Asymptotics and confluence for some singular nonlinear q-difference-differential Cauchy problem

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Abstract

We examine a family of nonlinear $q-$difference-differential Cauchy problems obtained as a coupling of linear Cauchy problems containing dilation $q-$difference operators, recently investigated by the author, and quasi-linear Kowalevski type problems that involve contraction $q-$difference operators. We build up local holomorphic solutions to these problems. Two aspects of these solutions are explored. One facet deals with asymptotic expansions in the complex time variable for which a mixed type Gevrey and $q-$Gevrey structure is exhibited. The other feature concerns the problem of confluence of these solutions as $q > 1$ tends to 1.

Key words: asymptotic expansion, confluence, formal power series, partial differential equation, $q-$difference equation. 2010 MSC: 35R10, 35C10, 35C15, 35C20.

1 Introduction

In this paper we study a particular family of nonlinear $q-$difference-differential Cauchy problems displayed as follows

$$P(t^{k+1}\partial_t)\partial_z^{\tilde{S}}v(t,z) = \mathfrak{G}\Big(t,z,\{(\partial_t^{r_1}\partial_z^{r_2}v)(q^{r_3}t,z)\}_{\underline{r}=(r_1,r_2,r_3)\in\mathfrak{H}}\Big) \tag{1}$$

for prescribed Cauchy data

$$(\partial_z^j v)(t,0) = \vartheta_j(t) \quad , \quad 0 \leq j \leq \tilde{S}-1 \tag{2}$$

where $\tilde{S}, k \geq 1$ are integers, $q > 1$ is a real number and where the symbol $P(T)$ from the leading term of the equation (1) stands for a non constant element of $\mathbb{C}[T]$ and $\mathfrak{H}$ represents some well chosen finite subset of $\mathbb{N}^2 \times \mathbb{Z}$. The right part $\mathfrak{G}(t,z,\{V_{\underline{r}}\}_{\underline{r}\in\mathfrak{H}})$ of (1) is a polynomial in the variables $V_{\underline{r}}$ for $\underline{r} \in \mathfrak{H}$, holomorphic relatively to z on some disc in $\mathbb{C}$ centered at the origin and which depends polynomially on t along with the data (2). The precise shape of (1), (2) is stated in Corollary 1 in Subsection 3.3.

The present work is the sequel of the investigation initiated in [13] that focused on some linear $q-$difference-differential Cauchy problems outlined as

$$P(t^{k+1}\partial_t)\partial_z^{S}u(t,z) = \mathcal{P}(t,z,\sigma_{q;t},t^{k+1}\partial_t,\partial_z)u(t,z) \tag{3}$$

for given Cauchy data

$$(\partial_z^j u)(t,0) = \varphi_j(t) \quad , \quad 0 \leq j \leq S-1 \tag{4}$$

where $S \geq 1$ is a suitable natural number, the piece $P(T)$ from the leading term of (3) is the polynomial appearing in (1), $k \geq 1$ is the integer stemming from (1), $\sigma_{q;t}$ stands for the dilation operator $t \mapsto qt$ for $q > 1$ arising in (1) and the right handside $\mathcal{P}(t,z,V_1,V_2,V_3)$ together with the data (4) represent fittingly selected polynomials. As sumed up in Theorem 1 of this work, under strong restrictions on the shape of (3) (not discussed in this paper but listed in [13]), a finite set $\{u_p(t,z)\}_{0 \leq p \leq \varsigma-1}$, for some integer $\varsigma \geq 2$, of holomorphic solutions to (3), (4) could be modeled on products $\mathcal{T}_p \times D$, where D stands for some small disc centered at 0 in $\mathbb{C}$ and where $\underline{\mathcal{T}} = \{\mathcal{T}_p\}_{0 \leq p \leq \varsigma-1}$ is a suitable set of bounded sectors whose union covers some neigborhood of 0 in $\mathbb{C}^*$, see Definition 1. These solutions were expressed through Laplace transforms of order k,

$$u_p(t,z) = k \int_{L_{\gamma_p}} w_p(u,z) \exp(-(u/t)^k) du/u \tag{5}$$

along halflines $L_{\gamma_p} = [0,+\infty)e^{\sqrt{-1}\gamma_p}$ in convenient directions $\gamma_p \in \mathbb{R}$ where the *Borel map* $w_p(u,z)$ is holomorphic relatively to z on D and is compelled to bear $q-$exponential growth rate (17) w.r.t u on some unbounded sector $\mathcal{U}_p$. In [13], we addressed two important features of these solutions

- Asymptotic behaviour w.r.t the time variable t as $t \to 0$.
- Confluence property as $q > 1$ tends to 1.

Regarding the first point, a fine structure of mixed Gevrey and $q-$Gevrey type was disclosed. Namely, as expounded in Theorem 3 and 4, all the partial maps $t \mapsto u_p(t,z)$ share a common formal power series $\hat{u}(t,z) = \sum_{n \geq 0} u_n(z)t^n$, where $u_n(z)$ are bounded holomorphic on D, as so-called Gevrey asymptotic expansion of mixed order $(1/k;(q,1))$ on $\mathcal{T}_p$, meaning that two constants $C, M > 0$ can be pinpointed with

$$\sup_{z \in D} |u_p(t,z) - \sum_{n=0}^{N} u_n(z)t^n| \leq CM^{N+1}\Gamma(\frac{N+1}{k})q^{(N+1)^2/2}|t|^{N+1} \tag{6}$$

whenever $t \in \mathcal{T}_p$, for all integers $N \geq 0$.

Concerning the second item, discussed in Subsections 5.1 and 5.5, we have shown that for any prescribed sector $\mathcal{T}$ from the covering $\underline{\mathcal{T}}$, the corresponding solution $u_{;q}(t,z)$ (whose reliance on the parameter q is flagged by an index $;q$) to (3), (4) merges uniformly on $\mathcal{T} \times D$, as $q \in (1,q_0]$ tends to 1 for some fixed $q_0 > 0$, to a holomorphic map $u_{;1}(t,z)$ on $\mathcal{T} \times D$ which itself solves some linear PDE Cauchy problem given by (175), (176). More precisely, some constant $K > 0$ (unrelated to q) could be singled out with

$$\sup_{t \in \mathcal{T}, z \in D} |u_{;q}(t,z) - u_{;1}(t,z)| \leq K(q-1) \tag{7}$$

provided that $q \in (1,q_0]$.

The problem (1), (2) examined in this work is actually obtained by means of a procedure (described in Subsection 2.2) which consists in *coupling* the singular linear Cauchy problem (3), (4) with a quasi-linear Kowalevski type problem which involves the contraction operator $\sigma_{q^{-1};t} : t \mapsto q^{-1}t$, framed as

$$\partial_z^{\kappa} v(t,z) = \mathcal{P}_1(t,z,\sigma_{q^{-1};t},t^{k+1}\partial_t,\partial_z)v(t,z) + a(t,z)v^2(t,z) + u(t,z) \tag{8}$$

for assigned Cauchy data

$$(\partial_z^j v)(t,0) = \tilde{\varphi}_j(t) \quad , \quad 0 \leq j \leq \kappa - 1 \tag{9}$$

where $\kappa \geq 1$ is an appropriate integer, $a(t,z)$ is some polynomial in t with holomorphic coefficients on D, the linear piece $\mathcal{P}_1(t,z,V_1,V_2,V_3)$ as well as the data (9) stand for properly chosen polynomials and where the forcing term $u(t,z)$ is required to solve the linear problem (3), (4). Notice that the appearance of the $q-$difference operator $\sigma_{q^{-1};t}$ is mandatory when some time derivative ∂_t occurs in the equation (8), according to the constraints (20).

Our objectives remain similar to those in [13] and concern

- The construction of local holomorphic solutions to (1), (2).
- Asymptotic expansions of these solutions as time t borders the origin.
- Confluence aspects as $q \to 1$.

The first item is completed in Subsection 3.3 (Theorem 2) where a finite set $\{v_p(t,z)\}_{0 \leq p \leq \varsigma - 1}$ of holomorphic solutions to (1), (2) is built up on domains $\mathcal{T}_p \times D$, provided that the radii of $\mathcal{T}_p$ and D are taken small enough. Furthermore, the solutions can be represented as Laplace transforms of order k,

$$v_p(t,z) = k \int_{L_{\gamma_p}} \Theta_p(u,z) \exp(-(u/t)^k) \frac{du}{u}$$

along the same halflines L_{γ_p} as in (5) where the *Borel map* $\Theta_p(u,z)$ remains holomorphic w.r.t z on D but suffers now (at most) exponential growth rate (89) of order k relatively to u on $\mathcal{U}_p$ (and not in general $q-$exponential increase as it was the case for w_p).

The second item is achieved in Subsection 4.5 (Theorem 5) where the existence of a formal power series $\hat{v}(t,z) = \sum_{l \geq 0} h_l(z) t^l$, with holomorphic coefficients $h_l(z)$ on D, is established which stands for the common Gevrey asymptotic expansion of mixed order $(1/k;(q,1))$ on $\mathcal{T}_p$ of the partial maps $t \mapsto v_p(t,z)$, $0 \leq p \leq \varsigma - 1$, satisfying therefore similar estimates to (6).

The last item is explained in Subsection 5.7 (Theorem 6). For any given sector $\mathcal{T}$ from the covering $\underline{\mathcal{T}}$, the related solution $v_{;q}(t,z)$ (whose dependence on q is marked by the index $;q$) to (1), (2) converges uniformly on $\mathcal{T} \times D$, as $q \to 1$, to a holomorphic map $v_{;1}(t,z)$ on $\mathcal{T} \times D$ which is the solution of some nonlinear PDE Cauchy problem, stated in (180), (181). Factually, comparable bounds to (7) hold, see (287).

We draw attention to the fact that the proofs of our three main statements Theorems 2, 5 and 6 lean on statements established in [13]. In essence, the features of the solutions $v_p(t,z)$ of (1), (2) reached in this paper are identical to those of the solutions $u_p(t,z)$ of (3), (4) achieved in [13]. However, their proofs differ fondamentally. Indeed, the construction of the sectorial local holomorphic solutions $v_p(t,z)$ to (1), (2) is performed by means of a fixed point argument in suitably selected Banach spaces (Subsections 3.1 and 3.2) when a mere induction principle and elementary estimates were only required to reach the local solutions $u_p(t,z)$ of (3), (4). The asymptotic features relatively to time t of the solutions $u_p(t,z)$ were obtained by dint of a version the so-called Ramis-Sibuya theorem which banks on sharp estimates of the differences $u_{p+1} - u_p$. This approach *fails* to be applied in our nonlinear context. We use instead a majorant series method that reduces the problem to the construction of formal power series solutions to a related Cauchy problem in appropriate Banach spaces (Section 4). Regarding the confluence properties, both works hinge on an auxiliary result which studies the action of $q-$difference operators on the Borel maps of the limit maps $u_{;1}(t,z)$ and $v_{;1}(t,z)$ (see Propositions 17 and 18) but the proofs of the present work are again based on functional analytic arguments and the use of accurate bounds in Banach spaces, while induction principle was favored in [13].

Observe that, by construction, the nonlinear $q-$difference differential equation (1) involves both dilations and contractions w.r.t the time variable t by means of the presence of operators $\sigma_{q;t}^{l}$ for both $l > 0$ and $l < 0$. The same property arises in the framework of nonlinear $q-$difference equation in the study of the so-called $q-$Painlevé equations. Indeed, the $q-$discrete versions of the first and second Painlevé equations are expressed through the next two equations

$$w(qx)w(\frac{x}{q}) = \frac{1}{w(x)} - \frac{1}{xw^2(x)}$$

and

$$g(qx)g(\frac{x}{q}) = \frac{\alpha x^2(g(x)+x^2)}{g(x)(g(x)-1)}$$

for $x = x_0q^n$ with $n \in \mathbb{Z}$ for some $x_0 \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0,1\}$ and α some parameter, see for instance [6], [8]. For an excellent comprehensive and introductive book to q-Painlevé equations and more generally to integrable discrete dynamical systems, we mention [5].

Regarding the existence of local holomorphic solutions to nonlinear $q-$difference equations, we may refer to some recent works. Indeed, for meromorphic or holomorphic solutions around the origin for special type of nonlinear $q-$difference equations such as the $q-$Painlevé equations, we can mention [7], [21]. Some category of nonlinear $q-$difference equations of the form

$$xy(qx) = y(x) + b(y(x), x)$$

where b is some polynomial has been investigated by F. Menous in the paper [19] who gave assumptions under which such equations can be analytically conjugated to well studied models of linear $q-$difference equations with so-called irregular singularity at the origin, $xz(qx) = z(x)$ or $xz(qx) = z(x) + x$. In the recent work [4], R. Gontsov, I. Goryuchkina and A. Lastra provide sufficient conditions for the convergence of so-called generalized power series

$$\varphi(z) = \sum_{j \geq 0} c_j z^{\lambda_j}$$

with complex coefficients $c_j \in \mathbb{C}$ and *complex* exponents $\lambda_j \in \mathbb{C}$, that are solutions of algebraic $q-$difference equations

$$F(z, \varphi(z), \varphi(qz), \ldots, \varphi(q^n z)) = 0$$

where F stands for some polynomial, providing in particular local sectorial holomorphic solutions to these equations.

