole	BNA
	Banque Attijari De Tunisie	BS
	Banque De Tunisie	BT
	Banque De Tunisie Et Des Emirats	BTEI
	Placement et Developpement Industriel et Touristique	SPDI
	Société Tunisienne d'Assurances et de Réassurances	STAR
	Societe Tunisienne De Banque (STB)	STB
	Tuninvest SICAR	TINV
	Société Tunisienne de Réassurance	TRE
	Union Bancaire pour le Commerce et L'Industrie	UBCI
	Union Internationale de Banques	UIB
	Wifack International Bank and leasing	WIFAK
Services	Automobile Réseau Tunisien et services Renault	ARTES
	Arab Tunisian lease	ATL
	BH Modern Leasing	BHL
	Cellcom	CELL
	Compagnie Internationale De Leasing	CIL
	City Cars	CITY
	Hannibal Lease	HANL
	Société Tunisienne des Marchés de Gros	MGR
	Monoprix	MNP
	Ennakl Automobiles	NAKL
	Société Immobilière et de Participations	SIMP
	Société Immobilière Tunisio Saoudienne	SITS
	Societe Magasin General	SMG
	Tawasol Group Holding	TGH
	Attijari Leasing	TJL
	Tunisie Leasing & Factoring	TLS
	Société HEXABYTE (internet)	XABYT

The choice of this market is motivated by the fact that Tunisia is a representative country of the so-called Arab spring, as target period of study which concerns the first 5-years-after the revolution and depends strongly on the socio-econo-political stability in the revolutionary countries. In the end of such period, the situation of the country has been affected as the rest of the world by the appearance of the pandemic COVID-19 which has a severe influence on the economic situation in all the world, especially the Arab spring countries where the political situation remained unstable also. Besides, and although many factors influenced the economic development in Tunisia, the strategic geographic position of the country in the mid of the south Mediterranean sea, allows it to be a central point in the inter-changing between the north and the south rivers, and thus plays a great role in the economic, social and political movements in this region of the world. Thus, a long-standing co-operation has been established with the European council since 2011, just after the revolution, to develop many sectors such as human rights, democracy, combating violence, improving the functioning of justice, the fight against corruption, which were immediately reflected in the partnership with Tunisia especially in the period 2015–2021. These priorities will have surely a great effect on the stability of the country, and thus permit the investment and the development to be pursued. It is also worth to notice that compared to other Arab spring countries, Tunisia is the most stabilized country that could pass the revolutionary period without a great instability as in some others where after-revolution wars have been broken out between

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the regimes and opposite parts. Such situations have now influenced quasi all the world by inducing great movement of illegal immigration, refugees movements, appearance and spread of terrorist organizations, ..., etc.

Now, the starting step consists in acting a wavelet (uniform or non-uniform) method to reconstruct missing data. On the total basis we recorded a total number of 206 missing values dispersed on the whole market, on different time dates (daily) as in Table 2. A complete and adjusted basis on the period of study is thus obtained and applied by the next.

Table 2: Missing data.

Stock	Number of missing values
ALKM	15
AST	Omitted
BL	Omitted
BTEI	03
CREAL	25
HANL	12
MIP	Omitted
PLAST	35
PLTU	Omitted
SAMAA	17
SIPHA	5
SMD	Omitted
SOTEM	53
TVAL	Omitted
UMED	23
WIFAK	18
Total	206 Missing values

We propose to study the relationship between excess return on each individual stock and the time scales of market portfolio using (23). The daily return of each stock is estimated as a log-price difference

$$R_{it} = \log P_{i,t} - \log P_{i,t-1}, \tag{35}$$

where $P_{i,t}$ is the price of asset i at day t. The market return R_{mt} is similarly evaluated as a log-difference

$$R_{mt} = \log I_t - \log I_{t-1}, \tag{36}$$

where I_t is the index value at day t.

Table 3 below shows the descriptive statistics of the market. The statistics correspond precisely to the return excess for each company relatively to the risk-free.

Table 3: Descriptive statistics of excess returns.

Stocks	Mean	Maximum	Minimum	Std.	Skewness	Kurtosis
Tunindex	0.0002	-0.0419	0.0268	0.0048	-1.4085	15.2260
AB	0.0087	-2.4888	13.2143	0.4131	25.0862	842.1409
ADWYA	-0.0000	-0.0678	0.0985	0.0168	0.5349	5.3468
ATEC	-0.0019	-0.2136	0.1727	0.0277	-0.4266	8.7224
AL	-0.0010	-0.1050	0.0879	0.0187	-0.3908	7.0118
ALKM	0.0004	-0.3394	0.2179	0.0373	-1.4224	19.6617
AMS	-0.0014	-0.2447	0.1620	0.0283	0.2387	10.3742
ARTES	-0.0003	-0.0776	0.0581	0.0124	-0.3018	7.7441
ASSAD	-0.0001	-0.1389	0.1390	0.0173	0.3309	9.9634
ATB	-0.0004	-0.0859	0.0705	0.0163	0.0910	6.0719
ATL	-0.0004	-0.1178	0.0635	0.0157	-0.2162	7.8302
BH	-0.0001	-0.0613	0.0592	0.0166	0.3649	4.0558
BHASS	0.0005	-0.1676	0.1313	0.0235	-0.3755	7.8264
BHL	-0.0013	-0.2333	0.1204	0.0236	-0.5241	11.7126
BIAT	0.0002	-0.0628	0.0591	0.0119	-0.0987	7.1844
BNA	-0.0002	-0.0945	0.1008	0.0149	0.5917	7.9965
BS	-0.0000	-0.0638	0.0586	0.0131	-0.4141	7.6677
BT	-0.0002	-0.2234	0.0812	0.0119	-5.0655	107.1594
BTEI	-0.0010	-0.1022	0.1313	0.0217	-0.0025	6.8876
CC	0.0002	-0.4079	0.1244	0.0253	-2.9116	57.6041
CELL	-0.0004	-0.1468	0.1004	0.0242	0.0561	5.1432
CIL	0.0001	-0.0929	0.0574	0.0152	-0.6240	7.5173
CITY	-0.0005	-0.2813	0.0548	0.0155	-4.9836	89.8893
CREAL	0.0002	-0.0886	0.0843	0.0222	0.0205	3.6709
DH	-0.0000	-0.0640	0.0575	0.0141	0.0287	4.2832
ECYCL	0.0000	-0.0762	0.0876	0.0179	0.2912	5.2231
GIF	-0.0007	-0.0712	0.2624	0.0240	2.0608	18.9931
HANL	-0.0009	-0.2944	0.0788	0.0184	-3.7005	57.9889
ICF	0.0003	-0.1473	0.1967	0.0266	0.3366	8.1322
LNDOR	0.0004	-0.0754	0.1015	0.0206	0.2629	4.3643
LSTR	-0.0011	-0.0953	0.2537	0.0305	1.0498	8.5629
MGR	0.0003	-0.0954	0.0591	0.0157	-0.0687	5.4146
MNP	-0.0006	-0.0609	0.0585	0.0162	0.1608	3.8115
MPBS	-0.0004	-0.1181	0.0598	0.0198	-0.0302	4.6772
NAKL	0.0001	-0.1130	0.0411	0.0101	-1.7371	23.5821
NBL	-0.0006	-0.0987	0.0582	0.0159	-0.3225	6.2855
OTH	0.0005	-0.0533	0.0575	0.0125	0.4109	5.1508
PLAST	-0.0005	-0.2926	0.2551	0.0368	-2.3356	22.3834
POULA	0.0006	-0.0600	0.0733	0.0143	0.4850	6.1390

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SAH	0.0003	-0.0834	0.0570	0.0130	-0.2102	6.8012
SAMAA	-0.0002	-0.1863	0.5857	0.0248	10.2429	252.4786
SCB	-0.0010	-0.1699	0.0994	0.0223	-0.4036	8.3289
SERVI	-0.0012	-0.6222	0.4215	0.0382	-2.2380	70.8611
SFBT	-0.0000	-0.2233	0.0581	0.0147	-6.5972	99.4471
SIAM	0.0004	-0.0734	0.0591	0.0162	0.3084	4.6604
SIMP	-0.0005	-0.0917	0.0884	0.0178	-0.0846	4.7251
SIPHA	-0.0009	-0.2283	0.1644	0.0344	-0.8407	8.3417
SITS	-0.0001	-0.1738	0.1101	0.0206	-0.4421	11.5821
SMG	-0.0004	-0.1064	0.0872	0.0201	-0.1493	5.1333
SOKNA	-0.0007	-0.0854	0.0580	0.0187	-0.2385	4.1779
SOMOC	-0.0006	-0.0588	0.1325	0.0186	0.7649	6.1544
SOPAT	-0.0005	-0.0762	0.1495	0.0254	0.7514	4.9379
SOTE	0.0010	-0.0935	0.1459	0.0249	0.7980	5.6197
SOTEM	0.0007	-0.2538	0.1178	0.0266	-1.2298	16.6829
SPDI	0.0001	-0.0963	0.0864	0.0167	-0.2026	6.8780
STAR	-0.0002	-0.1068	0.0631	0.0176	-0.0117	5.0213
STB	-0.0004	-0.0785	0.0709	0.0178	0.2621	4.2858
STPAP	0.0001	-0.1058	0.0587	0.0167	-0.0425	5.9996
STPIL	0.0003	-0.0975	0.0593	0.0168	-0.0744	5.3405
STVR	0.0005	-0.2045	0.0819	0.0168	-0.9566	21.3257
TAIR	-0.0002	-0.0592	0.1586	0.0202	1.0954	7.7621
TGH	0.0003	-0.0690	0.1710	0.0239	0.8023	6.9059
TINV	0.0004	-0.1261	0.2003	0.0236	0.1690	9.3758
TJL	-0.0008	-0.1799	0.0591	0.0190	-0.6743	10.1483
TLNET	0.0007	-0.0920	0.0591	0.0171	0.3615	4.9269
TLS	-0.0006	-0.0596	0.0590	0.0172	0.1369	3.7408
TPR	0.0001	-0.1193	0.0580	0.0140	-0.4542	9.6766
TRE	-0.0001	-0.0739	0.0541	0.0132	-0.1659	4.8217
UADH	-0.0010	-0.1045	0.1493	0.0254	0.7522	6.1941
UBCI	-0.0005	-0.2169	0.0965	0.0221	-0.8370	12.1977
UIB	0.0000	-0.0627	0.0590	0.0119	-0.0748	6.8758
UMED	-0.0000	-0.0596	0.1255	0.0143	0.8344	8.8368
WIFAK	-0.0006	-0.9776	0.1110	0.0314	-3.9557	746.9421
XABYT	-0.0002	-0.0872	0.0654	0.0187	-0.2181	4.2306

In the present case, an approximately zero median value is obtained. The flatness and distortion features of all stocks' returns are different from each other. Moreover, the Jarque-Bera test leads to $JB = 1$ which rejected the null hypothesis at the 5% significance level, and 0 otherwise.

