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A Complete Proof Of The Riemann Hypothesis Based On A New Expression Of $\xi(s)$

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Abstract Based on Hadamard product, a new expression of the completed zeta function $\xi(s)$ is obtained, i.e.,

$$\xi(s)=\xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i}$$

where $\rho_i=\alpha_i+j\beta_i, \bar{\rho}_i=\alpha_i-j\beta_i$ are complex conjugate zeros of $\xi(s)$, $d_i\geq 1$ (natural numbers) are the multiplicities of ρ_i, $0<\alpha_i<1$ and $\beta_i\neq 0$ are real numbers, $i\in\mathbb{N}$ are natural numbers from 1 to infinity. β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1|\leq|\beta_2|\leq|\beta_3|\leq,\cdots$. According to the functional equation $\xi(s)=\xi(1-s)$, We have

$$\xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i}=\xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i}$$

i.e.,

$$\prod_{i=1}^{\infty}\Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i}=\prod_{i=1}^{\infty}\Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i}$$

which, by Lemma 3, is equivalent to

$$\alpha_i=\frac{1}{2}, i\in\mathbb{N},\ \text{from 1 to infinity.}$$

Thus, we conclude that the Riemann Hypothesis is true.

Keywords Riemann Hypothesis (RH) · Proof · Completed zeta function

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1 Introduction

It has been almost 163 years since the Riemann Hypothesis (RH) was proposed in 1859 [1]. Many efforts and achievements have been made towards proving this celebrated hypothesis, but it is still an open problem [2−3].

The Riemann zeta function is the function of complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_1^{\infty}(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot (\frac{\theta(x)-1}{2})dx\} \tag{2}$$

where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}$ being the Jaccobi theta function, Γ being the Gamma function in the following Weierstrass expression (Meanwhile, there are also Gauss expression, Euler expression, and integral expression of the Gamma function.)

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-s/n} \tag{3}$$

where γ is the Euler-Mascheroni constant.

The connection between the Riemann zeta function and prime numbers can be established through the well-known Euler product.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1-p^{-s})^{-1}, \Re(s) > 1 \tag{4}$$

the product being over the prime numbers p.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: $-2, -4, -6, -8, \cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

In 1896, Hadamard [4] and Poussin [5] independently proved that no zeros could lie on the line $\Re(s) = 1$. Together with the functional equation and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the critical strip $0 < \Re(s) < 1$. This

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was a key step in their first proofs of the famous **Prime Number Theorem**.

Later on, Hardy (1914) [6], Hardy and Littlewood (1921) [7] showed that there are infinitely many zeros on the critical line $\Re(s) = \frac{1}{2}$, which was an astonishing result at that time.

As a summary of the properties of the non-trivial zeros of $\zeta(s)$, we have the following results [8−9].
Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho, \bar{\rho}, 1 - \bar{\rho}, 1 - \rho$ are all non-trivial zeroes.

As further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite polynomial, in the whole complex plane $\mathbb{C}$.
In addition, replacing s with $1 - s$ in Eq.(6), and combining Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, and recalling Eq.(3), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s - 1$ and the pole of $\zeta(s)$ cancel; the zero $s = 0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [9−10]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\zeta(s)$. Then we have the following Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with non-trivial zeros of $\zeta(s)$.

According to Lemma 2, the following two statements for the RH are equivalent.

Statement 1 of the RH: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of the RH: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, the natural thinking is to estimate the number of zeros of $\zeta(s)$ in some certain closed areas according to the Argument Principle. Along this train of thought, there are some famous research works. Let $N(T)$ denote the number of zeros of $\zeta(s)$ inside the rectangle: $0 < \alpha < 1, 0 < \beta \leq T$, and

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let $N_0(T)$ denote the number of zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 < \beta \leq T$. Selberg proved that there exist positive constants c and T_0, such that $N_0(T) > cN(T), (T > T_0)$ [11], later on, Levinson proved that $c \geq \frac{1}{3}$ [12], Lou and Yao proved that $c \geq 0.3484$ [13], Conrey proved that $c \geq \frac{2}{5}$ [14], Bui, Conrey and Young proved that $c \geq 0.41$ [15], Feng proved that $c \geq 0.4128$ [16].

On the other hand, many zeros have been calculated by hand or by computers. Among others, Riemann found the first three non-trivial zeros [17]. Gram found the first 15 zeros based on Euler-Maclaurin summation [18]. Titchmarsh calculated the 138^{th} to 195^{th} zeros using the Riemann-Siegel formula [19−20]. Here are the first three zeros: $\frac{1}{2} \pm j14.1347251$; $\frac{1}{2} \pm j21.0220396$; $\frac{1}{2} \pm j25.0108575$. Then we certainly have $\alpha_1 = \frac{1}{2}$.