In the context of nonlinear $q-$difference-differential equations, the literature concerning local existence of solutions is less profuse. Nevertheless, the important result by H. Yamazawa [22] ought to be quoted in that trend. The author constructs holomorphic and singular solutions of logarithmic type near the origin to equations of the form

$$u(qt, x) = u(t, x) + F(t, x, \{\partial_x^{\alpha} u\}_{|\alpha| \leq m})$$

for $t \in \mathbb{C}$, $x \in \mathbb{C}^n$, $n \geq 1$, $q > 1$ and where F is some well prepared analytic function in its arguments.

On the subject of confluence for linear $q-$difference equations, some recent references have been pointed out on our lastest contribution [13]. The confluence in the framework of nonlinear $q-$difference equations has been much less examined and represents a propitious direction for upcoming research. For instance, some aspects of confluence for the so-called $q-$Painlevé VI equation have been recently undertaken by T. Dreyfus and V. Heu in [3]. They construct some well prepared $q-$analog Hamiltonian system

$$\begin{cases} \partial_{q,t}\mathbf{y} = & \partial_{q,\mathbf{Z}}H(\mathbf{y},\mathbf{Z},t) + O(q-1) \\ \partial_{q,t}\mathbf{Z} = & -\partial_{q,\mathbf{y}}H(\mathbf{y},\mathbf{Z},t) + O(q-1) \end{cases}$$

where $\partial_{q,x} = \frac{\sigma_{q;x}-1}{qx-x}$ and show that its *discret solution* given in the form of two sequences $\mathbf{y}_l = \mathbf{y}(q^l t_0)$, $\mathbf{Z}_l = \mathbf{Z}(q^l t_0)$ for given $t_0 \in \mathbb{C}^*$ and $(\mathbf{y}_0, \mathbf{Z}_0) \in (\mathbb{C} \setminus \{0,1,t_0\}) \times \mathbb{C}$ encodes the Taylor series coefficients of the holomorphic solution $(y(t), Z(t))$ to the (*formal limit as $q > 1$ tends to 1*) non autonomous Hamiltonian system

$$\begin{cases} y'(t) = & \frac{\partial H}{\partial Z}(y,Z,t) \\ Z'(t) = & -\frac{\partial H}{\partial y}(y,Z,t) \end{cases}$$

with initial condition $y(t_0) = \mathbf{y}_0$ and $Z(t_0) = \mathbf{Z}_0$, which defines the sixth Painlevé equation for some prescribed rational map $H \in \mathbb{C}(y,Z,t)$.

2 The main problem outlined

2.1 A finite set of holomorphic solutions to a singular linear Cauchy problem

In this subsection, we remind the reader parts of the results obtained in our previous work [13] that will be used within the present section 2. We first describe the linear Cauchy problem we have considered in that study.

Let $k, S \geq 1$ be integers and $q > 1$ be a real number. We set $P(\tau) \in \mathbb{C}[\tau]$ a polynomial with complex coefficients such that

$$P(0) \neq 0 \quad (10)$$

Let $\mathcal{A}$ be a finite subset of $\mathbb{N}^4$. For all $\underline{l} \in \mathcal{A}$ and all $0 \leq j \leq S-1$, we fix polynomials $c_{\underline{l}}(z)$ and $\varphi_j(t)$ with complex coefficients.

We focus on the next singular linear Cauchy problem with polynomial coefficients in time,

$$P(t^{k+1}\partial_t)\partial_z^S u(t,z) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}} c_{\underline{l}}(z) t^{l_0} \left((t^{k+1}\partial_t)^{l_1} \partial_z^{l_2} u \right)(q^{l_3}t, z) \quad (11)$$

for given Cauchy data

$$(\partial_z^j u)(t,0) = \varphi_j(t) \quad , \quad 0 \leq j \leq S-1. \quad (12)$$

In order to describe a set of solutions to (11), (12), we need to recall the definitions of good coverings and admissible sets of sectors introduced in Section 6 of [13].

Definition 1 *Let $\varsigma \geq 2$ be an integer. For all $0 \leq p \leq \varsigma - 1$, we select open sectors $\mathcal{T}_p$ centered at 0 (and do not contain 0) with given radius $r_{\mathcal{T}}$ that fulfill the next three features:*
i) *The intersection of any two consecutive sectors of the family $\underline{\mathcal{T}} = \{\mathcal{T}_p\}_{0 \leq p \leq \varsigma-1}$ is non empty, namely*

$$\mathcal{T}_p \cap \mathcal{T}_{p+1} \neq \emptyset$$

for all $0 \leq p \leq \varsigma - 1$, *with the convention that* $\mathcal{T}_{\varsigma} = \mathcal{T}_0$.
ii) *The intersection of any three elements in* $\underline{\mathcal{T}}$ *is empty.*
iii) *The union of the sectors* $\mathcal{T}_p$ *covers some punctured neighborhood* $\dot{\mathcal{U}}$ *of the origin in* $\mathbb{C}^*$,

$$\cup_{p=0}^{\varsigma-1} \mathcal{T}_p = \dot{\mathcal{U}} = \mathcal{U} \setminus \{0\}$$

The family $\underline{\mathcal{T}}$ *is then named a good covering in* $\mathbb{C}^*$.

The notion of admissible set of sectors is depicted in the next

Definition 2 *We set* $\varsigma \geq 2$ *as an integer and set* $\underline{\mathcal{T}} = \{\mathcal{T}_p\}_{0 \leq p \leq \varsigma-1}$ *as a good covering in* $\mathbb{C}^*$. *We consider a set* $\underline{\mathcal{U}} = \{\mathcal{U}_p\}_{0 \leq p \leq \varsigma-1}$ *of unbounded sectors* $\mathcal{U}_p$ *centered at 0, that endorse the next two properties:*
1) *Each sector* $\mathcal{U}_p$ *does not contain any of the roots of the polynomial* $u \mapsto P(ku^k)$, *for* $0 \leq p \leq \varsigma - 1$.
2) *For all* $0 \leq p \leq \varsigma - 1$, *there exists a constant* $\Delta_p > 0$ *such that for all* $t \in \mathcal{T}_p$, *one can single out a direction* $\gamma_p \in \mathbb{R}$ *(that may depend on* t*) such that both conditions*

$$L_{\gamma_p} = [0, +\infty) \exp(\sqrt{-1}\gamma_p) \subset \mathcal{U}_p \cup \{0\} \tag{13}$$

and

$$\cos(k(\gamma_p - \arg(t))) > \Delta_p \tag{14}$$

hold.
We say that the set of sectors $\underline{\mathcal{D}} = \{\underline{\mathcal{T}}, \underline{\mathcal{U}}\}$ *represents an admissible set of sectors.*

In Section 6 of the paper [13], we have obtained the following result.

Theorem 1 *Let us assume that all the requirements asked in Section 2.1 of [13] hold true. Fix a good covering* $\underline{\mathcal{T}} = \{\mathcal{T}_p\}_{0 \leq p \leq \varsigma-1}$ *in* $\mathbb{C}^*$ *and a set* $\underline{\mathcal{U}} = \{\mathcal{U}_p\}_{0 \leq p \leq \varsigma-1}$ *of unbounded sectors chosen in a way that the data* $\underline{\mathcal{D}} = \{\underline{\mathcal{T}}, \underline{\mathcal{U}}\}$ *forms an admissible set of sectors.*

Then, for all $0 \leq p \leq \varsigma - 1$, *one can construct a solution* $u_p(t, z)$ *to the Cauchy problem (11), (12) that is bounded and holomorphic on* $\mathcal{T}_p \times D_{\frac{1}{2C_4}}$ *and that can be expressed through a Laplace transform of order* k,

$$u_p(t, z) = k \int_{L_{\gamma_p}} w_p(u, z) \exp(-(u/t)^k) du/u \tag{15}$$

for $(t, z) \in \mathcal{T}_p \times D_{\frac{1}{2C_4}}$ *where* $D_{\frac{1}{2C_4}}$ *stands for the disc centered at 0 with radius* $\frac{1}{2C_4}$ *for some well chosen constant* $C_4 > 0$. *The Borel map* $w_p(u, z)$ *represents a holomorphic function on the domain* $\mathcal{U}_p \times D_{\frac{1}{2C_4}}$ *whose Taylor expansion*

$$w_p(u, z) = \sum_{\beta \geq 0} w_{p,\beta}(u) \frac{z^\beta}{\beta!} \tag{16}$$

is subjected to the next bounds

$$|w_{p,\beta}(u)| \leq C_3 (C_4)^\beta \beta! |u| \exp\left(k_1 \log^2(|u| + u_0) + \alpha \log(|u| + u_0)\right) \tag{17}$$

for all $\beta \geq 0$, *all* $u \in \mathcal{U}_p$, *for suitably chosen constants* $C_3, k_1 > 0$ *and* $u_0 > 1$, $\alpha \geq 0$.

2.2 Setup of the main nonlinear Cauchy problem

Let $\kappa \geq 1$ be an integer and k, q as defined in the previous subsection. Let $\mathcal{C}$ be a finite subset of $\mathbb{N}^4$. For all $\underline{h} \in \mathcal{C}$, we fix a bounded and analytic function $d_{\underline{h}}(z)$ on a disc D_R centered at 0 with some radius $R > 0$. Furthermore, we define a polynomial

$$a(t,z) = \sum_{h=0}^{A} a_h(z)t^h$$

for some integer $A \geq 0$, where the coefficients $a_h(z)$ are bounded and holomorphic on D_R and for all $0 \leq j \leq \kappa - 1$, we denote $\check{\varphi}_j(t)$ polynomials with complex coefficients written in the form

$$\check{\varphi}_j(t) = \sum_{h \in \check{J}_j} \check{p}_{j,h}\Gamma(h/k)t^h \tag{18}$$

where $\check{J}_j$ stands for a finite subset of $\mathbb{N} \setminus \{0\}$.

The finite set $\mathcal{C}$ is compelled to fulfill to the next list of requirements:
A1) There exists a real number $b > 1$ for which

$$\kappa \geq bh_1 + h_2 \quad , \quad \kappa > h_2 \tag{19}$$

for all $\underline{h} = (h_0, h_1, h_2, h_3) \in \mathcal{C}$.
A2) The next inequalities

$$\kappa \geq (\frac{1}{k} + 1)h_1 + h_2 \quad , \quad \kappa > h_2 \quad , \quad h_3 \geq h_1 \tag{20}$$

hold provided that $\underline{h} = (h_0, h_1, h_2, h_3) \in \mathcal{C}$.

We consider the next nonlinear nonhomogeneous Cauchy problem

$$\partial_z^{\kappa} v_p(t,z) = \sum_{\underline{h}=(h_0,h_1,h_2,h_3) \in \mathcal{C}} d_{\underline{h}}(z) t^{h_0} \Big((t^{k+1}\partial_t)^{h_1} \partial_z^{h_2} v_p \Big)(q^{-h_3}t, z) + a(t,z)v_p^2(t,z) + u_p(t,z) \tag{21}$$

for given Cauchy data

$$(\partial_z^j v_p)(t,0) = \check{\varphi}_j(t) \quad , \quad 0 \leq j \leq \kappa - 1 \tag{22}$$

where the forcing term $u_p(t,z)$ is the holomorphic solution of the linear Cauchy problem (11), (12) disclosed in Theorem 1 of the former subsection.

We now unveil our main roadmap that will lead later on to the construction of suitable sets of solutions to our problem. We search for solutions to (21), (22) in the form of a Laplace transform of order k, namely

$$v_p(t,z) = k \int_{L_{\gamma_p}} \Theta_p(u,z) \exp(-(u/t)^k) du/u \tag{23}$$

along the halfline $L_{\gamma_p} = [0, +\infty)e^{\sqrt{-1}\gamma_p}$ appearing in the representation (15). So far, the so-called Borel map $\Theta_p(u,z)$ is supposed to be holomorphic with respect to u on the unbounded sector $\mathcal{U}_p$ and analytic w.r.t z on some small disc D_r centered at 0 with radius $r > 0$. For the Laplace transform to be well defined, we make the further assumption that $\Theta_p(u,z)$ has a most exponential growth of order k w.r.t u on $\mathcal{U}_p$, uniformly in z on D_r, meaning the existence of two constants $C, K > 0$ with

$$\sup_{z \in D_r} |\Theta_p(u,z)| \leq C|u| \exp(K|u|^k)$$

for all $u \in \mathcal{U}_p$. Once we assume that such solutions exists, we will derive some functional equations that the Borel map $\Theta_p(u,z)$ will be asked to solve at a formal level only. Such equations will be described in the next subsection. Later on, in Subsection 3.2, these *convolution* equations will be solved in some Banach space of holomorphic functions, see Proposition 9, producing a genuine holomorphic map $\Theta_p(u,z)$ satisfying the above requirements.