Notice from Table 3 that as in the majority of studies of financial markets, return excess of actions relatively to the risk-free as well as the return excess of the market relatively to its risk-free have always low skewness and high kurtosis.

Our analysis acts by projecting equation (23) relatively to time scales to test the effect of time on the systematic risk beta. This will be conducted by splitting the market returns into crystals or horizons relative to different time scales instead of using the classical periods such as weeks, moths, years.

The coefficients of the linear regressions will be estimated by the usual ordinary least square (OLS) estimates of the returns $(R_{it} - r_f)^j$ on the one of the market $(R_{mt} - r_f)^j$ for each level j. This leads to a j-level mathematical formulation as

$$(R_i - r_f)^j = \alpha_i^j + \beta_i^j (R_m - r_f)^j + \varepsilon_i^j = \alpha_i^j + \alpha_i D_m^j + \varepsilon_i^j. \tag{37}$$

The correspondence scale and dynamic days applied here are resumed in Table 4.

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Table 4: Time scales.

The time scale law	Number of dynamic days	
2-scale law classical method	J=1	2-4 dynamic days
	J=2	4-8 dynamic days
	J=3	8-16 dynamic days
	J=4	16-32 dynamic days
	J=5	32-64 dynamic days
	J=6	64-132 dynamic days
2-scale law uniform wavelets	J=1	2-4 dynamic days
	J=2	4-8 dynamic days
	J=3	8-16 dynamic days
	J=4	16-32 dynamic days
	J=5	32-64 dynamic days
	J=6	64-132 dynamic days
Scale law for non-uniform wavelets	J=1	$a_1 - a_2$ dynamic days
	J=2	$a_2 - a_3$ dynamic days
	J=3	$a_3 - a_4$ dynamic days
	J=4	$a_4 - a_5$ dynamic days
	J=5	$a_5 - a_6$ dynamic days
	J=6	$a_6 - a_7$ dynamic days
	...	...
	$J = J_{max}$	$a_{J_{max}} - a_{J_{max}+1}$ dynamic days

$(a_j)_j$ is an integer-valued random sequence.
J_{max} is chosen in such a way that we cover approximately one year.

6.1. The original CAPM processing

The first step is concerned with the estimation of the different regressions of the return excess of actions relatively to the one of the market at different scales $j = 1, 2, 3, 4, 5, 6$ obtained by the OLS estimates methods due to the classical method based on the dyadic scales. These estimates are resumed in Table 5.

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Table 5: 2-scale law estimations of the return excess of actions on the market.

STOCKS	Scale 1	Scale 2	Scale 3	Scale 4	Scale 5	Scale 6
AB	-0.0002	0.0591	-0.2180	-0.3776	17.8485	0.2997
ADWYA	-0.4076	-0.2126	-0.4743	0.2611	1.3677	-0.1445
ATEC	4.9438	-0.3863	-1.3218	-1.1750	-2.1399	1.0033
AL	-0.3212	0.1046	-0.2566	0.1659	0.6005	-0.1617
ALKM	1.5106	-0.9457	-0.9728	0.0689	2.0120	-0.4014
AMS	-0.0383	0.6879	-1.0509	1.2349	-0.7006	-1.9689
ARTES	6.2642	0.1509	-0.6865	0.1292	0.2132	-0.3064
ASSAD	9.6228	-0.5458	1.5148	-0.2662	-0.2275	0.1833
ATB	0	-1.1645	-0.0226	0.4832	-0.3619	0.1468
ATL	5.1037	0.5277	-0.1794	0.4683	-0.4400	0.0506
BH	-5.6918	-1.6956	0.3970	0.4731	0.5624	-0.4528
BHASS	2.6988	-1.8498	-0.2609	-1.0331	0.0107	-0.7261
BHL	4.5956	1.8404	-4.4186	-0.2500	-0.5940	-0.9149
BIAT	-5.2716	-1.6206	0.8517	1.3981	0.3008	0.7195
BNA	-4.4661	-0.3022	2.9580	-1.0840	-0.7884	-0.2922
BS	0.0775	-0.1653	0.1535	0.3048	0.7507	0.2246
BT	-3.9811	0.7353	0.1474	-1.1155	-0.4233	0.3121
BTEI	0.9986	-0.8905	0.2280	0.0113	-0.6924	-0.0826
CC	-16.7293	-1.1723	-0.0524	-0.6406	-0.6297	-0.7147
CELL	-5.0292	2.7989	0.6042	0.8133	1.5268	-0.1628
CIL	-3.7381	0.0878	0.0264	0.2987	-0.1531	0.1260
CITY	-1.3873	3.0872	0.4538	-0.0802	0.5985	-0.3514
CREAL	0.2708	0.0931	0.2473	-0.0153	0.0622	-0.2224
DH	0.3341	0.2288	0.0997	-0.8825	0.0088	-0.7322
ECYCL	2.4704	-0.8962	1.1489	-0.7773	0.6348	-0.8599
GIF	-7.7808	-5.1153	-1.3321	-0.3068	-1.2748	0.8006
HANL	-3.8630	0.4530	-0.3252	0.1629	0.4729	0.3479
ICF	-2.4487	-0.1790	0.9225	-1.8262	1.5302	-0.1800
LNDOR	-2.2172	0.5354	0.7557	-0.6854	3.0528	-1.1988
LSTR	-3.4115	4.5299	0.2364	-2.9891	-0.0588	-2.1696
MGR	-12.2621	-1.6788	0.7993	-0.1839	-0.3054	-0.0730
MNP	2.8838	0.2526	0.9393	-0.0207	-0.2951	-0.1613
MPBS	-1.6569	2.8330	-0.1618	0.1275	-0.8233	0.9380
NAKL	2.0220	0.3266	-0.0735	0.2530	-0.2231	0.5620
NBL	-2.6698	2.3679	1.0564	0.0298	-0.9252	1.0133
OTH	0	-0.0247	-0.0741	0.1825	0.2460	0.1813
PLAST	24.7637	3.0722	4.8113	2.9020	3.5928	0.2959
POULA	0.0456	-0.1827	-0.9960	0.7201	-0.4406	-0.0554
SAH	-6.0135	3.2193	1.0731	0.5983	0.1785	0.9718

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SAMAA	-6.8998	0.5338	1.0301	0.1694	-0.7055	0.6076
SCB	4.1182	0.9101	0.3385	-0.0799	-0.6495	0.6318
SERVI	-6.0288	-1.7968	-0.4037	-1.2213	-0.7486	1.0757
SFBT	-0.0969	-0.9255	2.1951	0.8171	0.1873	0.7301
SIAM	13.4455	-2.0938	0.6366	1.6688	0.9618	-0.1210
SIMP	1.5868	0.1124	0.1382	0.7009	-0.7430	0.6222
SIPHA	1.7590	0.8249	0.6905	-1.3421	0.8054	1.0517
SITS	1.0800	0.3311	-0.5158	0.2688	-0.2104	-0.1411
SMG	0.1202	0.9981	-0.7714	-0.3344	0.4702	-0.0715
SOKNA	1.8296	0.9958	-0.3290	1.0725	1.1269	-0.0022
SOMOC	-4.2680	0.5659	0.4614	0.6776	0.1037	-0.0181
SOPAT	3.4854	-1.3874	1.3892	0.7178	0.1555	0.7017
SOTE	-2.0371	-2.2244	-0.2732	-1.8874	-0.1659	-0.1844
SOTEM	6.7572	-3.7027	1.3099	-1.6350	-1.0877	0.5220
SPDI	9.7634	0.3961	0.4404	-0.4555	0.0582	0.1018
STAR	-0.7808	-0.9420	0.8378	-0.6899	-0.4044	-0.626
7 STB	-1.7339	-3.3515	1.4811	0.7048	0.1236	-0.3066
STPAP	5.2770	1.4540	-0.1801	0.6719	0.3612	0.0973
STPIL	4.7181	2.5081	0.9031	-0.4841	-0.6429	0.1618
STVR	0.2440	-4.1733	0.2800	0.1239	0.6635	-0.1412
TAIR	-8.7601	0.8458	0.7183	2.0956	-0.4845	1.0164
TGH	1.8449	0.4060	1.4901	1.2457	-0.1457	0.3129
TINV	5.2113	-2.5005	0.6575	-0.2680	-1.7215	0.5694
TJL	0.6007	-0.6622	-0.8262	-0.3960	0.9857	-0.4978
TLNET	0.1318	-0.5631	0.6544	-0.2373	0.2614	0.1005
TLS	-3.1588	-1.1106	-0.1840	0.5625	-0.1896	0.0433
TPR	3.5238	1.8559	0.0309	0.8883	0.2713	0.5553
TRE	-2.7765	-4.1528	-0.4250	-0.3026	0.1116	-0.0639
UADH	-3.6072	0.2348	-0.0796	-1.2127	0.6951	-0.8228
UBCI	9.5095	-1.9248	1.1071	-2.5089	0.1628	-0.8903
UIB	3.0589	0.2199	-0.5420	0.0428	0.1087	-0.0029
UMED	0.8953	0.0313	0.0958	-0.0257	0.4525	0.0611
WIFAK	-15.6149	-0.4211	-0.1457	-5.4381	0.0236	0.1471
XABYT	2.4190	2.8599	0.3539	0.1379	0.4281	-0.3605

It is noticeable from Table 5 some quite but perturbed linearity between multi-scale return and systematic risk. Some linear relations are almost positive, some others are moderately positive. But, no universal law of positivity is dominating for the whole market. A great number of actions are negatively related to the market index. The global view could not permit to conclude a strong interpretation of the evolution of the market. The looking at the individual results indicates a perturbation to conclude if the relationship becomes stronger at the lower, medium or higher scales. The only conclusion that may be deduced is the existence of the linear relationship, although weak, and some times opposite from horizon to another. The perturbation may be of course a cause of the non-resistance of the market according to time indicator. It may be also due to the lack of many hidden factors that are not involved in the mathematical model such as policy actions, local economic policy, etc. Besides, the perturbation if it is not re-studied and denounced by other methods, means that some crisis is always present in the market explained by an opposite variation of the actions and the market. Mathematically and empirically speaking, Table 5 shows that no strong or confirmed law may be expected simultaneously for all the contribution of the D_m^i relative to all components of the market based on the time scale. To confirm or denounce, or to understand more the findings in Table 5, the corresponding determination coefficient R^2 is computed and presented in Table 6.