The idea of this paper is originated from Euler's work on proving that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \tag{8}$$

This interesting and famous result is deduced from two types of infinite expansions, i.e., infinite polynomial and infinite product as follows

$$\begin{aligned} \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots \end{aligned} \tag{9}$$

This paper is also motivated by the fact that $\xi(s)$ can be expressed by the following Hadamard product, which was first proposed by Riemann. However, it was Hadamard [21] who showed the validity of this infinite product expansion.

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) \tag{10}$$

where ρ are precisely the non-trivial zeros of the Riemann zeta function $\zeta(s)$, or in another word, ρ runs over the zeros of the completed zeta function $\xi(s)$. Besides, it is well-known that $\xi(0) = \frac{1}{2}$.

2 A Complete Proof of the RH

This section is planned to present a proof of the Riemann Hypothesis. We first prove that Statement 2 of the RH is true, and then by Lemma 2 we know that Statement 1 of the RH is also true. For this purpose, we need the following Lemma 3.

Lemma 3: Given two infinite products

$$f(s) = \prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{11}$$

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and

$$f(1-s)=\prod_{i=1}^{\infty}\Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i} \tag{12}$$

where s is a complex variable, $\alpha_i \neq 0, \beta_i \neq 0$ are real numbers, $d_i \geq 1$ are natural numbers, $i \in \mathbb{N}$ are natural numbers from 1 to infinity, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq, \cdots$.
Then we have

$$f(s)=f(1-s) \Leftrightarrow \alpha_i=\frac{1}{2}, i \in \mathbb{N}, \text{ from 1 to infinity.}$$

where " $\Leftrightarrow$ " is the equivalent sign.

Proof: The proof is mainly based on Mathematical Induction. We first prove that $P(n)$ holds for all natural numbers $n \in \mathbb{N}$, and then prove that $P(\infty)$ holds. It should be noted that we prefer to use elementary method instead of Transfinite Induction to complete the proof.

First of all, we have the following proposition:

$$(1+\frac{(s-\alpha)^2}{\beta^2})^{d_i}=(1+\frac{(1-s-\alpha)^2}{\beta^2})^{d_i} \Leftrightarrow (s-\alpha)^2=(1-s-\alpha)^2 \Leftrightarrow \alpha=\frac{1}{2} \tag{13}$$

Let $P(n)$ be:

$$\begin{aligned}
&\prod_{i=1}^{n}\Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i}=\prod_{i=1}^{n}\Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i}\\
&\Leftrightarrow\\
&\begin{cases}
(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}\\
\cdots\\
(1+\frac{(s-\alpha_n)^2}{\beta_n^2})^{d_n}=(1+\frac{(1-s-\alpha_n)^2}{\beta_n^2})^{d_n}
\end{cases}\\
&\Leftrightarrow \text{(by Eq.(13)}\\
&\alpha_i=\frac{1}{2}, i=1,2,3,\cdots,n
\end{aligned} \tag{14}$$

Also by Eq.(13), $P(1)$ is an obvious fact as the base case, i.e.,

$$\begin{aligned}
&\prod_{i=1}^{1}\Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{d_1}=\prod_{i=1}^{1}\Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{d_1}\\
&\Leftrightarrow\\
&(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}\\
&\Leftrightarrow\\
&\alpha_1=\frac{1}{2}
\end{aligned} \tag{15}$$

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For the inductive step, assume that $P(n)$ holds, i.e., Eq.(14) holds, we need to show that $P(n+1)$ holds too. We have

$$
\begin{aligned}
&\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\
&\Leftrightarrow\\
&\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}\\
&\Leftrightarrow \text{(by Lemma 4}\\
&\begin{cases}\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\ \left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}\end{cases}\\
&\Leftrightarrow \text{(by Eq.(14)}\\
&\begin{cases}\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1}=\left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1}\\ \cdots\\ \left(1+\frac{(s-\alpha_n)^2}{\beta_n^2}\right)^{d_n}=\left(1+\frac{(1-s-\alpha_n)^2}{\beta_n^2}\right)^{d_n}\\ \left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}\end{cases}\\
&\Leftrightarrow \text{(by Eq.(13)}\\
&\alpha_i=\frac{1}{2}, i=1,2,3,\cdots,n,n+1
\end{aligned}
\tag{16}
$$

That is to say $P(n+1)$ holds when $P(n)$ is true.
Hence by Mathematical Induction, $P(n)$ is true for all natural numbers n.
Further, from the above process, we see that

$$P(n+1)\Leftrightarrow P(n)\wedge P(1)\Leftrightarrow P(n-1)\wedge P(1)\wedge P(1)\Leftrightarrow\cdots\Leftrightarrow \underbrace{P(1)\wedge\cdots\wedge P(1)}_{(n+1)\text{ times}}\Leftrightarrow P(1)$$

Then we also have

$$P(\infty)\Leftrightarrow \underbrace{P(1)\wedge\cdots}_{\text{infinity times}}\Leftrightarrow P(1)$$

Recalling Eq.(13), we know that $P(1)$ holds without regard to $\alpha_i, i\in\mathbb{N}$ from 1 to ∞. Hence we conclude that $P(\infty)$ holds.
That completes the proof of Lemma 3.