2.3 An auxiliary Cauchy problem satisfied by the Borel map Θ_p

We first need to remind the reader the next proposition which is a slightly modified version of Proposition 1 of [13].

Proposition 1 *We set* $(\mathbb{E}, ||.||_{\mathbb{E}})$ *as a complex Banach space. Let* $k \geq 1$ *be an integer and let* $w : S_{d,\delta} \to \mathbb{E}$ *be a holomorphic function on the open unbounded sector* $S_{d,\delta} = \{u \in \mathbb{C}^* : |d - \arg(u)| < \delta\}$, *continuous on* $S_{d,\delta} \cup \{0\}$. *The existence of two constants* $C > 0$ *and* $K > 0$ *such that*

$$||w(u)||_{\mathbb{E}} \leq C|u|e^{K|u|^k} \tag{24}$$

is assumed for all $u \in S_{d,\delta}$. *Then, the Laplace transform of order* k *of* w *in the direction* d *is defined by*

$$\mathcal{L}_k^d(w(u))(t) = k \int_{L_\gamma} w(u) e^{-(u/t)^k} \frac{du}{u},$$

along a half-line $L_\gamma = \mathbb{R}_+ e^{i\gamma} \subset S_{d,\delta} \cup \{0\}$, *where* γ *depends on* t *and is chosen in such a way that* $\cos(k(\gamma - \arg(t))) \geq \delta_1 > 0$, *for some fixed* δ_1. *The function* $\mathcal{L}_k^d(w(u))(t)$ *is well defined, holomorphic and bounded in any sector*

$$S_{d,\theta,R^{1/k}} = \{t \in \mathbb{C}^* : |t| < R^{1/k} \quad , \quad |d - \arg(t)| < \theta/2\}, \tag{25}$$

where $\frac{\pi}{k} < \theta < \frac{\pi}{k} + 2\delta$ *and* $0 < R < \delta_1/K$.
A) The action of the Laplace transform on entire functions is described as follows: If w *is an entire function on* $\mathbb{C}$, *with growth estimates (24) and with Taylor expansion* $w(u) = \sum_{n \geq 1} b_n u^n$, *then* $\mathcal{L}_k^d(w(u))(t)$ *defines an analytic function near the origin w.r.t* t *with convergent Taylor expansion* $\sum_{n \geq 1} \Gamma(\frac{n}{k}) b_n t^n$.
B) The actions of the irregular operator $t^{k+1}\partial_t$ *and the monomial* t^m *on the Laplace transform are expressed through the next formulas*

$$\mathcal{L}_k^d(ku^k w(u))(t) = t^{k+1}\partial_t\Big(\mathcal{L}_k^d(w(u))(t)\Big) \quad , \quad t^m \mathcal{L}_k^d(w(u))(t) = \mathcal{L}_k^d\Big(u \mapsto (u^m \star_k w(u))\Big)(t), \tag{26}$$

for every integers $m \geq 1$, *and for all* $t \in S_{d,\theta,R^{1/k}}$ *with* $0 < R < \delta_1/K$. *Here,* $u^m \star_k w(u)$ *stands for the convolution product*

$$u^m \star_k w(u) := \frac{u^k}{\Gamma(\frac{m}{k})} \int_0^{u^k} (u^k - s)^{\frac{m}{k} - 1} w(s^{1/k}) \frac{ds}{s}.$$

C) Let $w_1, w_2 : S_{d,\delta} \to \mathbb{E}$ be holomorphic maps with the same feature (24) as w above. Assume moreover, that $\mathbb{E}$ is equipped with a product $*$ in a way that $(\mathbb{E}, *)$ becomes a Banach algebra. Then, the next multiplicative formula

$$\mathcal{L}_k^d(w_1(u))(t) \times \mathcal{L}_k^d(w_2(u))(t) = \mathcal{L}_k^d(w_1(u) \star_k w_2(u))(t) \tag{27}$$

holds for all $t \in S_{d,\theta,R^{1/k}}$ with $0 < R < \delta_1/K$, where $w_1(u) \star_k w_2(u)$ represents the convolution product

$$w_1(u) \star_k w_2(u) := u^k \int_0^{u^k} w_1((u^k - s)^{1/k}) * w_2(s^{1/k}) \frac{1}{(u^k - s)s} ds$$

D) The action of the dilation q^δ commutes with the Laplace transform, for any integer $\delta \geq 1$, namely

$$\mathcal{L}_k^d(w(u))(q^\delta t) = \mathcal{L}_k^d(w(q^\delta u))(t) \tag{28}$$

holds for all $t \in S_{d,\theta,R_1^{1/k}}$ for $0 < R_1 < \delta_1/(K q^{k\delta})$.

The point A) allows the Cauchy data (22) to be expressed through Laplace transforms of order k,

$$\check{\varphi}_j(t) = k \int_{L_\gamma} \check{P}_j(u) \exp(-(u/t)^k) du/u \tag{29}$$

of polynomials given by $\check{P}_j(u) = \sum_{h \in \check{J}_j} \check{p}_{j,h} u^h$, for $0 \leq j \leq \kappa - 1$.

Owing to the above identities (26), (27) and (28), we observe that the Laplace representation (23) solves the Cauchy problem (21), (22) if the Borel map $\Theta_p(u,z)$ is subjected to the next nonlinear and nonhomogeneous convolution Cauchy problem

$$\begin{aligned}
\partial_z^\kappa \Theta_p(u,z) = &\sum_{\underline{h}=(h_0,h_1,h_2,h_3) \in \mathcal{C}; h_0 = 0} d_{\underline{h}}(z) (k(q^{-h_3}u)^k)^{h_1} \left(\partial_z^{h_2} \Theta_p \right) (q^{-h_3}u, z) \\
&+ \sum_{\underline{h}=(h_0,h_1,h_2,h_3) \in \mathcal{C}; h_0 \geq 1} d_{\underline{h}}(z) \frac{u^k}{\Gamma(h_0/k)} \int_0^{u^k} (u^k - s)^{\frac{h_0}{k} - 1} (k(q^{-h_3} s^{1/k})^k)^{h_1} \left(\partial_z^{h_2} \Theta_p \right) (q^{-h_3} s^{1/k}, z) \frac{ds}{s} \\
&+ a_0(z) u^k \int_0^{u^k} \Theta_p((u^k - s)^{1/k}, z) \Theta_p(s^{1/k}, z) \frac{1}{(u^k - s)s} ds \\
&+ \sum_{h=1}^{A} a_h(z) \frac{u^k}{\Gamma(h/k)} \int_0^{u^k} (u^k - s)^{\frac{h}{k} - 1} \left(s \int_0^s \Theta_p((s - s_1)^{1/k}, z) \Theta_p(s_1^{1/k}, z) \frac{1}{(s - s_1)s_1} ds_1 \right) \frac{ds}{s} \\
&+ w_p(u,z)
\end{aligned} \tag{30}$$

for prescribed Cauchy data

$$(\partial_z^j \Theta_p)(u,0) = \check{P}_j(u) \quad , \quad 0 \leq j \leq \kappa - 1. \tag{31}$$

3 Solving the main nonlinear Cauchy problem

Within this section we construct actual holomorphic solutions to our main problem (21), (22) as Laplace transforms which boils down to build up actual holomorphic solutions to the Cauchy problem (30), (31) in suitable Banach spaces.

3.1 Some Banach spaces of analytic functions

In this subsection, we disclose the definition and properties of the Banach spaces in which we search for solutions to the Cauchy problem (30), (31).

Definition 3 *a) Let $b > 1$ be a real number. We set*

$$r_b(\beta) = \sum_{n=0}^{\beta} \frac{1}{(n+1)^b} \tag{32}$$

for all integers $\beta \geq 0$. We fix $\sigma > 0$ a real number. We denote $E^k_{(\beta,\sigma,\mathcal{U}_p)}$ the vector space of all functions $\Theta(\tau)$ that are holomorphic on the unbounded sector $\mathcal{U}_p$ (determined in Theorem 1) for which the norm

$$||\Theta(\tau)||_{(\beta,\sigma,\mathcal{U}_p)} := \sup_{\tau \in \mathcal{U}_p} \frac{1+|\tau|^{2k}}{|\tau|} \exp(-\sigma r_b(\beta)|\tau|^k)|\Theta(\tau)| \tag{33}$$

is finite.
b) Let $Z_0 > 0$ be a real number. We set $G^k_{(\sigma,Z_0,\mathcal{U}_p)}$ as the vector space of all holomorphic functions

$$\Theta(\tau,z) = \sum_{\beta \geq 0} \Theta_\beta(\tau) \frac{z^\beta}{\beta!}$$

near $z = 0$ with holomorphic coefficients on $\mathcal{U}_p$ such that the norm

$$||\Theta(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} := \sum_{\beta \geq 0} ||\Theta_\beta(\tau)||_{(\beta,\sigma,\mathcal{U}_p)} \frac{Z_0^\beta}{\beta!} \tag{34}$$

is finite. The normed space $(G^k_{(\sigma,Z_0,\mathcal{U}_p)}, ||.||_{(\sigma,Z_0,\mathcal{U}_p)})$ turns out to be a Banach space.

It is worth noticing that these Banach spaces are slight modifications of the Banach spaces introduced by the author and C. Stenger in the work [18] and by O. Costin, S. Tanveer in [2]. Similar Banach spaces have been used by the author and his colleagues in related works, see [9], [11], [15].

In the next list of propositions, we analyze the continuity of linear and nonlinear maps acting on these Banach spaces that will show to be useful in the next subsection.

Proposition 2 *Let $h_1, h_2, h_3 \geq 0$ be integers such that*

$$h_2 \geq bh_1 \tag{35}$$

Then, one can find a constant $M_1 > 0$ (relying on $k, q, h_1, h_2, h_3, \sigma, b$) such that

$$||\tau^{kh_1}\left(\partial_z^{-h_2} f\right)(q^{-h_3}\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq M_1 Z_0^{h_2} ||f(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \tag{36}$$

for all $f(\tau,z) \in G^k_{(\sigma,Z_0,\mathcal{U}_p)}$, where $\partial_z^{-h_2}$ stands for the h_2-times iteration of the integration map $\partial_z^{-1} f(z) = \int_0^z f(s)ds$.

Proof Let $f(\tau,z) = \sum_{\beta \geq 0} f_{\beta}(\tau) z^{\beta}/\beta!$ with $f_{\beta}(\tau) \in E^{k}_{(\beta,\sigma,\mathcal{U}_p)}$ for all $\beta \geq 0$. We check that

$$\tau^{kh_1}\left(\partial_z^{-h_2} f\right)(q^{-h_3}\tau, z) = \sum_{\beta \geq h_2} \tau^{kh_1} f_{\beta - h_2}(q^{-h_3}\tau)\frac{z^{\beta}}{\beta!}$$

In the next lemma, we provide bounds for the coefficients of this last series.

Lemma 1 *The next bounds*

$$||\tau^{kh_1} f_{\beta-h_2}(q^{-h_3}\tau)||_{(\beta,\sigma,\mathcal{U}_p)} \leq q^{(2k-1)h_3}\left(\frac{h_1}{\sigma h_2}\right)^{h_1} e^{-h_1}(\beta+1)^{bh_1}||f_{\beta-h_2}(\tau)||_{(\beta-h_2,\sigma,\mathcal{U}_p)} \quad (37)$$

hold for all $\beta \geq h_2$.