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Table 6: The 2-scale law determination coefficient R^2 relative to Table 5.

	R^2 for each scale					
STOCKS	Scale 1	Scale 2	Scale 3	Scale 4	Scale 5	Scale 6
AB	0.7003	0.2840	0.0395	0.0210	0.0021	0.0015
ADWYA	0.0332	0.0625	0.0423	0.1978	0.0000	0.0013
ATEC	0.6264	0.0070	0.0840	0.1962	0.0407	0.0429
AL	0.2246	0.0805	0.0065	0.0133	0.0076	0.0017
ALKM	0.1500	0.1414	0.0178	0.1542	0.0269	0.0020
AMS	0.0000	0.0671	0.0774	0.0008	0.1178	0.0793
ARTES	0.9693	0.0025	0.1212	0.0014	0.0037	0.0062
ASSAD	0.9724	0.0822	0.2100	0.0438	0.0000	0.0033
ATB	NaN	0.4003	0.0001	0.1929	0.0006	0.0016
ATL	0.6609	0.0795	0.0072	0.0236	0.0056	0.0004
BH	0.3103	0.3304	0.0062	0.1436	0.0488	0.0128
BHASS	0.3768	0.9656	0.0061	0.0111	0.0145	0.0676
BHL	0.9482	0.2947	0.2033	0.0106	0.0019	0.0408
BIAT	0.8996	0.6675	0.2500	0.2339	0.0940	0.0921
BNA	0.2540	0.0058	0.3824	0.3391	0.0092	0.0042
BS	0.0176	0.0122	0.0071	0.2007	0.0042	0.0139
BT	0.8582	0.2393	0.0026	0.0288	0.0191	0.0108
BTEI	0.9388	0.8321	0.0122	0.0069	0.0004	0.0005
CC	0.5648	0.0913	0.0003	0.0057	0.0035	0.0163
CELL	0.1990	0.4494	0.0608	0.0928	0.0010	0.0016
CIL	0.9085	0.0018	0.0024	0.0160	0.0140	0.0026
CITY	0.9757	0.5729	0.0519	0.0055	0.0180	0.0118
CREAL	0.0287	0.1457	0.0151	0.0507	0.0030	0.0051
DH	0.6257	0.0103	0.0036	0.0231	0.0132	0.0750
ECYCL	0.0565	0.0663	0.1028	0.1182	0.0365	0.0290
GIF	0.4342	0.6332	0.1233	0.0643	0.0200	0.0383
HANL	0.2507	0.1381	0.0151	0.0491	0.0100	0.0051
ICF	0.0679	0.0120	0.0334	0.1624	0.0003	0.0014
LNDOR	0.2902	0.0334	0.0204	0.0526	0.0041	0.0350
LSTR	0.7423	0.3440	0.0107	0.0021	0.0374	0.0966
MGR	0.9956	0.2100	0.0494	0.0009	0.0082	0.0005
MNP	0.9207	0.0073	0.1081	0.0147	0.0002	0.0039
MPBS	0.9969	0.8156	0.0041	0.0871	0.0200	0.0287
NAKL	0.9816	0.0552	0.0240	0.0078	0.0563	0.0829
NBL	0.4828	0.4368	0.1332	0.0125	0.0421	0.0811
OTH	NaN	0.0013	0.0024	0.0164	0.0058	0.0044
PLAST	0.9946	0.5496	0.1802	0.0251	0.0023	0.0003
POULA	0.0029	0.0769	0.1340	0.1441	0.0003	0.0003
SAH	0.5062	0.8546	0.3050	0.0283	0.0539	0.0828

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SAMAA	0.9495	0.0548	0.1719	0.0093	0.0050	0.0322
SCB	0.9999	0.0932	0.0713	0.0071	0.0437	0.0697
SERVI	0.8756	0.2142	0.0209	0.0276	0.0113	0.0031
SFBT	0.0031	0.4723	0.2860	0.3348	0.0663	0.0401
SIAM	0.9225	0.3322	0.0441	0.3514	0.0117	0.0006
SIMP	0.2038	0.0335	0.0021	0.0317	0.0058	0.0351
SIPHA	0.2039	0.0501	0.0297	0.2947	0.0882	0.0483
SITS	0.2993	0.2304	0.0385	0.0000	0.0055	0.0023
SMG	0.0068	0.5108	0.1011	0.0003	0.0006	0.0002
SOKNA	0.2085	0.3592	0.0116	0.0462	0.0214	0.0000
SOMOC	0.6960	0.3516	0.0491	0.0834	0.0001	0.0000
SOPAT	0.2278	0.0524	0.1416	0.0127	0.0012	0.0096
SOTE	0.1083	0.5722	0.0059	0.0192	0.0002	0.0005
SOTEM	0.5056	0.7964	0.0658	0.1461	0.0323	0.0060
SPDI	0.2929	0.0341	0.0744	0.3858	0.0270	0.0026
STAR	0.1902	0.0925	0.2389	0.0229	0.0191	0.0382
STB	0.0697	0.6182	0.2045	0.0395	0.0134	0.0093
STPAP	0.3598	0.1938	0.0035	0.0466	0.0031	0.0005
STPIL	0.9240	0.7733	0.1090	0.0090	0.0066	0.0025
STVR	0.0011	0.7426	0.0049	0.0735	0.0005	0.0011
TAIR	0.9952	0.0837	0.0559	0.0063	0.0000	0.0372
TGH	0.2038	0.0822	0.0823	0.0255	0.0017	0.0063
TINV	0.6976	0.5765	0.0325	0.0183	0.0009	0.0075
TJL	0.0023	0.3057	0.0914	0.0036	0.0162	0.0152
TLNET	0.0031	0.0883	0.0174	0.0496	0.0047	0.0009
TLS	0.3366	0.1541	0.0374	0.1243	0.0153	0.0002
TPR	0.4286	0.5080	0.0002	0.1443	0.0017	0.0169
TRE	0.2593	0.4923	0.0542	0.0553	0.0002	0.0007
UADH	0.5193	0.0291	0.0191	0.0383	0.0017	0.0702
UBCI	0.1316	0.4816	0.1277	0.0016	0.0085	0.0324
UIB	0.2104	0.0324	0.0777	0.0182	0.0000	0.0000
UMED	0.9651	0.0108	0.0036	0.0248	0.0011	0.0002
WIFAK	0.9907	0.0567	0.0634	0.0364	0.0001	0.0101
XABYT	0.0432	0.5415	0.0127	0.0261	0.0490	0.0071

It is easily noticeable from Table 6 some quite decreasing aspect of the coefficient R^2 according to the time scale. However, it already presented some perturbation also in such a monotony, which is indeed not conserved along all time horizons, especially in the fourth (16-32 dynamic days) and the fifth (32-64 dynamic days) horizons. Globally, at low horizons the market is going down. Economically speaking this is a bad information for small companies and/or short investments.

The next step permits to explain more and visually the situation of the market via the eventual link between the actions' returns and the one of the market. Figure 1 below illustrates the 2-scale time law recomposed crystal of the excess return on the stock versus the corresponding crystal on the market portfolio.

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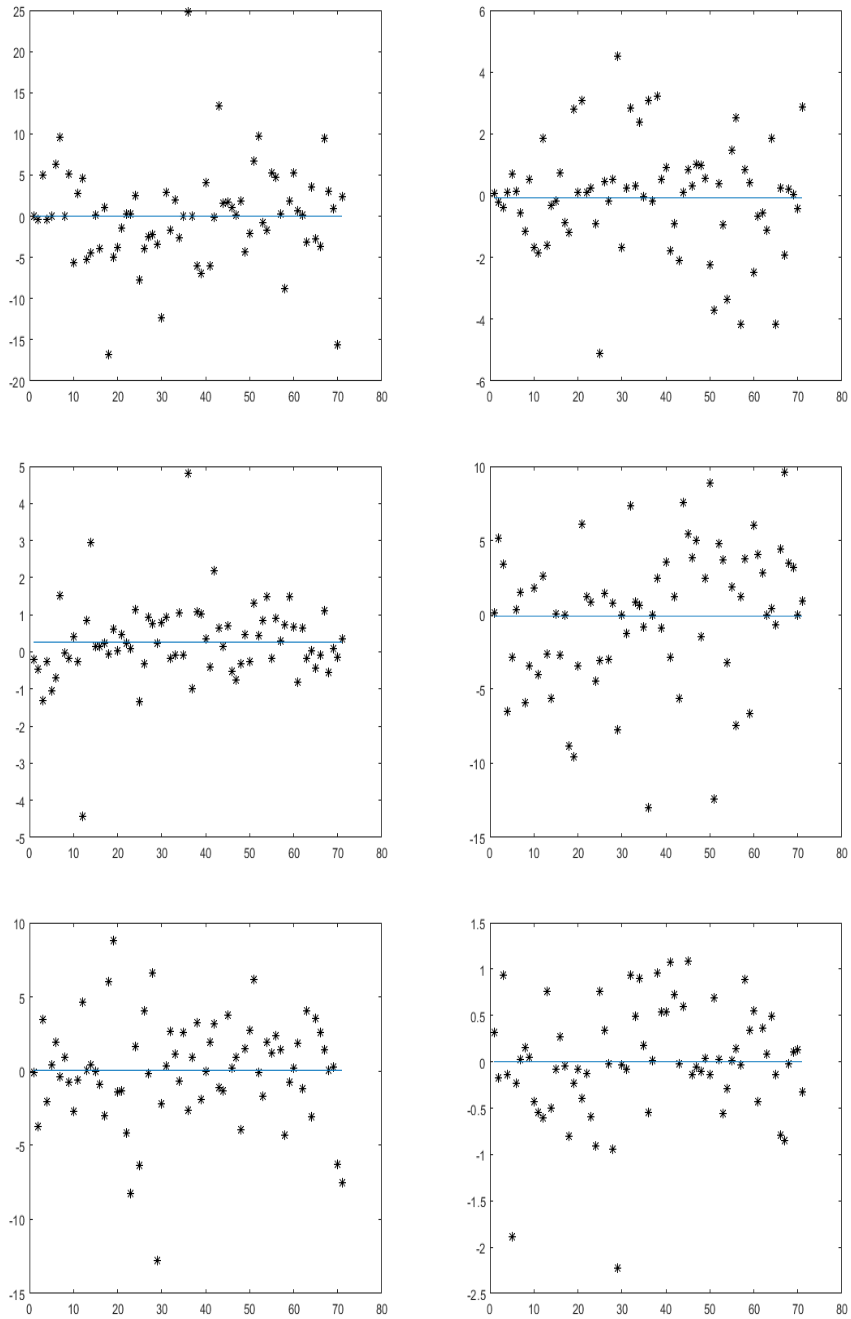

Figure 1. Excess market return (horizontal axis) versus excess return of the action (vertical axis) for different time scales with the 2-scale law.