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Lemma 4: We have the following fact that

$$\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$$
$$\Leftrightarrow$$
$$\begin{cases}\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\ \left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}\end{cases}$$

where s is a complex variable, $\alpha_i \neq 0, \beta_i \neq 0$ are real numbers, $i, n \geq 1, d_i \geq 1$ are natural numbers, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq , \cdots$.

Proof: Let's consider

$$\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{17}$$

i.e.,

$$\begin{aligned}&\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{dn+1}\\ =&\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}\end{aligned} \tag{18}$$

Leaving out the trivial cases

$$\begin{aligned}&\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=0\\ &\Leftrightarrow\\ &(s-\alpha_i)^2=(1-s-\alpha_i)^2=-\beta_i^2, i=1,2,3,\cdots,n,n+1\end{aligned} \tag{19}$$

which directly imply Lemma 4 is true, Eq.(18) is then equivalent to the following Eq.(20).

$$\frac{\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}=\frac{\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}{\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}} \tag{20}$$

Another possibility $\frac{\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}=\frac{\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}$ is excluded due to that means $\beta_{n+1}=\beta_n, \alpha_{n+1}=1-\alpha_n, 1-\alpha_{n+1}=\alpha_n, d_{n+1}=d_n$, i.e.,

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the $(n+1)^{th}$ zeros has already been considered in previous zeros.
Without loss of generality, we set

$$\frac{\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}=\frac{\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}{\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}=k\neq 0 \tag{21}$$

where k is a constant to be determined.
Therefore Eq.(20) is equivalent to the following Eq.(22)

$$\begin{cases}\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=k\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\ \left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=k\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}\end{cases} \tag{22}$$

By comparing the like terms of polynomials in both sides of Eq.(22), we know that Eq.(22) holds if and only if $k=1$, that means Eq.(20) is equivalent to the following Eq.(23)

$$\begin{cases}\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\ \left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}\end{cases} \tag{23}$$

Recalling that Eq.(17) $\Leftrightarrow$ Eq.(18) $\Leftrightarrow$ Eq.(20), then we have

$$\begin{aligned}&\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\ &\Leftrightarrow\\ &\begin{cases}\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\ \left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}\end{cases}\end{aligned} \tag{24}$$

That completes the proof of Lemma 4.

Proof of the RH: The details are delivered in three steps as follows.

Step 1: Based on the Hadamard product in Eq.(10), we have

$$\begin{aligned}\xi(s)&=\xi(0)\prod_{\rho}(1-\frac{s}{\rho})\\ &=\xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i})\\ &=\xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\alpha_i+j\beta_i})(1-\frac{s}{\alpha_i-j\beta_i})\\ &=\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})\end{aligned} \tag{25}$$

The convergence of the infinite product in Eq.(25) in the form

$$\xi(s)=\xi(0)\prod_{\rho}(1-\frac{s}{\rho})(1-\frac{s}{\bar{\rho}})=\xi(0)\prod_{\rho}\Big(1-\frac{s(2\alpha-s)}{|\rho|^2}\Big),0<\alpha=\Re(\rho)<1$$

can be obtained by the similar way that was used in Chapter 2.5 of Reference [22] to prove the convergence of

$$\xi(s)=\xi(0)\prod_{\rho}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho})=\xi(0)\prod_{\rho}\Big(1-\frac{s(1-s)}{\rho(1-\rho)}\Big)$$

In fact, the convergence of these two infinite products depends on the convergence of infinite series: $\sum_{\rho}\frac{1}{|\rho|^2}$, and $\sum_{\rho}\frac{1}{|\rho(1-\rho)|}$, respectively. The proof details of the convergence of these two infinite series can be found on pages 42-43 in Ref. [22], as well as on page 156 in Ref. [9].

Further, taking into account the possibility of multiple zeros, we have

$$\xi(s)=\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{26}$$

where $d_i\geq 1$ are natural numbers.
Step 2: Replacing s with $1-s$ in Eq.(26), we obtain the infinite product expression of $\xi(1-s)$

$$\xi(1-s)=\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{27}$$

Step 3: We have by $\xi(s)=\xi(1-s)$ that

$$\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i}=\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{28}$$

which is equivalent to

$$\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{d_i}=\prod_{i=1}^{\infty}(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{d_i} \tag{29}$$

And that β_i can be certainly arranged in order of increasing $|\beta_i|$, i.e., $|\beta_1|\leq|\beta_2|\leq|\beta_3|\leq,\cdots$.
Then according to Lemma 3, Eq.(29) is equivalent to $\alpha_i=\frac{1}{2}, i\in\mathbb{N}$ from 1 to infinity.
Thus we conclude that all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$, i.e., Statement 2 of the RH is true; According to Lemma 2, the Statement 1 of the RH is also true, i.e., All the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.
That completes the proof of the RH.