Proof We observe that

$$\begin{aligned} ||\tau^{kh_1} f_{\beta-h_2}(q^{-h_3}\tau)||_{(\beta,\sigma,\mathcal{U}_p)} &= \sup_{\tau \in \mathcal{U}_p} \frac{1+|\tau|^{2k}}{|\tau|} \exp(-\sigma r_b(\beta)|\tau|^k)|\tau^{kh_1} f_{\beta-h_2}(q^{-h_3}\tau)| \\ &\leq \sup_{\tau \in \mathcal{U}_p} \left\{ \frac{1+|q^{-h_3}\tau|^{2k}}{|q^{-h_3}\tau|} \exp(-\sigma r_b(\beta-h_2)|q^{-h_3}\tau|^k)|f_{\beta-h_2}(q^{-h_3}\tau)| \right\} \times \mathcal{A}(|\tau|,\beta) \end{aligned} \quad (38)$$

where

$$\mathcal{A}(|\tau|,\beta) = \frac{|q^{-h_3}\tau|}{|\tau|}\frac{1+|\tau|^{2k}}{1+|q^{-h_3}\tau|^{2k}}|\tau|^{kh_1}\exp(-\sigma r_b(\beta)|\tau|^k + \sigma r_b(\beta-h_2)|q^{-h_3}\tau|^k)$$

for all $\beta \geq h_2$. Besides, we notice that

$$\frac{1+|\tau|^{2k}}{1+|q^{-h_3}\tau|^{2k}} = \frac{1+q^{2kh_3}|q^{-h_3}\tau|^{2k}}{1+|q^{-h_3}\tau|^{2k}} \leq \sup_{x \geq 0} \frac{1+q^{2h_3k}x}{1+x} = q^{2h_3k}$$

for all $\tau \in \mathcal{U}_p$, since the function $h(x) = \frac{1+q^{2kh_3}x}{1+x}$ is increasing on $[0,+\infty)$ provided that $q > 1$. We deduce that

$$\mathcal{A}(|\tau|,\beta) \leq \psi(|\tau|,\beta) \quad (39)$$

where

$$\psi(|\tau|,\beta) = q^{(2k-1)h_3}|\tau|^{kh_1}\exp\Big(-\sigma(r_b(\beta)-r_b(\beta-h_2))|\tau|^k\Big)$$

for all $\tau \in \mathcal{U}_p$, all $\beta \geq h_2$. In the next step, we supply bounds for the map ψ. We check that

$$r_b(\beta) - r_b(\beta-h_2) = \sum_{n=\beta-h_2+1}^{\beta} \frac{1}{(n+1)^b} \geq \frac{h_2}{(\beta+1)^b}$$

for all $\beta \geq h_2$ and in a row with the classical estimates

$$\sup_{x \geq 0} x^{m_1}e^{-m_2 x} = (\frac{m_1}{m_2})^{m_1}e^{-m_1}$$

for any given integers $m_1 \geq 0$, $m_2 \geq 1$, we deduce that

$$\begin{aligned} \psi(|\tau|,\beta) \leq q^{(2k-1)h_3}|\tau|^{kh_1}\exp\Big(-\frac{\sigma h_2}{(\beta+1)^b}|\tau|^k\Big) &\leq q^{(2k-1)h_3}\sup_{x \geq 0} x^{h_1}\exp\Big(-\frac{\sigma h_2}{(\beta+1)^b}x\Big) \\ &= q^{(2k-1)h_3}\Big(\frac{h_1}{\sigma h_2}\Big)^{h_1}e^{-h_1}(\beta+1)^{bh_1} \end{aligned} \quad (40)$$

for all $\tau \in \mathcal{U}_p$, all $\beta \geq h_2$.

At last, collecting (38), (39) and (40) gives rise to the forcast bounds (37). □

Owing to the above lemma, we deduce the next bounds

$$(41)\quad ||\tau^{kh_1}\left(\partial_z^{-h_2} f\right)(q^{-h_3}\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq \sum_{\beta\geq h_2} ||\tau^{kh_1}f_{\beta-h_2}(q^{-h_3}\tau)||_{(\beta,\sigma,\mathcal{U}_p)}\frac{Z_0^{\beta}}{\beta!}$$
$$\leq q^{(2k-1)h_3}\left(\frac{h_1}{\sigma h_2}\right)^{h_1} e^{-h_1}\sum_{\beta\geq h_2}||f_{\beta-h_2}(\tau)||_{(\beta-h_2,\sigma,\mathcal{U}_p)}(\beta+1)^{bh_1}\frac{Z_0^{\beta}}{\beta!}$$
$$= q^{(2k-1)h_3}\left(\frac{h_1}{\sigma h_2}\right)^{h_1} e^{-h_1}\sum_{\beta\geq h_2}\left\{(\beta+1)^{bh_1}\frac{(\beta-h_2)!}{\beta!}Z_0^{h_2}\right\}||f_{\beta-h_2}(\tau)||_{(\beta-h_2,\sigma,\mathcal{U}_p)}\frac{Z_0^{\beta-h_2}}{(\beta-h_2)!}$$

Futhermore, we see that

$$(42)\quad (\beta+1)^{bh_1}\frac{(\beta-h_2)!}{\beta!}Z_0^{h_2} = \frac{(\beta+1)^{bh_1}}{\beta(\beta-1)\cdots(\beta-h_2+1)}Z_0^{h_2} \leq \frac{(\beta+1)^{bh_1}}{(\beta-h_2+1)^{h_2}}Z_0^{h_2} \leq \check{M}_1 Z_0^{h_2}$$

for some constant $\check{M}_1$ (relying on b, h_1, h_2), for all $\beta \geq h_2$ under the constraint (35). Finally, gathering (41) and (42) yields the expected bounds (36). □

Proposition 3 *There exists a constant $M_2 > 0$ (depending on k) such that*

$$(43)\quad ||\tau^k\int_0^{\tau^k} f((\tau^k-s)^{1/k},z)g(s^{1/k},z)\frac{1}{(\tau^k-s)s}ds||_{(\sigma,Z_0,\mathcal{U}_p)}$$
$$\leq M_2||f(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}||g(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}$$

for all $f,g \in G^k_{(\sigma,Z_0,\mathcal{U}_p)}$.

Proof Let

$$f(\tau,z) = \sum_{\beta\geq 0} f_\beta(\tau)\frac{z^\beta}{\beta!} \quad , \quad g(\tau,z) = \sum_{\beta\geq 0} g_\beta(\tau)\frac{z^\beta}{\beta!}$$

where $f_\beta(\tau), g_\beta(\tau) \in E^k_{(\beta,\sigma,\mathcal{U}_p)}$, for all $\beta \geq 0$. By construction, one can check that

$$(44)\quad \tau^k\int_0^{\tau^k} f((\tau^k-s)^{1/k},z)g(s^{1/k},z)\frac{1}{(\tau^k-s)s}ds$$
$$= \sum_{\beta\geq 0}\sum_{\beta_1+\beta_2=\beta}\left(\tau^k\int_0^{\tau^k}\beta!\frac{f_{\beta_1}((\tau^k-s)^{1/k})}{\beta_1!}\frac{g_{\beta_2}(s^{1/k})}{\beta_2!}\frac{1}{(\tau^k-s)s}ds\right)\frac{z^\beta}{\beta!}$$

In the next lemma, estimates for the coefficients of the latter series are disclosed.

Lemma 2 *There exists a constant $B_k > 0$ depending on k such that*

$$(45)\quad ||\tau^k\int_0^{\tau^k} f_{\beta_1}((\tau^k-s)^{1/k})g_{\beta_2}(s^{1/k})\frac{1}{(\tau^k-s)s}ds||_{(\beta,\sigma,\mathcal{U}_p)}$$
$$\leq B_k||f_{\beta_1}(\tau)||_{(\beta_1,\sigma,\mathcal{U}_p)}||g_{\beta_2}(\tau)||_{(\beta_2,\sigma,\mathcal{U}_p)}$$

for all $\beta \geq 0$, all $\beta_1, \beta_2 \geq 0$ such that $\beta_1 + \beta_2 = \beta$.

Proof Departing from the very definition of the norms, we can factorize the upper bounds as follows

$$(46)\quad ||\tau^k \int_0^{\tau^k} f_{\beta_1}((\tau^k - s)^{1/k}) g_{\beta_2}(s^{1/k}) \frac{1}{(\tau^k - s)s} ds||_{(\beta,\sigma,\mathcal{U}_p)} \leq \sup_{\tau \in \mathcal{U}_p} \frac{1+|\tau|^{2k}}{|\tau|} \exp(-\sigma r_b(\beta)|\tau|^k)$$
$$\times \left| \tau^k \int_0^{\tau^k} \left\{ \frac{1+|(\tau^k - s)^{1/k}|^{2k}}{|\tau^k - s|^{1/k}} \exp(-\sigma r_b(\beta_1)(|\tau^k - s|^{1/k})^k) f_{\beta_1}((\tau^k - s)^{1/k}) \right\} \right.$$
$$\times \left\{ \frac{1+|s^{1/k}|^{2k}}{|s^{1/k}|} \exp(-\sigma r_b(\beta_2)|s^{1/k}|^k) g_{\beta_2}(s^{1/k}) \right\} \times \left[\frac{|\tau^k - s|^{1/k}}{1+|(\tau^k - s)^{1/k}|^{2k}} \frac{|s^{1/k}|}{1+|s^{1/k}|^{2k}} \right.$$
$$\left. \left. \times \exp(\sigma r_b(\beta_1)(|\tau^k - s|^{1/k})^k) \exp(\sigma r_b(\beta_2)|s^{1/k}|^k) \frac{1}{(\tau^k - s)s} \right] ds \right|$$
$$\leq \mathcal{B}(\beta_1,\beta_2) ||f_{\beta_1}(\tau)||_{(\beta_1,\sigma,\mathcal{U}_p)} ||g_{\beta_2}(\tau)||_{(\beta_2,\sigma,\mathcal{U}_p)}$$

where

$$(47)\quad \mathcal{B}(\beta_1,\beta_2) := \sup_{\tau \in \mathcal{U}_p} \frac{1+|\tau|^{2k}}{|\tau|} \exp(-\sigma r_b(\beta)|\tau|^k)$$
$$\times |\tau|^k \int_0^{|\tau|^k} \frac{(|\tau|^k - u)^{1/k}}{1+(|\tau|^k - u)^2} \frac{u^{1/k}}{1+u^2} \exp(\sigma r_b(\beta_1)(|\tau|^k - u))$$
$$\times \exp(\sigma r_b(\beta_2)u) \frac{1}{(|\tau|^k - u)u} du$$

for all $\beta_1, \beta_2 \geq 0$ with $\beta_1 + \beta_2 = \beta$. Since $\beta \mapsto r_b(\beta)$ is an increasing sequence, we observe that $r_b(\beta_1) \leq r_b(\beta)$ together with $r_b(\beta_2) \leq r_b(\beta)$ and we get

$$\sigma r_b(\beta_1)(|\tau|^k - u) + \sigma r_b(\beta_2)u \leq \sigma r_b(\beta)|\tau|^k$$

for all $0 \leq u \leq |\tau|^k$. We deduce that

$$(48)\quad \mathcal{B}(\beta_1,\beta_2) \leq B_k$$

with

$$B_k = \sup_{\tau \in \mathcal{U}_p} \frac{1+|\tau|^{2k}}{|\tau|} |\tau|^k \int_0^{|\tau|^k} \frac{(|\tau|^k - u)^{\frac{1}{k}-1}}{1+(|\tau|^k - u)^2} \frac{u^{\frac{1}{k}-1}}{1+u^2} du$$

From now on, we perform computations based on the ones already done in our previous work [10]. We first assume that $k \geq 2$.

We make the change of variable $u = |\tau|^k x$, $0 \leq x \leq 1$ in the above integral part

$$(49)\quad \int_0^{|\tau|^k} \frac{(|\tau|^k - u)^{\frac{1}{k}-1}}{1+(|\tau|^k - u)^2} \frac{u^{\frac{1}{k}-1}}{1+u^2} du$$
$$= \int_0^1 \frac{1}{1+|\tau|^{2k}(1-x)^2} \frac{1}{1+|\tau|^{2k}x^2} \frac{1}{(1-x)^{1-\frac{1}{k}}} \frac{1}{x^{1-\frac{1}{k}}} dx \times |\tau|^{-k+2}$$

Besides, using a partial fraction decomposition, we can split the next integral along $[0,1]$ as

$$(50)\quad \int_0^1 \frac{1}{1+|\tau|^{2k}(1-x)^2}\frac{1}{1+|\tau|^{2k}x^2}\frac{1}{(1-x)^{1-\frac{1}{k}}}\frac{1}{x^{1-\frac{1}{k}}}dx$$
$$= \frac{1}{4+|\tau|^{2k}}\Big(\int_0^1 \frac{3-2x}{1+|\tau|^{2k}(1-x)^2}\frac{1}{(1-x)^{1-\frac{1}{k}}}\frac{1}{x^{1-\frac{1}{k}}}dx$$
$$+ \int_0^1 \frac{2x+1}{1+|\tau|^{2k}x^2}\frac{1}{(1-x)^{1-\frac{1}{k}}}\frac{1}{x^{1-\frac{1}{k}}}dx\Big)$$

for all $\tau \in \mathcal{U}_p$. Furthermore, the change of variable $x_1 = |\tau|^k x$, for $0 \leq x \leq 1$ enables us to reach the bounds

$$(51)\quad \int_0^1 \frac{2x+1}{1+|\tau|^{2k}x^2}\frac{1}{(1-x)^{1-\frac{1}{k}}}\frac{1}{x^{1-\frac{1}{k}}}dx \leq 3\int_0^1 \frac{1}{1+|\tau|^{2k}x^2}\frac{1}{(1-x)^{1-\frac{1}{k}}}\frac{1}{x^{1-\frac{1}{k}}}dx$$
$$= \frac{3}{|\tau|}\int_0^{|\tau|^k} \frac{1}{1+x_1^2}\frac{1}{(1-\frac{x_1}{|\tau|^k})^{1-\frac{1}{k}}}\frac{1}{x_1^{1-\frac{1}{k}}}dx_1 \leq \frac{B_{k,1}}{|\tau|}$$

for some constant $B_{k,1} > 0$ provided that $\tau \in \mathcal{U}_p$ with $|\tau| \geq 1$. In a similar way, one can find a constant $B_{k,2} > 0$ for which

$$(52)\quad \int_0^1 \frac{3-2x}{1+|\tau|^{2k}(1-x)^2}\frac{1}{(1-x)^{1-\frac{1}{k}}}\frac{1}{x^{1-\frac{1}{k}}}dx \leq \frac{B_{k,2}}{|\tau|}$$

as long as $\tau \in \mathcal{U}_p$ with $|\tau| \geq 1$.