In Figure 1 the daily stock returns versus corresponding stock beta at different time scales are plotted. By inspecting such a representation, it is noticeable clearly that the linearity is confirmed between the average betas of stocks and average returns at every scale (no matter being strong positive or not). This reinforces the usual finding about the CAPM as linear relationship. However, such figure shows clearly the suffering of the market from a serious crisis. Indeed, the market did not conserve the same resistant linearity along the whole period for the majority of actions. At law horizons (scale 1) the linearity is clear, and becomes perturbed at the scale 2, to return acceptable at scale 3, perturbed at scale 4 and quietly acceptable at scales 5 and 6. This evidence supports

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the proposition to act more sophisticated tools to zoom out the market and to discover the hidden behavior. Besides, the figure confirms our conclusion previously about the market being non encouraging for small companies and short investments. The next section essays to overcome these problems by adopting wavelet tools.

6.2. The (wavelet) multifractal processing

In financial case, for example, the studies have shown that the tail distribution is not leptokurtic but in the contrary, it has a kurtosis exceeding the normal case. In the present case, a kurtosis value over-crossing the normal one is detected in quasi all the statistical series. Moreover, the skewness induces for many times some negative values which means that the data are spread out more to the left relatively to the mean of the series than to the right.

To show how clearly the high volatile aspect of the market, a wavelet analysis is conducted in order to appear eventual fluctuations. Such analysis yielded the following illustrations in Figure 2 relative to the wavelet decomposition at the level 6 by means of a uniform wavelet decomposition 2.

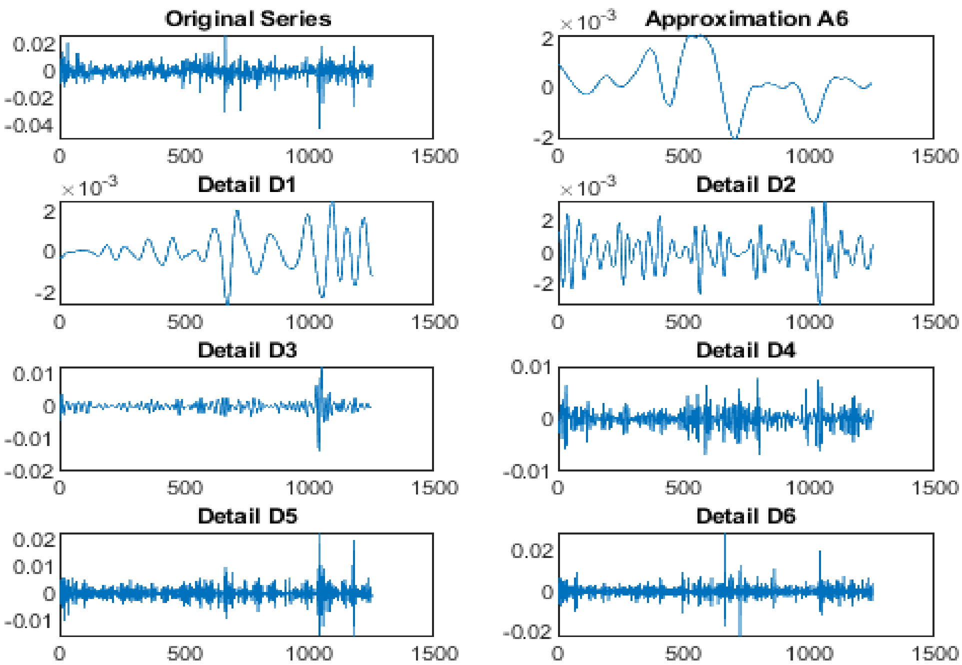

Figure 2. Wavelet decomposition of Tunindex return series at level 6.

Thew volatile behavior of the TUNINDEX is clearly shown, which confirms and The idea is to prove the existance of a fractal/multifractal scaling law in the TUNINDEx series, and/or its components. We thus proceed as mentioned previously by plotting its Hurst–H¨older exponent or its spectrum of singularities. This is illustrated by Figure 3 below.

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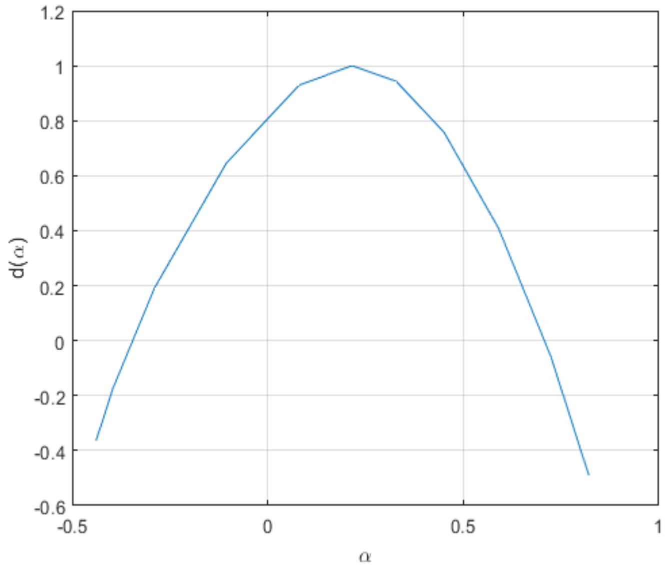

Figure 3. The multifractal spectrum of Tunindex return series.

Notice from these illustrations that although the wavelet applied is not necessarily non uniform, the market shows a concave nonlinear scaling function, or equivalently a nonlinear concave multifractal spectrum. This means that the market is multifractally volatile and it represents a point-wise scaling behavior with a point-wise changing Hurst–Holder exponent. This motivated the application of wavelet tools to explore its structure as well as its behavior by means of the CAPM.

6.3. The classical wavelet CAPM processing

The CAPM wavelet approach estimates the systematic risk beta on a scale-by-scale wavelet decomposition of the financial time series. The wavelet variance of the market index and the co-variance between the market and the components is computed to obtain a scale-by-scale estimate of stock's beta. Table 7 represents theses estimations for all the stock components using a uniform wavelet decomposition at the levels $J = 1, 2, 3, 4, 5, 6$.

Table 7: Uniform wavelet beta for each level.

STOCKS	D1	D2	D3	D4	D5	D6
AB	-0.0058	10.8224	8.8060	-2.5686	19.6819	-23.9260
ADWYA	-2.2882	1.6573	0.7644	0.0784	1.4485	0.2894
ATEC	-0.3098	-2.4031	-2.1237	0.6262	-0.9401	0.1602
AL	0.1467	0.4682	0.4818	-0.1080	0.9225	0.4316
ALKM	-0.4293	-0.3348	-0.0901	-3.1556	2.6938	0.1907
AMS	-1.1777	0.4045	0.4021	1.3881	-0.6886	-1.1341
ARTES	-1.6414	0.2592	0.1544	-0.7078	0.4698	0.3621
ASSAD	1.3090	-0.4502	0.0548	-4.5340	-0.3007	-0.0681
ATB	0.3447	0.0946	0.2000	0.3809	-0.3448	0.1018
ATL	-1.0770	-0.3199	-0.4179	0.3464	-0.5067	-0.1905
BH	3.0369	0.2605	0.4742	2.9209	0.9372	-1.1997
BHASS	0.0656	-0.4992	-0.9534	-1.1077	-0.0282	1.2911
BHL	-2.2572	0.2245	-0.2871	3.3041	-0.6511	0.7858
BIAT	0.6392	1.1340	1.0688	3.7823	0.2847	1.7848
BNA	1.2234	-1.1696	-0.7282	0.6407	-0.2140	0.5112
BS	0.9569	0.7049	0.6793	-0.5439	0.7257	0.3802
BT	-0.4848	-0.4604	-0.7623	-0.3860	-0.2200	-0.8652
BTEI	0.9866	-0.3991	-0.2950	0.1450	-0.7518	-1.0038
CC	-2.1892	-0.2356	-0.6133	4.2276	-0.3530	0.4163
CELL	-0.0301	1.2574	1.2746	1.0497	1.6922	-0.7457
CIL	0.3133	0.1787	0.0899	0.9712	0.3231	0.5030
CITY	0.6182	0.3557	0.2724	0.0902	0.6256	-0.6090
CREAL	0.0907	-0.1389	0.0118	0.1834	-0.4967	2.2908
DH	0.4328	0.1575	0.1694	-0.7097	0.0022	0.1035
ECYCL	1.3425	0.1364	0.4144	0.6597	0.9613	-0.5872
GIF	-1.8486	-1.1806	-1.1797	0.5990	-1.5362	-0.6681
HANL	-0.3796	0.2790	0.0687	0.8390	0.3842	-0.6035
ICF	0.4396	-1.0868	-0.7471	-7.0126	1.8237	0.7698
LNDOR	1.9304	2.0894	2.6104	-2.1298	2.5789	-1.3952
LSTR	0.9055	-0.5440	0.0586	0.3257	-1.0613	1.1838
MGR	1.5076	0.5316	0.8817	0.7304	-0.7121	0.4132
MNP	0.6616	0.4269	0.2874	0.5109	-0.7035	-0.6606
MPBS	0.7480	-0.4173	-0.1181	-0.7203	-0.8307	-2.4648
NAKL	-0.0509	-0.1784	-0.1213	-0.2713	-0.2425	0.5236
NBL	2.3168	-0.1255	0.2475	2.3258	-1.3143	0.3416
OTH	-0.7798	0.1069	-0.0084	-1.2417	-0.0557	-0.5968
PLAST	2.8624	2.7317	2.8280	16.0361	4.7460	-0.7449
POULA	-1.4599	-0.1276	-0.2870	0.5253	-1.0117	0.9684
SAH	0.8431	-0.0013	0.1688	-3.1832	0.3483	0.3253