3 Conclusion

The Riemann Hypothesis is proved based on a new expression of the completed zeta function $\xi(s)$, i.e., the infinite product of quadratic factors determined by complex conjugate roots of $\xi(s) = 0$, i.e., $\rho_i = \alpha_i + j\beta_i, \bar{\rho}_i = \alpha_i - j\beta_i, 0 < \alpha_i < 1, \beta_i \neq 0, i \in \mathbb{N}$, from 1 to infinity.

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References

1. Riemann B. (1859), Über die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberichte der Deutschen Akademie der Wissenschaften zu Berlin, 2, 671-680.
2. Bombieri E. (2000), Problems of the millennium: The Riemann Hypothesis, CLAY
3. Peter Sarnak (2004), Problems of the Millennium: The Riemann Hypothesis, CLAY
4. Hadamard J. (1896), Sur la distribution des zros de la fonction $\zeta(s)$ et ses consquences arithmtiques, Bulletin de la Socit Mathmatique de France, 14: 199-220, doi:10.24033/bsmf.545 Reprinted in (Borwein et al. 2008).
5. de la Valle-Poussin Ch. J. (1896), Recherches analytiques sur la thorie des nombers premiers, Ann. Soc. Sci. Bruxelles, 20: 183-256
6. Hardy G. H. (1914), Sur les Zros de la Fonction $\zeta(s)$ de Riemann, C. R. Acad. Sci. Paris, 158: 1012-1014, JFM 45.0716.04 Reprinted in (Borwein et al. 2008).
7. Hardy G. H., Littlewood J. E. (1921), The zeros of Riemann's zeta-function on the critical line, Math. Z., 10 (3-4): 283-317.
8. Tom M. Apostol (1998), Introduction to Analytic Number Theory, New York: Springer.
9. Chengdong Pan, Chengbiao Pan (2016), Basic Analytic Number Theory (in Chinese), 2nd Edition, Harbin Institute of Technology Press.
10. Reyes E. O. (2004), The Riemann zeta function, Master Thesis of California State University, San Bernardino, Theses Digitization Project. 2648. https: //scholarworks.lib.csusb.edu /etd-project/2648
11. A. Selberg (1942), On the zeros of the zeta-function of Riemann, Der Kong. Norske Vidensk. Selsk. Forhand. 15, 59-62; also, Collected Papers, Springer- Verlag, Berlin - Heidelberg - New York 1989, Vol. I, 156-159.
12. N. Levinson (1974), More than one-third of the zeros of the Riemann zeta function are on $\sigma = \frac{1}{2}$, Adv. Math. 13, 383-436.
13. S. Lou and Q. Yao (1981), A lower bound for zeros of Riemanns zeta function on the line $\sigma = \frac{1}{2}$, Acta Mathematica Sinica (in chinese), 24, 390-400.
14. J. B. Conrey (1989), More than two fifths of the zeros of the Riemann zeta function are on the critical line, J. reine angew. Math. 399, 1-26.
15. H. M. Bui, J. B. Conrey and M. P. Young (2011), More than 41% of the zeros of the zeta function are on the critical line, http://arxiv.org/abs/1002.4127v2.
16. Feng S. (2012), Zeros of the Riemann zeta function on the critical line, Journal of Number Theory, 132(4), 511-542.
17. Siegel, C. L. (1932), Über Riemanns Nachlaß zur analytischen Zahlentheorie, Quellen Studien zur Geschichte der Math. Astron. Und Phys. Abt. B: Studien 2: 45-80, Reprinted in Gesammelte Abhandlungen, Vol. 1. Berlin: Springer-Verlag, 1966.
18. Gram, J. P. (1903), Note sur les zéros de la fonction $\zeta(s)$ de Riemann, Acta Mathematica, 27: 289-304.
19. Titchmarsh, Edward Charles (1935), The Zeros of the Riemann Zeta-Function, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, The Royal Society, 151 (873): 234-255.
20. Titchmarsh, Edward Charles (1936), The Zeros of the Riemann Zeta-Function, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, The Royal Society, 157 (891): 261-263.

 doi:10.20944/preprints202108.0146.v17

21. Hadamard J. (1893), Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. Journal de mathématiques pures et appliquées, 9, 171-216.
22. Edwards, H. M. (1974), Riemann's Zeta Function, New York: Dover Publications.