Gathering (49), (50), (51) and (52) yields that

$$(53)\quad B_k \quad \text{is a finite quantity}$$

provided that $k \geq 2$.

It remains to check the case $k = 1$. In that situation, the quantity B_1 can be computed explicitely

$$(54)\quad B_1 = \sup_{\tau\in\mathcal{U}_p}(1+|\tau|^2)\int_0^{|\tau|}\frac{1}{(1+(|\tau|-u)^2)(1+u^2)}du$$
$$= \sup_{x\geq 0}(1+x^2)2\frac{\log(1+x^2)+x\arctan(x)}{x(x^2+4)}$$

and turns out to be a finite positive real number.

Finally, collecting the intermediate upper estimates (46), (48), (53) and (54) gives rise to Lemma 2. □

From the expansion (44) along with the lemma 2, we obtain the next bounds

$$(55)\quad ||\tau^k\int_0^{\tau^k} f((\tau^k-s)^{1/k},z)g(s^{1/k},z)\frac{1}{(\tau^k-s)s}ds||_{(\sigma,Z_0,\mathcal{U}_p)}$$
$$\leq \sum_{\beta\geq 0}\sum_{\beta_1+\beta_2=\beta}\beta!\Big(B_k\frac{||f_{\beta_1}(\tau)||_{(\beta_1,\sigma,\mathcal{U}_p)}}{\beta_1!}\frac{||g_{\beta_2}(\tau)||_{(\beta_2,\sigma,\mathcal{U}_p)}}{\beta_2!}\Big)\frac{Z_0^\beta}{\beta!}$$
$$= B_k||f(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}||g(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}$$

from which Proposition 3 follows. □

Proposition 4 *Let $h_0 \geq 1$ and $h_1, h_2, h_3 \geq 0$ natural numbers such that*

$$h_2 \geq bh_1 \tag{56}$$

Then, there exists a constant $M_3 > 0$ (depending upon $k, q, h_0, h_1, h_2, h_3, \sigma, b$) such that

$$||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1-1} \left(\partial_z^{-h_2} f\right)(q^{-h_3} s^{1/k}, z) ds||_{(\sigma,Z_0,\mathcal{U}_p)} \leq M_3 Z_0^{h_2} ||f(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \tag{57}$$

for all $f \in G^k_{(\sigma,Z_0,\mathcal{U}_p)}$.

Proof We check that Proposition 4 is a direct consequence of Proposition 2 and Proposition 3. Indeed, we set

$$\tilde{g}(\tau,z) = \tau^{kh_1} \left(\partial_z^{-h_2} f\right)(q^{-h_3}\tau, z)$$

for a given $f \in G^k_{(\sigma,Z_0,\mathcal{U}_p)}$. According to Proposition 2, we observe that $\tilde{g}(\tau,z)$ belongs to $G^k_{(\sigma,Z_0,\mathcal{U}_p)}$ and that

$$||\tilde{g}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq M_1 Z_0^{h_2} ||f(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \tag{58}$$

for some constant $M_1 > 0$ provided that (56) holds. Furthermore, we set $\tilde{f}(\tau,z) = \tau^{h_0}$. By construction, we notice that $\tilde{f}(\tau,z) \in G^k_{(\sigma,Z_0,\mathcal{U}_p)}$. Then, we get that

$$\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1-1} \left(\partial_z^{-h_2} f\right)(q^{-h_3} s^{1/k}, z) ds = \tau^k \int_0^{\tau^k} \tilde{f}((\tau^k - s)^{1/k}, z) \tilde{g}(s^{1/k}, z) \frac{1}{(\tau^k - s)s} ds$$

Owing to Proposition 3, we get the next bounds

$$||\tau^k \int_0^{\tau^k} \tilde{f}((\tau^k - s)^{1/k}, z) \tilde{g}(s^{1/k}, z) \frac{1}{(\tau^k - s)s} ds||_{(\sigma,Z_0,\mathcal{U}_p)} \leq M_2 ||\tilde{f}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} ||\tilde{g}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq M_2 M_1 Z_0^{h_2} ||\tau^{h_0}||_{(\sigma,Z_0,\mathcal{U}_p)} ||f(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \tag{59}$$

for some constant M_2 (relying on k). At last, we set $M_3 = M_2 M_1 ||\tau^{h_0}||_{(\sigma,Z_0,\mathcal{U}_p)}$ which yields the result. □

Proposition 5 *Let $a(z) = \sum_{\beta \geq 0} a_\beta z^\beta/\beta!$ be a holomorphic function on a disc D_R with radius $R > Z_0$. Then, the next bounds*

$$||a(z) f(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq |a|(Z_0) ||f(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \tag{60}$$

hold for all $f \in G^k_{(\sigma,Z_0,\mathcal{U}_p)}$, where

$$|a|(Z_0) = \sum_{\beta \geq 0} |a_\beta| \frac{Z_0^\beta}{\beta!}$$

Proof Let

$$f(\tau,z) = \sum_{\beta \geq 0} f_{\beta}(\tau) z^{\beta}/\beta!$$

with $f_{\beta}(\tau) \in E^{k}_{(\beta,\sigma,\mathcal{U}_p)}$ for all $\beta \geq 0$. We first expand the product

$$a(z)f(\tau,z) = \sum_{\beta \geq 0} \Big(\sum_{\beta_1+\beta_2=\beta} \beta! \frac{a_{\beta_1}}{\beta_1!} \frac{f_{\beta_2}(\tau)}{\beta_2!} \Big) \frac{z^{\beta}}{\beta!}$$

which allows to control its norm

$$||a(z)f(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq \sum_{\beta \geq 0} \Big(\sum_{\beta_1+\beta_2=\beta} \beta! \frac{|a_{\beta_1}|}{\beta_1!} \frac{||f_{\beta_2}(\tau)||_{(\beta,\sigma,\mathcal{U}_p)}}{\beta_2!} \Big) \frac{Z_0^{\beta}}{\beta!}$$

Besides, since $r_b(\beta_2) \leq r_b(\beta)$ for $\beta_2 \leq \beta$, we notice that

$$||f_{\beta_2}(\tau)||_{(\beta,\sigma,\mathcal{U}_p)} \leq ||f_{\beta_2}(\tau)||_{(\beta_2,\sigma,\mathcal{U}_p)}$$

for $0 \leq \beta_2 \leq \beta$. Hence,

$$\begin{aligned} ||a(z)f(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} &\leq \sum_{\beta \geq 0} \Big(\sum_{\beta_1+\beta_2=\beta} \beta! \frac{|a_{\beta_1}|}{\beta_1!} \frac{||f_{\beta_2}(\tau)||_{(\beta_2,\sigma,\mathcal{U}_p)}}{\beta_2!} \Big) \frac{Z_0^{\beta}}{\beta!} \\ &= \Big(\sum_{\beta \geq 0} \frac{|a_{\beta}|}{\beta!} Z_0^{\beta} \Big) \Big(\sum_{\beta \geq 0} \frac{||f_{\beta}(\tau)||_{(\beta,\sigma,\mathcal{U}_p)}}{\beta!} Z_0^{\beta} \Big) \end{aligned}$$

which confirms the statement of Proposition 5. □

In the next proposition, we show that the Cauchy data (31) belong to the Banach spaces considered in Definition 3.

Proposition 6 *We set*

$$\check{\Psi}(\tau,z) = \sum_{j=0}^{\kappa-1} \check{P}_j(\tau) \frac{z^j}{j!} \quad (61)$$

where the polynomials $\check{P}_j(\tau)$ are given by the Cauchy data (31). Then,
a) For all integers $h_1, h_3 \geq 0$, $0 \leq h_2 \leq \kappa-1$, the maps

$$(\tau,z) \mapsto \tau^{kh_1} \left(\partial_z^{h_2} \check{\Psi} \right) (q^{-h_3}\tau, z)$$

belong to $G^{k}_{(\sigma,Z_0,\mathcal{U}_p)}$, for all $\sigma, Z_0 > 0$.
b) For all integers $h_0 \geq 1$, $h_1, h_3 \geq 0$, $0 \leq h_2 \leq \kappa-1$, the maps

$$(\tau,z) \mapsto \tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1-1} \left(\partial_z^{h_2} \check{\Psi} \right) (q^{-h_3} s^{1/k}, z) ds$$

appertain to $G^{k}_{(\sigma,Z_0,\mathcal{U}_p)}$, for all $\sigma, Z_0 > 0$.

Proof We first make the Taylor expansion

$$\left(\partial_z^{h_2}\check{\Psi}\right)(q^{-h_3}\tau,z) = \sum_{j=0}^{\kappa-1-h_2} \check{P}_{j+h_2}(q^{-h_3}\tau)\frac{z^j}{j!}$$

explicit provided that $0 \leq h_2 \leq \kappa - 1$.

We focus on a). By the very definition of the norm, we get

$$||\tau^{kh_1}\left(\partial_z^{h_2}\check{\Psi}\right)(q^{-h_3}\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} = \sum_{j=0}^{\kappa-1-h_2} ||\tau^{kh_1}\check{P}_{j+h_2}(q^{-h_3}\tau)||_{(j,\sigma,\mathcal{U}_p)}\frac{Z_0^j}{j!}$$

and we can find a constant $\check{\Delta}_j$ (relying on j,h_1,h_2,h_3,k) for which

$$(62)\quad ||\tau^{kh_1}\check{P}_{j+h_2}(q^{-h_3}\tau)||_{(j,\sigma,\mathcal{U}_p)} = \sup_{\tau\in\mathcal{U}_p}\frac{1+|\tau|^{2k}}{|\tau|}\exp(-\sigma r_b(j)|\tau|^k)|\check{P}_{j+h_2}(q^{-h_3}\tau)|$$
$$\leq \sup_{x\geq 0}\frac{1+x^{2k}}{x}x^{kh_1}|\check{P}_{j+h_2}|(q^{-h_3}x)\exp(-\sigma x^k) := \check{\Delta}_j$$

since $r_b(j) \geq 1$, for all $j \geq 1$, where

$$(63)\qquad |\check{P}_{j+h_2}|(x) = \sum_{h\in\check{J}_{j+h_2}} |\check{p}_{j+h_2,h}|x^h$$

and $\check{J}_{j+h_2} \subset \mathbb{N}\setminus\{0\}$. As a consequence,

$$(64)\qquad ||\tau^{kh_1}\left(\partial_z^{h_2}\check{\Psi}\right)(q^{-h_3}\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq \sum_{j=0}^{\kappa-1-h_2}\check{\Delta}_j\frac{Z_0^j}{j!}$$

which is a finite quantity.

We turn our attention to b). The definition of the norm yields

$$||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1-1}\left(\partial_z^{h_2}\check{\Psi}\right)(q^{-h_3}s^{1/k},z)ds||_{(\sigma,Z_0,\mathcal{U}_p)}$$
$$= \sum_{j=0}^{\kappa-1-h_2}||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1-1}\check{P}_{j+h_2}(q^{-h_3}s^{1/k})ds||_{(j,\sigma,\mathcal{U}_p)}\frac{Z_0^j}{j!}$$

We needs bounds for the coefficients of this latter polynomial in Z_0. Indeed,

$$\mathcal{C}_j := ||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1-1}\check{P}_{j+h_2}(q^{-h_3}s^{1/k})ds||_{(j,\sigma,\mathcal{U}_p)}$$
$$\leq \sup_{\tau\in\mathcal{U}_p}\frac{1+|\tau|^{2k}}{|\tau|}\exp(-\sigma r_b(j)|\tau|^k)|\tau|^k|\int_0^{|\tau|^k}(|\tau|^k-h)^{\frac{h_0}{k}-1}h^{h_1-1}|\check{P}_{j+h_2}|(q^{-h_3}h^{1/k})dh$$

where $|\check{P}_{j+h_2}|(x)$ is defined in (63). We make the change of variable $h = |\tau|^k x$, for $0 \leq x \leq 1$ in the above integral and observe that $r_b(j) \geq 1$, $j \geq 1$. This gives rise to a constant $\check{\nabla}_j$ (depending on j,h_0,h_1,h_2,h_3,k) such that

$$(65)\quad \mathcal{C}_j \leq \sup_{y\geq 0}\frac{1+y^{2k}}{y}\exp(-\sigma y^k)y^{h_0+kh_1}|\check{P}_{j+h_2}|(q^{-h_3}y)\left(\int_0^1(1-x)^{\frac{h_0}{k}-1}x^{h_1-1}dx\right) := \check{\nabla}_j$$

Finally,

$$||\tau^{k}\int_{0}^{\tau^{k}}(\tau^{k}-s)^{\frac{h_0}{k}-1}s^{h_1-1}\left(\partial_z^{h_2}\check{\Psi}\right)(q^{-h_3}s^{1/k},z)ds||_{(\sigma,Z_0,\mathcal{U}_p)} \leq \sum_{j=0}^{\kappa-1-h_2}\check{\nabla}_j\frac{Z_0^j}{j!}$$

which represents a positive real number. □

In the following proposition, we claim that the forcing term w_p of the Cauchy problem (21), (22) belong to the Banach space described in Definition 3.