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SAMAA	1.4727	-0.7483	-0.4252	-0.5312	-1.0167	2.6624
SCB	0.7012	-0.7450	-0.4810	-1.1657	-0.8208	0.7585
SERVI	-1.5955	-0.5592	-0.4832	-0.1775	-1.4936	0.0792
SFBT	-0.1345	1.5196	1.2794	0.3316	0.1586	1.8079
SIAM	1.4304	1.0006	0.7132	1.7451	0.8960	0.5927
SIMP	0.0549	0.0296	0.0234	-0.7653	-0.2101	0.4337
SIPHA	0.0095	-0.2565	-0.2455	0.3606	1.5488	0.6600
SITS	0.4031	-0.2085	-0.1832	-0.6524	-0.0307	0.6124
SMG	0.7395	0.3061	-0.1039	-1.0045	1.0950	-0.0382
SOKNA	-1.5430	1.2272	0.8952	2.7420	1.5712	-0.1190
SOMOC	-0.2167	0.1739	0.1093	-0.5376	-0.1098	-0.0857
SOPAT	3.6454	-0.4350	-0.0597	-5.7672	0.2579	0.8665
SOTE	0.0351	-3.1662	-1.7651	-5.1409	-0.6427	-0.3849
SOTEM	3.1962	-1.5652	-0.9122	1.8140	-0.3950	1.6369
SPDI	-0.4348	-0.1123	-0.1357	-3.0575	-0.0880	0.6584
STAR	0.3301	-0.1985	-0.0586	0.3669	-0.2771	-0.9198
STB	3.7127	0.7791	0.8423	1.1379	0.1722	-0.1455
STPAP	-1.0473	0.3928	-0.0324	1.3691	0.0614	0.6823
STPIL	0.6868	-1.1114	-0.6990	-1.2427	-0.7562	-1.1170
STVR	-0.2567	0.1736	0.3458	-0.4569	0.8693	-0.1652
TAIR	2.5230	-0.4463	0.2380	1.4587	-0.6266	0.0747
TGH	1.0990	0.5406	0.4120	5.1598	-1.9450	1.2579
TINV	1.9159	-1.8687	-1.0265	-4.9678	-1.3932	-1.3193
TJL	-1.2424	0.2993	0.0841	-3.2779	0.9461	-1.9180
TLNET	3.0938	-0.4201	-0.0828	-1.3405	0.2932	-0.0528
TLS	0.0417	-0.1633	-0.2220	0.2447	-0.2910	-0.0308
TPR	-1.5041	0.1185	-0.1120	0.7167	0.6093	0.5873
TRE	-0.6456	-0.0019	-0.2737	0.6331	0.1419	-0.9789
UADH	0.1182	0.5612	0.3026	-2.0140	0.7947	0.3833
UBCI	1.3628	-0.1010	0.1815	-1.9463	0.3181	1.2316
UIB	-0.2409	0.2689	0.1748	0.5254	0.0044	0.0531
UMED	-0.0047	0.4276	0.3867	-2.0484	0.5677	-1.1359
WIFAK	0.2122	-2.9050	-2.8702	-0.0112	0.0732	0.0265
XABYT	0.1916	-0.0540	-0.4878	-2.7606	0.8138	1.4183

Notice from Table 7 that the linear dependence between the market index and the actions becomes more illustrated relative to the previous method. Some cases have yielded zero coefficient beta by means of the previous method, however, the wavelet procedure have explained better the dependence or the contribution of these actions in the market, although being negative for some cases. (see for example AB, ATB and OTH actions), and some times returns to be perturbed at high scales. This means that to explore really the situation of this type of markets, and/or the at-the-crisis situations, more macroscopic and microscopic tools should be applied to detect the hidden facts. To explore more the dependence detected via the uniform wavelets, and to confirm the efficiency of the wavelet tool, R^2 determination coefficient relative to the estimations in 7 is provided hereafter in Table 8. It represents precisely the determination coefficient R^2 relative to the betas of each stock component at the wavelet levels $J = 1, 2, 3, 4, 5, 6$.

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Table 8: Uniform wavelet R^2 relative to Table 7 for each level.

	R^2					
STOCKS	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$
AB	0.0000	0.3652	0.3048	0.0083	0.8816	0.0227
ADWYA	0.4748	0.2882	0.0519	0.0023	0.2414	0.0050
ATEC	0.0041	0.1757	0.1242	0.0210	0.0574	0.0018
AL	0.0050	0.0652	0.0453	0.0028	0.4130	0.0062
ALKM	0.0014	0.0017	0.0001	0.0948	0.3105	0.0003
AMS	0.1195	0.0099	0.0079	0.0275	0.0409	0.0297
ARTES	0.3072	0.0129	0.0036	0.0909	0.7013	0.0210
ASSAD	0.1148	0.0159	0.0002	0.2678	0.0193	0.0001
ATB	0.0179	0.0022	0.0064	0.2015	0.0285	0.0008
ATL	0.1516	0.0280	0.0368	0.0428	0.0802	0.0018
BH	0.2617	0.0084	0.0197	0.6577	0.1336	0.0828
BHASS	0.0003	0.0604	0.0805	0.0999	0.0001	0.0659
BHL	0.5680	0.0034	0.0040	0.2823	0.0928	0.0078
BIAT	0.3279	0.1482	0.1568	0.4355	0.2109	0.2234
BNA	0.0443	0.1042	0.0300	0.0238	0.0142	0.0279
BS	0.3263	0.1674	0.1503	0.0500	0.2404	0.0249
BT	0.1209	0.0444	0.0108	0.0583	0.0195	0.1114
BTEI	0.1326	0.0582	0.0217	0.0047	0.1412	0.0193
CC	0.4203	0.0023	0.0146	0.2238	0.0424	0.0015
CELL	0.0001	0.1467	0.1516	0.0761	0.5492	0.0086
CIL	0.2614	0.0116	0.0030	0.2233	0.1200	0.0158
CITY	0.0358	0.0329	0.0124	0.0009	0.3175	0.0181
CREAL	0.0058	0.0027	0.0000	0.0691	0.0364	0.0555
DH	0.0226	0.0044	0.0045	0.0356	0.0005	0.0009
ECYCL	0.2264	0.0008	0.0081	0.0135	0.0571	0.0331
GIF	0.1811	0.1130	0.0860	0.2950	0.1417	0.0045
HANL	0.0167	0.0148	0.0007	0.0543	0.0286	0.0169
ICF	0.4680	0.0638	0.0209	0.7304	0.1017	0.0070
LNDOR	0.1798	0.1103	0.1624	0.0817	0.5586	0.0110
LSTR	0.3151	0.0118	0.0002	0.0025	0.1600	0.0271
MGR	0.1347	0.0378	0.0616	0.0707	0.0624	0.0039
MNP	0.0613	0.0568	0.0107	0.1679	0.1202	0.0168
MPBS	0.0524	0.0217	0.0014	0.0364	0.2750	0.0640
NAKL	0.0054	0.0212	0.0109	0.0447	0.5672	0.1220
NBL	0.6402	0.0006	0.0027	0.2965	0.3513	0.0034
OTH	0.1204	0.0025	0.0000	0.2233	0.0067	0.0146
PLAST	0.1879	0.0373	0.0463	0.3026	0.1380	0.0032
POULA	0.3414	0.0017	0.0091	0.0435	0.1949	0.0551
SAH	0.4946	0.0000	0.0039	0.4980	0.1155	0.0095

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SAMAA	0.2631	0.1272	0.0223	0.0420	0.1863	0.1012
SCB	0.1995	0.0548	0.0270	0.0303	0.6222	0.0175
SERVI	0.3929	0.0120	0.0079	0.0007	0.5858	0.0001
SFBT	0.0055	0.0320	0.0272	0.0107	0.0022	0.3244
SIAM	0.1380	0.1214	0.0489	0.1089	0.2620	0.0097
SIMP	0.0002	0.0001	0.0000	0.0145	0.0069	0.0199
SIPHA	0.0000	0.0027	0.0025	0.0062	0.1787	0.0064
SITS	0.0644	0.0081	0.0045	0.0396	0.0002	0.0521
SMG	0.2673	0.0096	0.0011	0.0448	0.1866	0.0000
SOKNA	0.2224	0.0952	0.0525	0.5505	0.4317	0.0005
SOMOC	0.0171	0.0018	0.0009	0.0154	0.0032	0.0002
SOPAT	0.4077	0.0086	0.0002	0.5243	0.0165	0.0122
SOTE	0.0002	0.1000	0.0323	0.5654	0.0286	0.0038
SOTEM	0.1546	0.0991	0.0229	0.0806	0.0238	0.0098
SPDI	0.1063	0.0045	0.0058	0.6621	0.0084	0.0262
STAR	0.1069	0.0100	0.0007	0.0157	0.6342	0.0559
STB	0.6141	0.0665	0.0640	0.0589	0.0053	0.0009
STPAP	0.1600	0.0140	0.0001	0.0844	0.0005	0.0251
STPIL	0.0809	0.0637	0.0264	0.0272	0.0443	0.0163
STVR	0.0102	0.0046	0.0153	0.0269	0.1388	0.0006
TAIR	0.8716	0.0083	0.0027	0.0429	0.0405	0.0002
TGH	0.0376	0.0044	0.0027	0.7065	0.0463	0.0520
TINV	0.1695	0.1381	0.0400	0.2802	0.1676	0.0344
TJL	0.1081	0.0108	0.0007	0.5180	0.8903	0.0393
TLNET	0.3230	0.0290	0.0005	0.0976	0.0510	0.0001
TLS	0.0083	0.0021	0.0045	0.0017	0.0279	0.0000
TPR	0.1604	0.0017	0.0013	0.0450	0.3185	0.0226
TRE	0.0760	0.0000	0.0253	0.1583	0.2848	0.0298
UADH	0.1836	0.0799	0.0258	0.6651	0.4596	0.0017
UBCI	0.3116	0.0005	0.0016	0.0648	0.0089	0.0419
UIB	0.0170	0.0277	0.0086	0.0186	0.0000	0.0003
UMED	0.0000	0.0215	0.0193	0.1250	0.0633	0.0561
WIFAK	0.3618	0.0070	0.0087	0.0002	0.0204	0.0001
XABYT	0.0024	0.0002	0.0115	0.1243	0.0571	0.0438

Table 8 presents some acceptable coherence with the results on the uniform wavelet beta estimation illustrated by Table 7, although being quietly null in some cases. The market appears to be efficient at low and medium scales and the R^2 determination coefficients shows some decreasing behavior at low scales, however, it returns to be increasing at medium horizons. This yields that to invest in such a market, investors need may be to come over the microscopic and macroscopic scales of the market, and also to explore more adequately different panels to conclude about the best or good returns sectors.