Proposition 7 *For all $0 \leq p \leq \varsigma - 1$, the map $(\tau,z) \mapsto w_p(\tau,z)$ belongs to the space $G^k_{(\sigma,Z_0,\mathcal{U}_p)}$, for all $\sigma > 0$, provided that $Z_0 < \frac{1}{2C_4}$.*

Proof According to the Taylor expansion (16), the very definition of the norm yields

$$||w_p(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} = \sum_{\beta\geq 0}||w_{p,\beta}(\tau)||_{(\beta,\sigma,\mathcal{U}_p)}\frac{Z_0^{\beta}}{\beta!}$$

Then, we need to control the coefficients of this latter series. Namely, one can find a constant $\tilde{C}_3 > 0$ (relying on C_3, k_1, α, u_0 given in (17) and on k, σ) with

$$\begin{aligned} (66)\quad ||w_{p,\beta}(\tau)||_{(\beta,\sigma,\mathcal{U}_p)} &:= \sup_{\tau\in\mathcal{U}_p}\frac{1+|\tau|^{2k}}{|\tau|}\exp(-\sigma r_b(\beta)|\tau|^k)|w_{p,\beta}(\tau)| \\ &\leq C_3(C_4)^{\beta}\beta!\sup_{x\geq 0}(1+x^{2k})\exp(-\sigma x^k)\exp\left(k_1\log^2(x+u_0)+\alpha\log(x+u_0)\right) = \tilde{C}_3(C_4)^{\beta}\beta! \end{aligned}$$

since $r_b(\beta) \geq 1$ for all $\beta \geq 0$. Consequently,

$$||w_p(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq \tilde{C}_3\sum_{\beta\geq 0}(C_4Z_0)^{\beta} \leq 2\tilde{C}_3$$

whenever $Z_0 < \frac{1}{2C_4}$ and Proposition 7 ensues. □

3.2 Solving the main nonlinear convolution Cauchy problem

In the following, we search for a solution to the nonlinear convolution Cauchy problem (30), (31) expressed by means of the next shape

$$(67)\qquad \Theta_p(u,z) = \partial_z^{-\kappa}\Xi_p(u,z) + \check{\Psi}(u,z)$$

where the map $\check{\Psi}(u,z)$ is defined in (61), for some expression $\Xi_p(u,z)$.

We check that $\Theta_p(u,z)$ solves the problem (30), (31) if the quantity $\Xi_p(u,z)$ fulfills the next *fixed point equation*

$$
\begin{aligned}
(68)\quad \Xi_p(u,z) = & \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)(k(q^{-h_3}u)^k)^{h_1} \\
& \times \left[\left(\partial_z^{-(\kappa-h_2)}\Xi_p\right)(q^{-h_3}u,z) + \left(\partial_z^{h_2}\check{\Psi}\right)(q^{-h_3}u,z)\right] \\
& + \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{u^k}{\Gamma(h_0/k)}\int_0^{u^k}(u^k-s)^{\frac{h_0}{k}-1}(k(q^{-h_3}s^{1/k})^k)^{h_1} \\
& \times \left[\left(\partial_z^{-(\kappa-h_2)}\Xi_p\right)(q^{-h_3}s^{1/k},z) + \left(\partial_z^{h_2}\check{\Psi}\right)(q^{-h_3}s^{1/k},z)\right]\frac{ds}{s} \\
& + a_0(z)u^k\int_0^{u^k}\left[\partial_z^{-\kappa}\Xi_p((u^k-s)^{1/k},z) + \check{\Psi}((u^k-s)^{1/k},z)\right] \\
& \times \left[\partial_z^{-\kappa}\Xi_p(s^{1/k},z) + \check{\Psi}(s^{1/k},z)\right]\frac{1}{(u^k-s)s}ds + \sum_{h=1}^{A} a_h(z)\frac{u^k}{\Gamma(h/k)}\int_0^{u^k}(u^k-s)^{\frac{h}{k}-1} \\
& \times \left\{ s\int_0^s \left[\partial_z^{-\kappa}\Xi_p((s-s_1)^{1/k},z) + \check{\Psi}((s-s_1)^{1/k},z)\right]\right. \\
& \left.\times \left[\partial_z^{-\kappa}\Xi_p(s_1^{1/k},z) + \check{\Psi}(s_1^{1/k},z)\right]\frac{1}{(s-s_1)s_1}ds_1\right\}\frac{ds}{s} + w_p(u,z)
\end{aligned}
$$

Our next task will be to seek for a solution of this last equation (68) in the Banach space we have discussed in the previous subsection 3.1. We introduce the nonlinear map

$$
\begin{aligned}
(69)\quad \mathfrak{B}(\Xi(\tau,z)) := & \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)(k(q^{-h_3}\tau)^k)^{h_1} \\
& \times \left[\left(\partial_z^{-(\kappa-h_2)}\Xi\right)(q^{-h_3}\tau,z) + \left(\partial_z^{h_2}\check{\Psi}\right)(q^{-h_3}\tau,z)\right] \\
& + \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{\tau^k}{\Gamma(h_0/k)}\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}(k(q^{-h_3}s^{1/k})^k)^{h_1} \\
& \times \left[\left(\partial_z^{-(\kappa-h_2)}\Xi\right)(q^{-h_3}s^{1/k},z) + \left(\partial_z^{h_2}\check{\Psi}\right)(q^{-h_3}s^{1/k},z)\right]\frac{ds}{s} \\
& + a_0(z)\tau^k\int_0^{\tau^k}\left[\partial_z^{-\kappa}\Xi((\tau^k-s)^{1/k},z) + \check{\Psi}((\tau^k-s)^{1/k},z)\right] \\
& \times \left[\partial_z^{-\kappa}\Xi(s^{1/k},z) + \check{\Psi}(s^{1/k},z)\right]\frac{1}{(\tau^k-s)s}ds + \sum_{h=1}^{A} a_h(z)\frac{\tau^k}{\Gamma(h/k)}\int_0^{\tau^k}(\tau^k-s)^{\frac{h}{k}-1} \\
& \times \left\{ s\int_0^s \left[\partial_z^{-\kappa}\Xi((s-s_1)^{1/k},z) + \check{\Psi}((s-s_1)^{1/k},z)\right]\right. \\
& \left.\times \left[\partial_z^{-\kappa}\Xi(s_1^{1/k},z) + \check{\Psi}(s_1^{1/k},z)\right]\frac{1}{(s-s_1)s_1}ds_1\right\}\frac{ds}{s} + w_p(\tau,z)
\end{aligned}
$$

In the next proposition, we give sufficient conditions under which $\mathfrak{B}$ represents a shrinking map on some small ball centered at 0 in the space $G^k_{(\sigma,Z_0,\mathcal{U}_p)}$.

Proposition 8 *Under the assumption (19), there exists some small real number $\chi > 0$ such that if $0 < Z_0 < \chi$, one can select a radius $\upsilon > 0$ such that $\mathfrak{B}$ satisfies the next two properties:*

Let B_{υ} be the ball centered at 0 in $G^{k}_{(\sigma,Z_0,\mathcal{U}_p)}$ with radius υ.

1. *$\mathfrak{B}$ maps B_{υ} into B_{υ}, meaning that*

$$\mathfrak{B}(B_{\upsilon}) \subset B_{\upsilon} \tag{70}$$

2. *For all $\Xi_1, \Xi_2 \in B_{\upsilon}$, we have*

$$||\mathfrak{B}(\Xi_1) - \mathfrak{B}(\Xi_2)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq \frac{1}{2}||\Xi_1 - \Xi_2||_{(\sigma,Z_0,\mathcal{U}_p)} \tag{71}$$

Proof We discuss the first point 1. Let $\Xi(\tau,z)$ belong to $G^{k}_{(\sigma,Z_0,\mathcal{U}_p)}$ with $||\Xi(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq \upsilon$.

Under the constraint (19) and owing to Propositions 2, 5 and 6, we get a constant $M_1 > 0$ such that

$$\begin{aligned} &||d_{\underline{h}}(z)\tau^{kh_1}\Big[\Big(\partial_z^{-(\kappa-h_2)}\Xi\Big)(q^{-h_3}\tau,z) + \Big(\partial_z^{h_2}\check{\Psi}\Big)(q^{-h_3}\tau,z)\Big]||_{(\sigma,Z_0,\mathcal{U}_p)} \\ &\leq |d_{\underline{h}}|(Z_0)\Big(M_1 Z_0^{\kappa-h_2}||\Xi(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} + ||\tau^{kh_1}\Big(\partial_z^{h_2}\check{\Psi}\Big)(q^{-h_3}\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}\Big) \end{aligned} \tag{72}$$

The condition (19) and Propositions 4, 5 and 6 allow us to reach a constant $M_3 > 0$ for which

$$\begin{aligned} &||d_{\underline{h}}(z)\tau^{k}\int_0^{\tau^k}(\tau^k - s)^{\frac{h_0}{k}-1}s^{h_1-1} \\ &\times \Big[\Big(\partial_z^{-(\kappa-h_2)}\Xi\Big)(q^{-h_3}s^{1/k},z) + \Big(\partial_z^{h_2}\check{\Psi}\Big)(q^{-h_3}s^{1/k},z)\Big]ds||_{(\sigma,Z_0,\mathcal{U}_p)} \\ &\leq |d_{\underline{h}}|(Z_0)\Big(M_3 Z_0^{\kappa-h_2}||\Xi(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \\ &+ ||\tau^{k}\int_0^{\tau^k}(\tau^k - s)^{\frac{h_0}{k}-1}s^{h_1-1}\Big(\partial_z^{h_2}\check{\Psi}\Big)(q^{-h_3}s^{1/k},z)ds||_{(\sigma,Z_0,\mathcal{U}_p)}\Big) \end{aligned} \tag{73}$$

According to Propositions 2, 3, 5 and 6, we obtain constants $M_1, M_2 > 0$ with

$$\begin{aligned} &||a_0(z)\tau^{k}\int_0^{\tau^k}\Big[\partial_z^{-\kappa}\Xi((\tau^k - s)^{1/k},z) + \check{\Psi}((\tau^k - s)^{1/k},z)\Big] \\ &\times \Big[\partial_z^{-\kappa}\Xi(s^{1/k},z) + \check{\Psi}(s^{1/k},z)\Big]\frac{1}{(\tau^k - s)s}ds||_{(\sigma,Z_0,\mathcal{U}_p)} \\ &\leq |a_0|(Z_0)M_2||(\partial_z^{-\kappa}\Xi)(\tau,z) + \check{\Psi}(\tau,z)||^2_{(\sigma,Z_0,\mathcal{U}_p)} \\ &\leq |a_0|(Z_0)M_2\Big(||\partial_z^{-\kappa}\Xi(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}\Big)^2 \\ &\leq |a_0|(Z_0)M_2\Big(M_1 Z_0^{\kappa}||\Xi(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}\Big)^2 \end{aligned} \tag{74}$$

Propositions 2, 3, 4, 5 and 6 grant the existence of constants $M_1, M_2, M_3 > 0$ such that

$$
\begin{aligned}
(75)\quad ||a_h(z)\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h}{k}-1} \\
\times \left\{ s \int_0^s \left[\partial_z^{-\kappa}\Xi((s-s_1)^{1/k}, z) + \check{\Psi}((s-s_1)^{1/k}, z) \right] \right. \\
\left. \times \left[\partial_z^{-\kappa}\Xi(s_1^{1/k}, z) + \check{\Psi}(s_1^{1/k}, z) \right] \frac{1}{(s-s_1)s_1} ds_1 \right\} \frac{ds}{s} ||_{(\sigma,Z_0,\mathcal{U}_p)} \\
\leq |a_h|(Z_0) M_3 ||\tau^k \int_0^{\tau^k} \left[\partial_z^{-\kappa}\Xi((\tau^k-s_1)^{1/k}, z) + \check{\Psi}((\tau^k-s_1)^{1/k}, z) \right] \\
\times \left[\partial_z^{-\kappa}\Xi(s_1^{1/k}, z) + \check{\Psi}(s_1^{1/k}, z) \right] \frac{1}{(\tau^k - s_1)s_1} ds_1 ||_{(\sigma,Z_0,\mathcal{U}_p)} \\
\leq |a_h|(Z_0) M_3 M_2 ||\partial_z^{-\kappa}\Xi(\tau, z) + \check{\Psi}(\tau, z)||^2_{(\sigma,Z_0,\mathcal{U}_p)} \\
\leq |a_h|(Z_0) M_3 M_2 \Big(||\partial_z^{-\kappa}\Xi(\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} + ||\check{\Psi}(\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} \Big)^2 \\
\leq |a_h|(Z_0) M_3 M_2 \Big(M_1 Z_0^{\kappa} ||\Xi(\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} + ||\check{\Psi}(\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} \Big)^2
\end{aligned}
$$

Now, we choose the radius $\upsilon > 0$ and $\chi > 0$ sufficiently small with $0 < Z_0 < \chi$ in a way that the next inequality holds,

$$
\begin{aligned}
(76)\quad \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} (kq^{-h_3k})^{h_1} |d_{\underline{h}}|(Z_0) \Big(M_1 Z_0^{\kappa-h_2}\upsilon + ||\tau^{kh_1} \Big(\partial_z^{h_2}\check{\Psi}\Big)(q^{-h_3}\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} \Big) \\
+ \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} \frac{(kq^{-h_3k})^{h_1}}{\Gamma(h_0/k)} |d_{\underline{h}}|(Z_0) \\
\times \Big(M_3 Z_0^{\kappa-h_2}\upsilon + ||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1-1} \Big(\partial_z^{h_2}\check{\Psi}\Big)(q^{-h_3}s^{1/k}, z) ds ||_{(\sigma,Z_0,\mathcal{U}_p)} \Big) \\
+ |a_0|(Z_0) M_2 \Big(M_1 Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} \Big)^2 \\
+ \sum_{h=1}^{A} |a_h|(Z_0) \frac{1}{\Gamma(h/k)} M_3 M_2 \Big(M_1 Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} \Big)^2 + ||w_p(\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq \upsilon
\end{aligned}
$$

Collecting all the above estimates (72), (73), (74) and (75), under the latter constraint (76), sires the awaited property (70).