In the TUNINDEX market, it appears that banking sector at low/medium horizons may be encouraging, reminiscent of some risk at high scales, which may be really explained by the instability of the market due to the political and social situations. Food sector seems to be risky, and unstable, although it contains the daily needs of consumers. This is due may be to the high increase of food prices compared to the economic situation of the society where a big percentage of the people becomes more and more poor due to unemployment situation. We notice also that transportation sector is suffering, which is the most comprehensible task due to the COVID-19 pandemic which stroked the sector of air transportation severely. The sector of eal estate, construction and/or housing presents also a bit positive evolution, which may be explained by the fact that in the crisis, the prices in such a sector may decrease, and the facilities of purchasing may be offered by the banks and/or the government to maintain some liquidity in the market.

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Recall in this context that the migrants and over-broad citizens are the major mass active in this sector.

Besides, the most important notification is the fact that although some encouraging situations appeared in some horizons, such situation did not resist in all time scales. The evolution of the market according to the time scales present already some perturbations. There are actions and sectors that start to be positively progressing which by the next present some deficiency. The findings in Tables 7 and 8 are confirmed by the next by the graphical illustrations in Figure 4 hereafter which represents the overage of wavelet excess return of the actions versus the wavelet excess return of the market at the same levels $j = 1, 2, 3, 4, 5, 6$.

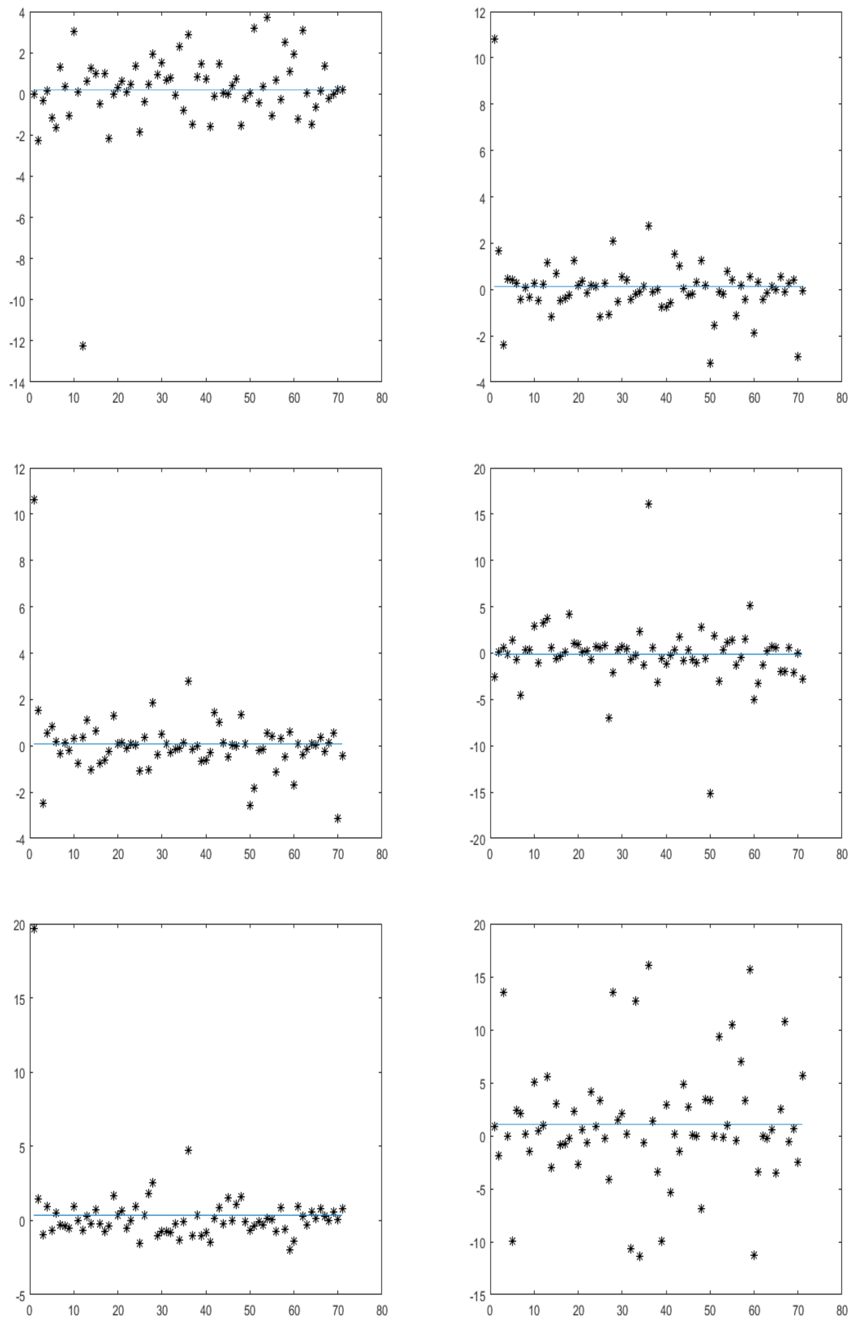

Figure 4. Uniform wavelet excess market return (horizontal axis) versus uniform wavelet excess return of the action (vertical axis) at levels j=1,2,3,4,5,6.

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Notice effectively from Figure 4 the linearity illustrated clearly. However, the perturbed aspect of the market is also clearly shown at the lower level, and the highest level. This may be a bad signal for investors who is/are preparing to a long time investment. This lead us to think about acting the non-uniform wavelet procedure in order to explore more the market evolution.

6.4. The non-uniform wavelet CAPM processing

In the present section, the purpose is to improve the WCAPm by acting the non-uniform wavelets. We pretend that such type of wavelets will lead to a best exploration of the market comprehension.

The idea empirically acts by similar way as the in the last part based on the dyadic wavelets. In the present part, we applied a non-uniform scaling law for the time intervals and the wavelets supports defined in section 3.2. By choosing suitable parameters for the subdivision Δ_m generated by means of a random process

$$\Delta_m = sort((b-a)*rand(m,1)+a,'ascend'),$$

relative an interval $[a,b]$. In our case, the parameters are fixed to $[0,T]$, where T is the size of the time series TUNINDEX, m is the maximum level, which is also the number of time interval sub-segments, fixed here to $m = 6$. The result of the non uniform wavelet estimation of the systematic risk beta results in 6 levels (chosen deliberately to be adequate with the previous methods). The estimations are gathered in Table 9 where for each scale, the non uniform wavelet variance of the market return and the non uniform wavelet co-variance between the market return and the stock return is computed to obtain a scale-by-scale estimate of stock's beta.

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Table 9: The non-uniform wavelet beta for each level.

STOCKS	D_1	D_2	D_3	D_4	D_5	D_6
AB	-5.0353	7.5403	-0.2631	-0.0058	9.6819	0.4989
ADWYA	-0.3050	0.9530	-0.1649	-2.2882	1.4485	1.6849
ATEC	-1.7371	-0.8598	-2.0838	-0.3098	-0.9401	-.2678
AL	0.0825	-0.1880	-0.4768	0.1467	0.9225	-.4015
ALKM	-0.9358	-0.0503	-2.0026	-0.4293	2.6938	-.9419
AMS	0.5631	-1.0126	-1.8366	-1.1777	-0.6886	1.3822
ARTES	0.0096	-0.1066	-0.2871	-1.6414	0.4698	0.0569
ASSAD	0.1596	-0.3465	1.7453	1.3090	-0.3007	0.7089
ATB	-1.2843	0.3100	-0.0068	0.3447	-0.3448	1.0966
ATL	0.6763	0.6325	0.0786	-1.0770	-0.5067	-.6581
BH	-0.6323	-1.4957	-0.1935	3.0369	0.9372	-.7629
BHASS	-0.7761	-0.7545	-0.3824	0.0656	-0.0282	0.1390
BHL	1.3667	0.3190	-1.0432	-2.2572	-0.6511	-.1183
BIAT	-1.3302	-0.0096	0.8501	0.6392	0.2847	1.2720
BNA	0.2924	-1.6610	4.1078	1.2234	-0.2140	-.0705
BS	-0.1458	0.3418	-0.6299	0.9569	0.7257	1.4221
BT	0.1121	-0.3470	1.0324	-0.4848	-0.2200	-.4777
BTEI	-0.6672	-0.1344	-0.1646	0.9866	-0.7518	-.0173
CC	-2.8705	-4.2232	1.2398	-2.1892	-0.3530	-.0183
CELL	2.2966	1.1852	1.3174	-0.0301	1.6922	0.5550
CIL	-0.0486	-0.1575	-0.0543	0.3133	0.3231	-.6778
CITY	1.9486	-0.2252	-0.2506	0.6182	0.6256	-.0100
CREAL	0.1003	0.0908	0.1039	0.0907	-0.4967	1.4672
DH	-0.4504	0.5241	0.1825	0.4328	0.0022	0.5505
ECYCL	-0.5451	0.5635	-0.8155	1.3425	0.9613	-.8662
GIF	-2.7725	-0.2349	-1.3705	-1.8486	-1.5362	0.6733
HANL	0.1466	-0.5392	-1.1041	-0.3796	0.3842	0.6311
ICF	0.7538	-0.5140	0.5490	0.4396	1.8237	-.6382
LNDOR	-0.6852	-0.8971	-1.5490	1.9304	2.5789	0.5386
LSTR	4.6647	-1.6185	0.0850	0.9055	-1.0613	-.5724
MGR	-0.2506	-0.2998	-0.4166	1.5076	-0.7121	0.8805
MNP	0.5342	0.6696	0.0865	0.6616	-0.7035	0.9216
MPBS	1.9565	0.0746	-0.8088	0.7480	-0.8307	-.5226
NAKL	0.6882	0.1085	-0.0940	-0.0509	-0.2425	-.1455
NBL	0.1017	1.2616	-0.6420	2.3168	-1.3143	-.6718
OTH	-0.5935	-0.4283	0.1610	-0.7798	-0.0557	1.5126
PLAST	4.1206	3.2116	6.1627	2.8624	4.7460	-.6910
POULA	-0.3940	0.9049	-1.0119	-1.4599	-1.0117	2.5275
SAH	2.9358	0.7437	1.6909	0.8431	0.3483	0.3105

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SAMAA	0.6585	-0.8032	1.1454	1.4727	-1.0167	-.0461
SCB	1.1502	1.5531	0.0194	0.7012	-0.8208	0.2875
SERVI	-1.7777	-1.1372	0.0427	-1.5955	-1.4936	-.8632
SFBT	-0.2028	-1.1407	2.5117	-0.1345	0.1586	-.2649
SIAM	-0.7879	-0.4308	0.1339	1.4304	0.8960	2.6744
SIMP	0.3952	1.3442	0.5205	0.0549	-0.2101	0.0033
SIPHA	0.2307	1.0122	2.4105	0.0095	1.5488	-.9649
SITS	0.1559	1.0619	-0.3050	0.4031	-0.0307	-.2903
SMG	-0.0133	0.8811	-2.2874	0.7395	1.0950	-.7470
SOKNA	1.0259	-0.9201	1.3230	-1.5430	1.5712	0.7248
SOMOC	-0.9663	0.7099	0.3357	-0.2167	-0.1098	1.8653
SOPAT	-1.3562	0.3721	0.8164	3.6454	0.2579	0.1467
SOTE	-3.0307	0.4245	-1.5603	0.0351	-0.6427	-.3470
SOTEM	-1.5077	5.5979	-0.4774	3.1962	-0.3950	-.6921
SPDI	0.1228	0.1178	0.0199	-0.4348	-0.0880	1.1054
STAR	-0.8572	-0.0639	1.4175	0.3301	-0.2771	-.4317
STB	-2.8912	0.9286	0.0575	3.7127	0.1722	0.9493
STPAP	0.3642	0.7775	0.0188	-1.0473	0.0614	1.1077
STPIL	2.0058	-0.4005	0.4575	0.6868	-0.7562	0.0698
STVR	-3.0913	0.4294	0.7957	-0.2567	0.8693	1.6345
TAIR	-1.1355	-0.4848	0.1436	2.5230	-0.6266	-.1926
TGH	0.5325	-0.1040	1.8528	1.0990	-1.9450	2.8173
TINV	-1.8065	-0.1180	0.8786	1.9159	-1.3932	-.0821
TJL	0.1178	-0.2724	-1.0938	-1.2424	0.9461	0.0816
TLNET	-1.7191	-0.3174	0.6720	3.0938	0.2932	-.9534
TLS	-0.4655	1.1347	0.0817	0.0417	-0.2910	-.0939
TPR	1.0499	-1.1573	1.0399	-1.5041	0.6093	-.4334
TRE	-4.4680	0.4078	-0.3849	-0.6456	0.1419	-.4406
UADH	-0.1535	-0.0047	0.0338	0.1182	0.7947	-.3445
UBCI	-1.5562	-0.3520	0.7204	1.3628	0.3181	1.0367
UIB	0.6015	1.9169	-0.6413	-0.2409	0.0044	-.0981
UMED	-0.0237	2.8104	1.2834	-0.0047	0.5677	-.0766
WIFAK	-1.0690	0.8824	-0.1407	0.2122	0.0732	0.4194
XABYT	3.4503	0.1040	1.3244	0.1916	0.8138	-.1719

Notice from Table 9 that contrarily to the previous cases, the linear dependence between the market and the actions is confirmed with no zero coefficients. Besides, such dependence is sometimes strongly negative, or strongly positive even for a same action at different time horizons (which may be observed since the first action AB). Compared to the previous methods, the market seems to suffer from unstable situation according to the time factor, which is conformed also in the previous sections. This makes the authorities manly to think about the real causes behind these fragile and ambiguous situation. The main causes in our opinion are, as we said in the abstract and the introduction

- the unstable political situation in the country during this period.
- the possible corruption in the governments during this period.
- the social movements and strikes which yielded a migration of many investors.

However, although many actions in the market have shown some perturbation, the majority or the global view of the market shows that some stability may occur from time to time, but at random periods distributed randomly on the time horizons. These facts allows the investors, the analysts and also the authorities to think and to chose intelligently about the time of investment, and also the time of surveys studies conducted to conclude about the situation of the market. In other words, the non uniform method tells us that the market is not progressing uniformly as we thought previously, but, in

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contrast, short horizons sometimes, may be good moments to act even in the moment of long crises.

Now, as in the previous studies, the estimations are followed by the computation of the R^2 coefficient of determination. This is the subject of Table 10 below which shows precisely the estimations of the determination coefficient R^2 relative to the betas of each stock component estimated in Table 9 at the non uniform wavelet levels $J = 1, 2, 3, 4, 5, 6$.

Table 10: Non-uniform wavelet R^2 relative to Table 9 for each level.

	R^2					
STOCKS	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$
AB	0.1930	0.3109	0.0790	0.0000	0.8816	0.4098
ADWYA	0.1196	0.1178	0.0513	0.4748	0.2414	0.1650
ATEC	0.1563	0.0942	0.2802	0.0041	0.0574	0.2471
AL	0.1583	0.0073	0.0129	0.0050	0.4130	0.0495
ALKM	0.1456	0.0021	0.6339	0.0014	0.3105	0.3013
AMS	0.0361	0.1425	0.3991	0.1195	0.0409	0.0988
ARTES	0.0000	0.0020	0.0505	0.3072	0.7013	0.0004
ASSAD	0.0044	0.0211	0.4001	0.1148	0.0193	0.1327
ATB	0.5294	0.2396	0.0000	0.0179	0.0285	0.3379
ATL	0.1426	0.1174	0.0034	0.1516	0.0802	0.0615
BH	0.0339	0.3191	0.0027	0.2617	0.1336	0.6233
BHASS	0.1014	0.2837	0.1241	0.0003	0.0001	0.0092
BHL	0.2123	0.0779	0.0807	0.5680	0.0928	0.0003
BIAT	0.4806	0.0000	0.1690	0.3279	0.2109	0.2847
BNA	0.0069	0.1576	0.7922	0.0443	0.0142	0.7113
BS	0.0148	0.0365	0.0651	0.3263	0.2404	0.3097
BT	0.0062	0.0376	0.0636	0.1209	0.0195	0.1175
BTEI	0.5943	0.0786	0.0605	0.1326	0.1412	0.0009
CC	0.2106	0.4105	0.1756	0.4203	0.0424	0.0001
CELL	0.2811	0.2861	0.3404	0.0001	0.5492	0.0209
CIL	0.0009	0.0465	0.0117	0.2614	0.1200	0.1324
CITY	0.3401	0.0123	0.0349	0.0358	0.3175	0.3161
CREAL	0.0485	0.0243	0.0414	0.0058	0.0364	0.1142
DH	0.0361	0.1702	0.0203	0.0226	0.0005	0.1109
ECYCL	0.0431	0.0354	0.1747	0.2264	0.0571	0.5451
GIF	0.1360	0.0433	0.2116	0.1811	0.1417	0.0642
HANL	0.0065	0.1054	0.7015	0.0167	0.0286	0.4777
ICF	0.0522	0.3983	0.0207	0.4680	0.1017	0.3747
LNDOR	0.0369	0.0882	0.1227	0.1798	0.5586	0.0101
LSTR	0.5822	0.2045	0.0013	0.3151	0.1600	0.0908
MGR	0.0022	0.0332	0.1623	0.1347	0.0624	0.1970
MNP	0.0266	0.1196	0.0132	0.0613	0.1202	0.1167
MPBS	0.4882	0.0473	0.1218	0.0524	0.2750	0.0328
NAKL	0.1829	0.0131	0.0981	0.0054	0.5672	0.2169
NBL	0.0005	0.5473	0.0904	0.6402	0.3513	0.0436
OTH	0.2258	0.1187	0.0234	0.1204	0.0067	0.4868
PLAST	0.3785	0.4808	0.1906	0.1879	0.1380	0.0052
POULA	0.2472	0.7316	0.1375	0.3414	0.1949	0.2025
SAH	0.5347	0.3574	0.4678	0.4946	0.1155	0.0677

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SAMAA	0.0887	0.2333	0.3965	0.2631	0.1863	0.0008
SCB	0.1400	0.3246	0.0003	0.1995	0.6222	0.0244
SERVI	0.1666	0.1787	0.0002	0.3929	0.5858	0.1001
SFBT	0.0138	0.1927	0.4251	0.0055	0.0022	0.5692
SIAM	0.0365	0.0557	0.0036	0.1380	0.2620	0.5937
SIMP	0.1228	0.6525	0.2266	0.0002	0.0069	0.0000
SIPHA	0.0041	0.3522	0.3231	0.0000	0.1787	0.8635
SITS	0.0446	0.8421	0.0181	0.0644	0.0002	0.0295
SMG	0.0000	0.1400	0.4179	0.2673	0.1866	0.2239
SOKNA	0.4516	0.1732	0.2247	0.2224	0.4317	0.0832
SOMOC	0.0940	0.2037	0.0201	0.0171	0.0032	0.2834
SOPAT	0.0419	0.1038	0.1367	0.4077	0.0165	0.0015
SOTE	0.6063	0.0342	0.2813	0.0002	0.0286	0.0261
SOTEM	0.1056	0.6665	0.1010	0.1546	0.0238	0.2926
SPDI	0.0005	0.0030	0.0001	0.1063	0.0084	0.7178
STAR	0.1925	0.0005	0.3813	0.1069	0.6342	0.2493
STB	0.6539	0.2091	0.0012	0.6141	0.0053	0.0700
STPAP	0.0125	0.1933	0.0000	0.1600	0.0005	0.0699
STPIL	0.6739	0.0359	0.3717	0.0809	0.0443	0.0028
STVR	0.5404	0.0668	0.0289	0.0102	0.1388	0.2230
TAIR	0.0641	0.0469	0.0071	0.8716	0.0405	0.0012
TGH	0.1347	0.0033	0.5790	0.0376	0.0463	0.0914
TINV	0.4030	0.0052	0.1109	0.1695	0.1676	0.0533
TJL	0.0013	0.0101	0.2020	0.1081	0.8903	0.0006
TLNET	0.3105	0.0085	0.0167	0.3230	0.0510	0.0683
TLS	0.0257	0.2849	0.0289	0.0083	0.0279	0.1046
TPR	0.1626	0.2751	0.2709	0.1604	0.3185	0.4618
TRE	0.5792	0.0496	0.0323	0.0760	0.2848	0.1266
UADH	0.0092	0.0003	0.2531	0.1836	0.4596	0.0646
UBCI	0.0483	0.0439	0.1180	0.3116	0.0089	0.0431
UIB	0.0967	0.9159	0.2047	0.0170	0.0000	0.0203
UMED	0.0072	0.5987	0.2792	0.0000	0.0633	0.0008
WIFAK	0.0762	0.2064	0.0131	0.3618	0.0204	0.0691
XABYT	0.4986	0.0044	0.4158	0.0024	0.0571	0.2147

Table 10 shows a coherence with results on the non uniform wavelet estimation of the systematic risk beta. The market is globally efficient but with some non monotone coefficient R^2. In fact, we may notice that in the majority of cases, the determination coefficient is going to be decreasing as the time scale increases. This may lead to some corrections and improvements on the previous interpretations. Indeed, although some random deficiencies appear for some actions, globally the market is going to reach some equilibrium at long time. This equilibrium has to be well predicted. This lead us to return to the previous idea of intelligently choosing the time of investment and surveys about the market. We think that during the unstable political and social movements, it would not be good to fix at advance the time scales and act according to them. In the contrary, predictions may be done at the basis of non uniform time scales. the global or major stability due to the non uniform time and wavelet scales is shown clearly in Figure 5 hereafter which illustrates the overage of non uniform wavelet excess return of the actions against the non uniform wavelet excess return of the market at the different wavelet levels $j = 1, 2, 3, 4, 5, 6$.