We now turn to the second feature 2. Let $\Xi_j(\tau, z) \in G^k_{(\sigma,Z_0,\mathcal{U}_p)}$ with $||\Xi_j(\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq \upsilon$, for $j = 1, 2$.

Under the condition (19), the propositions 2 and 5 yield a constant $M_1 > 0$ for which

$$
\begin{aligned}
(77)\quad ||d_{\underline{h}}(z)\tau^{kh_1} \Big(\partial_z^{-(\kappa-h_2)}(\Xi_1 - \Xi_2)\Big)(q^{-h_3}\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)} \\
\leq |d_{\underline{h}}|(Z_0) M_1 Z_0^{\kappa-h_2} ||(\Xi_1 - \Xi_2)(\tau, z)||_{(\sigma,Z_0,\mathcal{U}_p)}
\end{aligned}
$$

and the propositions 4 and 5 breed a constant $M_3 > 0$ such that

$$(78)\quad ||d_{\underline{h}}(z)\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1-1} \left(\partial_z^{-(\kappa-h_2)}(\Xi_1 - \Xi_2)\right)(q^{-h_3}s^{1/k}, z) ds||_{(\sigma,Z_0,\mathcal{U}_p)} \leq |d_{\underline{h}}|(Z_0)\left(M_3 Z_0^{\kappa-h_2}||\Xi_1(\tau,z) - \Xi_2(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}\right)$$

In order to deal with the nonlinear terms, we use the next identity $ab - cd = (a-c)b + c(b-d)$ which helps to factorize the next difference

$$(79)\quad \left[\partial_z^{-\kappa}\Xi_1((\tau^k - s)^{1/k}, z) + \check{\Psi}((\tau^k - s)^{1/k}, z)\right] \times \left[\partial_z^{-\kappa}\Xi_1(s^{1/k}, z) + \check{\Psi}(s^{1/k}, z)\right] - \left[\partial_z^{-\kappa}\Xi_2((\tau^k - s)^{1/k}, z) + \check{\Psi}((\tau^k - s)^{1/k}, z)\right] \times \left[\partial_z^{-\kappa}\Xi_2(s^{1/k}, z) + \check{\Psi}(s^{1/k}, z)\right] = \partial_z^{-\kappa}(\Xi_1 - \Xi_2)((\tau^k - s)^{1/k}, z)\left[\partial_z^{-\kappa}\Xi_1(s^{1/k}, z) + \check{\Psi}(s^{1/k}, z)\right] + \left[\partial_z^{-\kappa}\Xi_2((\tau^k - s)^{1/k}, z) + \check{\Psi}((\tau^k - s)^{1/k}, z)\right] \times \partial_z^{-\kappa}(\Xi_1 - \Xi_2)(s^{1/k}, z)$$

which leads, by means of Proposition 2, 3, 5 and 6, to constants $M_1, M_2 > 0$ for which

$$(80)\quad ||a_0(z)\tau^k \int_0^{\tau^k} \left[\partial_z^{-\kappa}\Xi_1((\tau^k - s)^{1/k}, z) + \check{\Psi}((\tau^k - s)^{1/k}, z)\right] \times \left[\partial_z^{-\kappa}\Xi_1(s^{1/k}, z) + \check{\Psi}(s^{1/k}, z)\right] \frac{1}{(\tau^k - s)s} ds - a_0(z)\tau^k \int_0^{\tau^k} \left[\partial_z^{-\kappa}\Xi_2((\tau^k - s)^{1/k}, z) + \check{\Psi}((\tau^k - s)^{1/k}, z)\right] \times \left[\partial_z^{-\kappa}\Xi_2(s^{1/k}, z) + \check{\Psi}(s^{1/k}, z)\right] \frac{1}{(\tau^k - s)s} ds||_{(\sigma,Z_0,\mathcal{U}_p)}$$
$$\leq |a_0|(Z_0)M_2\Big[||\partial_z^{-\kappa}(\Xi_1 - \Xi_2)(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \times ||\partial_z^{-\kappa}\Xi_1(\tau,z) + \check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} + ||\partial_z^{-\kappa}\Xi_2(\tau,z) + \check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \times ||\partial_z^{-\kappa}(\Xi_1 - \Xi_2)(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}\Big]$$
$$\leq |a_0|(Z_0)M_2\Big[M_1 Z_0^{\kappa}||\Xi_1(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} + M_1 Z_0^{\kappa}||\Xi_2(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} + 2||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}\Big] \times M_1 Z_0^{\kappa}||\Xi_1(\tau,z) - \Xi_2(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}$$

Furthermore, according to Proposition 4, we get a constant $M_3 > 0$ with

$$
\begin{aligned}
(81) \quad \mathcal{A} := ||a_h(z)\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h}{k}-1} \\
\times \Big\{ s \int_0^s \Big[\partial_z^{-\kappa}\Xi_1((s-s_1)^{1/k},z) + \check{\Psi}((s-s_1)^{1/k},z) \Big] \\
\times \Big[\partial_z^{-\kappa}\Xi_1(s_1^{1/k},z) + \check{\Psi}(s_1^{1/k},z) \Big] \frac{1}{(s-s_1)s_1} ds_1 \Big\} \frac{ds}{s} \\
- a_h(z)\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h}{k}-1} \\
\times \Big\{ s \int_0^s \Big[\partial_z^{-\kappa}\Xi_2((s-s_1)^{1/k},z) + \check{\Psi}((s-s_1)^{1/k},z) \Big] \\
\times \Big[\partial_z^{-\kappa}\Xi_2(s_1^{1/k},z) + \check{\Psi}(s_1^{1/k},z) \Big] \frac{1}{(s-s_1)s_1} ds_1 \Big\} \frac{ds}{s} ||_{(\sigma,Z_0,\mathcal{U}_p)} \\
\leq |a_h|(Z_0)M_3||\tau^k \int_0^{\tau^k} \Big\{ \Big[\partial_z^{-\kappa}\Xi_1((\tau^k-s_1)^{1/k},z) + \check{\Psi}((\tau^k-s_1)^{1/k},z) \Big] \\
\times \Big[\partial_z^{-\kappa}\Xi_1(s_1^{1/k},z) + \check{\Psi}(s_1^{1/k},z) \Big] \\
- \Big[\partial_z^{-\kappa}\Xi_2((\tau^k-s_1)^{1/k},z) + \check{\Psi}((\tau^k-s_1)^{1/k},z) \Big] \times \Big[\partial_z^{-\kappa}\Xi_2(s_1^{1/k},z) + \check{\Psi}(s_1^{1/k},z) \Big] \Big\} \\
\times \frac{1}{(\tau^k-s_1)s_1} ds_1 ||_{(\sigma,Z_0,\mathcal{U}_p)}
\end{aligned}
$$

Using the factorization (79), the propositions 2, 3, 5 and 6 grant the existence of constants $M_1, M_2 > 0$ such that

$$
\begin{aligned}
(82) \quad \mathcal{A} \leq |a_h|(Z_0)M_3M_2 \Big[||\partial_z^{-\kappa}(\Xi_1 - \Xi_2)(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \times ||\partial_z^{-\kappa}\Xi_1(\tau,z) + \check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \\
+ ||\partial_z^{-\kappa}\Xi_2(\tau,z) + \check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \times ||\partial_z^{-\kappa}(\Xi_1 - \Xi_2)(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \Big] \\
\leq |a_h|(Z_0)M_3M_2 \Big[M_1 Z_0^{\kappa} ||\Xi_1(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} + M_1 Z_0^{\kappa} ||\Xi_2(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \\
+ 2||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \Big] \times M_1 Z_0^{\kappa} ||\Xi_1(\tau,z) - \Xi_2(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)}
\end{aligned}
$$

From now one, we select the radius $\upsilon > 0$ and $\chi > 0$ small enough with $0 < Z_0 < \chi$ such that $0 < Z_0 < \chi$ in order that the next condition holds,

$$
\begin{aligned}
(83) \quad \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} (kq^{-h_3k})^{h_1} |d_{\underline{h}}|(Z_0) M_1 Z_0^{\kappa-h_2} \\
+ \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} \frac{(kq^{-h_3k})^{h_1}}{\Gamma(h_0/k)} |d_{\underline{h}}|(Z_0) M_3 Z_0^{\kappa-h_2} \\
+ |a_0|(Z_0)M_2 \Big[2M_1 Z_0^{\kappa}\upsilon + 2||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \Big] M_1 Z_0^{\kappa} \\
+ \sum_{h=1}^{A} |a_h|(Z_0) \frac{1}{\Gamma(h/k)} M_3 M_2 \Big[2M_1 Z_0^{\kappa}\upsilon + 2||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \Big] M_1 Z_0^{\kappa} \leq 1/2
\end{aligned}
$$

Gathering the list of bounds (77), (78), (80), (81), (82), subjected to (83) implies the shrinking condition (71) we are looking for.

Finally, in order to certify both properties (70) and (71), we impose on the constants $\chi > 0$ and $\upsilon > 0$ the conjoint constraints (76) and (83). The proposition 8 follows. □

In the next proposition, we solve the nonlinear Cauchy problem (30), (31) in the Banach spaces described in Subsection 3.1.

Proposition 9 *We take for granted that the assumption (19) holds. We fix the constants $\chi > 0$ and $\upsilon > 0$ as in the proposition 8. Then, the convolution Cauchy problem (30), (31) possesses a solution $\Theta_p(u,z)$ which belongs to the space $G^k_{(\sigma,Z_0,\mathcal{U}_p)}$, for any given $\sigma > 0$ provided that $0 < Z_0 < \chi$. Futhermore, one can single out a constant $M_1 > 0$ (relying on k,q,κ,σ,b) such that*

$$||\Theta_p(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq M_1 Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \tag{84}$$

Proof According to Proposition 8, we can apply the classical fixed point theorem for shrinking maps in complete metric spaces to the map $\mathfrak{B} : B_\upsilon \to B_\upsilon$. By construction, (B_υ, d) is a complete metric space for the distance $d(x,y) = ||x-y||_{(\sigma,Z_0,\mathcal{U}_p)}$ since the vector space $G^k_{(\sigma,Z_0,\mathcal{U}_p)}$ equipped with the norm $||.||_{(\sigma,Z_0,\mathcal{U}_p)}$ is a Banach space. Futhermore, under the constraints imposed in Proposition 8, the map $\mathfrak{B} : B_\upsilon \to B_\upsilon$ is shrinking of Lipschitz type 1/2. As a result, $\mathfrak{B} : B_\upsilon \to B_\upsilon$ has a unique fixed point, denoted $\Xi_p(\tau,z) \in B_\upsilon$, meaning that

$$\mathfrak{B}(\Xi_p) = \Xi_p$$

This means in particular that we obtain a solution (unique in the ball B_υ) $\Xi_p(\tau,z)$ for the equation (68). Moreover, owing to Proposition 2, we check that

$$||\partial_z^{-\kappa}\Xi_p(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq M_1 Z_0^{\kappa}||\Xi_p(\tau,z)||_{(\sigma,Z_0,\mathcal{U}_p)} \leq M_1 Z_0^{\kappa}\upsilon$$

The decomposition (67) then confirms that the map $\Theta_p(\tau,z) = \partial_z^{-\kappa}\Xi_p(\tau,z) + \check{\Psi}(\tau,z)$ belongs to $G^k_{(\sigma,Z_0,\mathcal{U}_p)}$, solves the convolution Cauchy problem (30), (31) and suffers the bounds (84). □

3.3 Construction of genuine solutions to the main Cauchy problem (21), (22)

In this subsection, we state the first main result of this work.