6.5. The sectorial CAPM

This last section is concerned to a setorial study of the market. In which, the classification into sectors is used to try to explore best the situation of the TUNINDEX stock market via the CAPM. Tables 11, 12 and 13 show effectively that the grouping of

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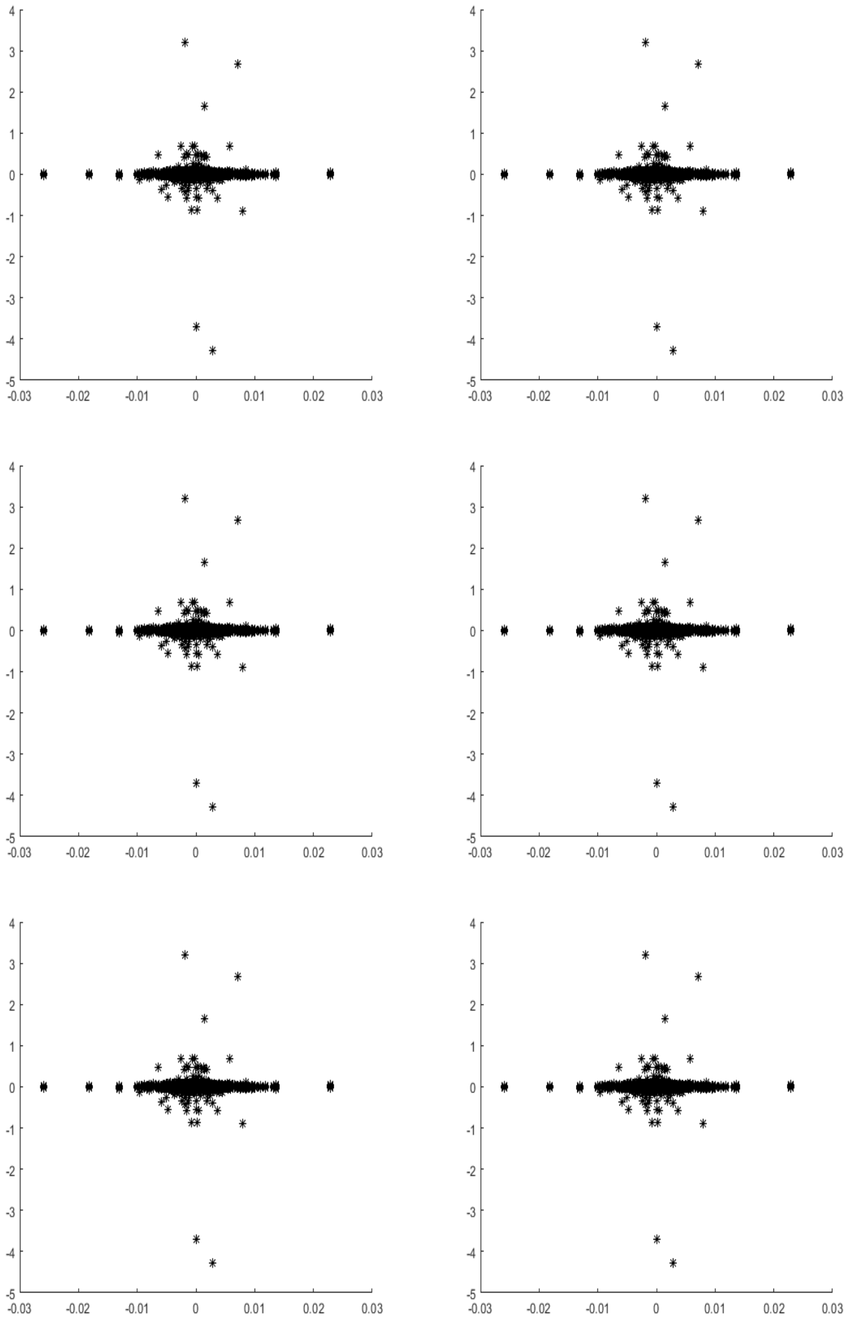

Figure 5. Non-uniform wavelet excess market return (horizontal axis) versus non-uniform wavelet excess return of the action (vertical axis) at levels j=1,2,3,4,5,6.

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the market components into sectors may allow us to carry a clear, out global, view of the market. Indeed, we notice that on 36 coefficients, a minimum of 29 ones are positive for the first method, 28 for the second and 32 coefficients are positive for the non uniform wavelet method. This somehow confirms that although these positivities did not resist on the same law, they explain some good hope, prospect and/or trust to the market. We notice also as in the previous sections that the sector of consumers, and finance are the most leading ones.

Table 11: The 2-scale mean beta for each sector.

	2-scale beta for levels $J = 1$ to $J = 6$.					
Sectors	Scale 1	Scale 2	Scale 3	Scale 4	Scale 5	Scale 6
Basic Materials	-1.0611	-0.3618	0.1180	-0.2719	0.5057	-0.1424
Capital Goods	0.5296	0.2936	0.0244	0.1330	-0.3134	0.2167
Consumer	0.2771	0.3885	0.8016	-0.0636	0.3380	-0.0256
Energy	2.1984	1.3064	0.3233	-0.1591	-0.0212	0.00005
Financial	-0.5293	-1.1512	0.4497	-0.5797	0.9217	-0.0543
Healthcare	0.7489	0.2145	0.1040	-0.3689	0.8752	0.3228
Services	-0.0540	0.6342	-0.1868	0.2221	0.0821	-0.0574
Technology	4.1210	-1.3169	-0.0760	-0.4077	-0.2706	0.1996
Transportation	-8.7601	0.8458	0.7183	2.0956	-0.4845	1.0164

Table 12: The uniform wavelet mean Beta for each sector.

	Uniform wavelet beta for levels $J = 1$ to $J = 6$.					
Sectors	Scale 1	Scale 2	Scale 3	Scale 4	Scale 5	Scale 6
Basic Materials	-0.5108	-0.2993	-0.2635	-1.0733	0.6119	0.0257
Capital Goods	-0.1739	-0.1703	-0.0875	0.5075	-0.2697	-0.0927
Consumer	0.8404	0.2327	0.3914	0.1848	0.2184	0.4292
Energy	0.4167	-0.3216	-0.1086	-0.6754	0.0832	-0.3427
Financial	0.7632	0.3735	0.3131	-0.2386	1.1190	-1.4306
Healthcare	-0.7611	0.6095	0.3019	-0.5365	1.1883	-0.0622
Services	-0.1230	0.2829	0.1241	0.3045	-0.0979	0.1179
Technology	1.0624	-1.2472	-0.8146	-1.0275	-0.0984	0.0788
Transportation	2.5230	-0.4463	0.2380	1.4587	-0.6266	0.0747

Table 13: The non-uniform wavelet mean Beta for each sector.

	Non-uniform wavelet beta for levels $J = 1$ to $J = 6$.					
Sectors	Scale 1	Scale 2	Scale 3	Scale 4	Scale 5	Scale 6
Basic Materials	-0.6239	-0.6299	0.1742	-0.5108	0.6119	0.3895
Capital Goods	0.1867	0.4635	-0.0453	-0.1739	-0.2697	0.2186
Consumer	0.3955	0.1305	0.4990	0.8404	0.2184	0.2561
Energy	1.0441	-0.2943	-0.0096	0.4167	0.0832	-0.1658
Financial	-4.3171	4.1476	0.3692	0.7632	1.1190	0.3270
Healthcare	-0.0327	1.5919	1.1763	-0.7611	1.1883	0.2145
Services	0.6789	0.3374	-0.0985	-0.0617	0.0622	0.1785
Technology	-1.8187	-0.2959	-0.7096	1.0624	-0.0984	0.2766
Transportation	-1.1355	-0.4848	0.1436	2.5230	-0.6266	-0.1926

7. Conclusion

In this paper, we proposed a wavelet multifractal procedure to estimate the systematic risk beta via the so-called CAPM. The main idea consists in testing the volatile

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behavior of the market to act next a non uniform version of wavelet analysis. Compared to the existing methods, the new process of non uniform wavelets has been proved to be more efficient and powerful in exploiting the CAPM to explore the market situation.

The theoretical methodology is applied on the case of the Tunisian TUNINDEX stock market as a representative market of the so-called Arab spring countries. The study is conducted during a critical period chosen deliberately to contain the after period of the Arab spring revolutions, and also the first COVID-19 pandemic wave appearance.

The study shows that contrarily to the existing studies, especially the old ones, the inclusion of the non uniform time scales law is adequate to understand the situation of the market during the unstable period and the crisis.

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Conceptualization, M. M. Sarraj and A. Ben Mabrouk; Methodology, M. M. Sarraj and A. Ben Mabrouk; Software, M. M. Sarraj and A. Ben Mabrouk; Validation, M. M. Sarraj and A. Ben Mabrouk; Formal analysis, M. M. Sarraj and A. Ben Mabrouk; Investigation, M. M. Sarraj and A. Ben Mabrouk; Resources, M. M. Sarraj and A. Ben Mabrouk; Data curation, M. M. Sarraj and A. Ben Mabrouk; Writing—original draft preparation, M. M. Sarraj and A. Ben Mabrouk; Writing—review and editing, M. M. Sarraj and A. Ben Mabrouk; Visualization, M. M. Sarraj and A. Ben Mabrouk; Supervision, M. M. Sarraj and A. Ben Mabrouk; Project administration, M. M. Sarraj and A. Ben Mabrouk; Funding acquisition, M. M. Sarraj and A. Ben Mabrouk. All authors have read and agreed to the published version of the manuscript.

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The authors declare no conflict of interest.

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