Theorem 2 *Assume that the condition (19) is granted. Assume that the radius $r_{\mathcal{T}}$ of each bounded sector $\mathcal{T}_p$ described in Theorem 1 fulfills*

$$r_{\mathcal{T}} < (\frac{\Delta_p}{\sigma\zeta(b)})^{1/k} \tag{85}$$

for $0 \leq p \leq \varsigma - 1$, where Δ_p is introduced in (14) and where $\zeta(b) = \sum_{n\geq 0}\frac{1}{(n+1)^b}$.

Then, the Cauchy problem (21), (22) possesses a holomorphic solution $v_p(t,z)$ on the product $\mathcal{T}_p \times D_{Z_0/2}$, for some radius $Z_0 > 0$ small enough. Furthermore, $v_p(t,z)$ can be expressed by means of a Laplace transform of order k,

$$v_p(t,z) = k\int_{L_{\gamma_p}} \Theta_p(u,z)\exp(-(u/t)^k)du/u \tag{86}$$

for $(t,z) \in \mathcal{T}_p \times D_{Z_0/2}$ *along a halfline* $L_{\gamma_p} = [0,+\infty)e^{\sqrt{-1}\gamma_p}$ *which appears in the representation (15). The Borel map* $\Theta_p(u,z)$ *represents a holomorphic function on the domain* $\mathcal{U}_p \times D_{Z_0/2}$ *whose Taylor expansion*

$$\Theta_p(u,z) = \sum_{\beta \geq 0} \Theta_{p,\beta}(u) \frac{z^\beta}{\beta!} \tag{87}$$

is submitted to the following estimates

$$|\Theta_{p,\beta}(u)| \leq K_1 \beta! (\frac{1}{Z_0})^{\beta} \frac{|u|}{1+|u|^{2k}} \exp(\sigma r_b(\beta)|u|^k) \tag{88}$$

for all $\beta \geq 0$*, all* $u \in \mathcal{U}_p$*, for well selected constants* $K_1 > 0$*, any given* $\sigma > 0$*, where* $r_b(\beta)$ *is the sequence defined by (32). In particular, the Borel map* $\Theta_p(u,z)$ *suffers the next bounds estimates*

$$|\Theta_p(u,z)| \leq 2K_1|u| \exp(\sigma\zeta(b)|u|^k) \tag{89}$$

for $(u,z) \in \mathcal{U}_p \times D_{Z_0/2}$*.*

Proof The proof is a direct consequence of Proposition 9 and of the construction made in Subsection 2.3. □

In the next corollary, we show that $v_p(t,z)$ turns out to solve a nonlinear Cauchy problem with analytic coefficients in space z near the origin and polynomial in time t, which involves both differential operators and dilatations/contractions $q-$difference operators acting on time.

Corollary 1 *The holomorphic map* $v_p(t,z)$ *solves on the product* $\mathcal{T}_p \times D_{Z_0/2}$ *a particular non-linear Cauchy problem which is polynomial in time* t *of the form*

$$P(t^{k+1}\partial_t)\partial_z^{S+\kappa} v_p(t,z) = \sum_{\underline{m}=(m_1,m_2,m_3)\in\mathcal{M}} f_{\underline{m}}(t,z)\Big(\partial_t^{m_1}\partial_z^{m_2} v_p\Big)(q^{m_3}t,z) + G(t,z,\Big\{\big(\partial_t^{r_1}\partial_z^{r_2} v_p\big)(q^{r_3}t,z)\Big\}_{\underline{r}=(r_1,r_2,r_3)\in\mathcal{R}}) \tag{90}$$

for given Cauchy data of the form

$$(\partial_z^j v_p)(t,0) = \check{\varphi}_j(t) \quad , \quad 0 \leq j \leq \kappa - 1 \tag{91}$$

together with

$$(\partial_z^j v_p)(t,0) = \check{\phi}_j(t) \quad , \quad \kappa \leq j \leq \kappa + S - 1 \tag{92}$$

for well chosen polynomials $\check{\phi}_j(t)$ *with complex coefficients for* $\kappa \leq j \leq \kappa + S - 1$ *(which depend on* q*).*

The set $\mathcal{M}$ *is a finite subset of* $\mathbb{N}^2 \times \mathbb{Z}$ *which satisfies in particular the constraint* $m_2 < S + \kappa$ *for all* $\underline{m} = (m_1,m_2,m_3) \in \mathcal{M}$ *and* $\mathcal{R}$ *is a finite subset of* $\mathbb{N}^3$ *with the property that* $r_2 \leq S$ *whenever* $(r_1,r_2,r_3) \in \mathcal{R}$*.*

For all $\underline{m} \in \mathcal{M}$*, the coefficients* $f_{\underline{m}}(t,z)$ *are polynomial in* t *and holomorphic w.r.t* z *the disc* D_R*. The map* $G(t,z,(u_{\underline{r}})_{\underline{r}\in\mathcal{R}})$ *is polynomial in the variables* t *and* $u_{\underline{r}}$ *for* $\underline{r} \in \mathcal{R}$ *and holomorphic relatively to* z *on* D_R*.*

Proof We introduce the next two $q-$difference differential operators

$$\mathcal{O}_1(t,z,\partial_t,\partial_z,\sigma_q) := P(t^{k+1}\partial_t)\partial_z^S - \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}} c_{\underline{l}}(z)t^{l_0}\sigma_{q;t}^{l_3}((t^{k+1}\partial_t)^{l_1}\partial_z^{l_2})$$

and

$$\mathcal{O}_2(t,z,\partial_t,\partial_z,\sigma_q)v := \partial_z^{\kappa}v - \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C}} d_{\underline{h}}(z)t^{h_0}\sigma_{q;t}^{-h_3}((t^{k+1}\partial_t)^{h_1}\partial_z^{h_2})v - a(t,z)v^2$$

where $\sigma_{q;t}^{l}$ stands for the action $t \mapsto q^l t$ on the time variable for $l \in \mathbb{Z}$. By construction, the holomorphic map v_p satisfies

$$\mathcal{O}_2(t,z,\partial_t,\partial_z,\sigma_q)v_p(t,z) = u_p(t,z) \tag{93}$$

and the map u_p fulfills

$$\mathcal{O}_1(t,z,\partial_t,\partial_z,\sigma_q)u_p(t,z) = 0 \tag{94}$$

for $(t,z) \in \mathcal{T}_p \times D_{Z_0/2}$. By coupling (93) and (94), we obtain that

$$\mathcal{O}_1(t,z,\partial_t,\partial_z,\sigma_q) \circ \mathcal{O}_2(t,z,\partial_t,\partial_z,\sigma_q)v_p(t,z) = 0 \tag{95}$$

which gives rise to the equation (90). Concerning the Cauchy data, the map v_p is compelled to the conditions (22) which are rewritten in (91). Furthermore, the fact that $u_p(t,z)$ is subjected to the constraints (12) at $z=0$ can be recast in the form

$$(\partial_z^j \circ \mathcal{O}_2(t,z,\partial_t,\partial_z,\sigma_q)v_p)(t,0) = \varphi_j(t)$$

for $0 \le j \le S-1$ which can be rephrased through the Cauchy conditions (92) for suitably selected polynomials $\check{\phi}_l(t)$ with complex coefficients for $\kappa \le l \le \kappa + S - 1$, since $j + \kappa > j + h_2$ for all $\underline{h} = (h_0,h_1,h_2,h_3) \in \mathcal{C}$ provided that $0 \le j \le S-1$. □

4 Asymptotic expansions in time variable

In order to simplify the notations throughout this section, we rewrite our main nonlinear Cauchy problem (21), (22) in the following form

$$\partial_z^{\kappa}v_p(t,z) = \sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}} e_{\underline{k}}(t,z)\left(\partial_t^{k_1}\partial_z^{k_2}v_p\right)(q^{-k_3}t,z) + a(t,z)v_p^2(t,z) + u_p(t,z) \tag{96}$$

for prescribed Cauchy data

$$(\partial_z^j v_p)(t,0) = \check{\varphi}_j(t) \quad , \quad 0 \le j \le \kappa - 1 \tag{97}$$

where $\mathcal{B}$ is a finite subset of $\mathbb{N}^3$ and where the coefficients $e_{\underline{k}}(t,z)$ are polynomial in t and holomorphic relatively to z on the disc D_R given in Subsection 2.2.

According to the condition (20) imposed on the set $\mathcal{C}$ in (21), the next feature holds for the set $\mathcal{B}$,

$$\kappa \ge k_2 + (\frac{1}{k}+1)k_1 \quad , \quad \kappa > k_2 \quad , \quad k_3 \ge k_1 \tag{98}$$

for all $\underline{k} = (k_1,k_2,k_3) \in \mathcal{B}$. Indeed, it is straight to check that for any given integer $h_1 \ge 1$, the decomposition

$$(t^{k+1}\partial_t)^{h_1} = \sum_{j=0}^{h_1} a_{j,h_1,k}(t)\partial_t^j$$

holds for suitable polynomials $a_{j,h_1,k}(t)$ with real coefficients that rely on j, h_1, k.

4.1 Reduction to an auxiliary Cauchy problem

We first write down the convergent Taylor expansions of the coefficients of (96) at $z = 0$,

$$e_{\underline{k}}(t,z) = \sum_{n\geq 0} e_{\underline{k},n}(t)\frac{z^n}{n!} \quad , \quad a(t,z) = \sum_{n\geq 0} a_n(t)\frac{z^n}{n!} \tag{99}$$

for all $t \in \mathbb{C}$, all $z \in D_R$. Furthermore, we expand the forcing term $u_p(t,z)$ at $z = 0$

$$u_p(t,z) = \sum_{n\geq 0} u_{p,n}(t)\frac{z^n}{n!} \tag{100}$$

that converges provided that $z \in D_{\frac{1}{2C_4}}$, for all $t \in \mathcal{T}_p$ according to Theorem 1. Finally, we recast the analytic solution $v_p(t,z)$ to (96), (97) obtained in Theorem 2 as Taylor series at $z = 0$,

$$v_p(t,z) = \sum_{n\geq 0} v_{p,n}(t)\frac{z^n}{n!} \tag{101}$$

which is convergent provided that $z \in D_{Z_0/2}$ and $t \in \mathcal{T}_p$. The constant $Z_0 > 0$ is in particular taken small enough in a way that $Z_0 < \frac{1}{C_4}$ and $Z_0 < 2R$. By plugging the above expressions in the main equation (96), we check that v_p solves (96), (97) if and only if the sequence of functions $v_{p,n}(t)$ satisfies the next recursion

$$\frac{v_{p,n+\kappa}(t)}{n!} = \Big[\sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}} \sum_{n_1+n_2=n} \frac{e_{\underline{k},n_1}(t)}{n_1!} \frac{\left(\partial_t^{k_1} v_{p,n_2+k_2}\right)(q^{-k_3}t)}{n_2!}\Big] + \Big[\sum_{n_1+n_2+n_3=n} \frac{a_{n_1}(t)}{n_1!}\frac{v_{p,n_2}(t)}{n_2!}\frac{v_{p,n_3}(t)}{n_3!}\Big] + \frac{u_{p,n}(t)}{n!} \tag{102}$$

for all $n \geq 0$, with prescribed conditions

$$v_{p,j}(t) = \check{\varphi}_j(t) \quad , \quad 0 \leq j \leq \kappa - 1 \tag{103}$$

In order to study the asymptotic behaviour of the holomorphic map $v_p(t,z)$ as t tends to 0 on the sector $\mathcal{T}_p$, we adopt a similar strategy as the one initiated by the author in [17].

We plan to reduce this problem, by means of a *majorant series approach* to an auxiliary problem disclosed in (114), (115) that will be solve in the forthcoming subsection 4.4.

At the onset, we apply the general differential operator ∂_t^l for all integers $l \geq 0$ to the latter recursion (102) and get

$$\frac{\partial_t^l v_{p,n+\kappa}(t)}{n!} = \Big[\sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}} \sum_{n_1+n_2=n} \sum_{l_1+l_2=l} l! \frac{\partial_t^{l_1} e_{\underline{k},n_1}(t)}{l_1!n_1!} \frac{\left(\partial_t^{l_2+k_1} v_{p,n_2+k_2}\right)(q^{-k_3}t)}{l_2!n_2!} q^{-k_3 l_2}\Big] + \Big[\sum_{n_1+n_2+n_3=n} \sum_{l_1+l_2+l_3=l} l! \frac{\partial_t^{l_1} a_{n_1}(t)}{l_1!n_1!}\frac{\partial_t^{l_2} v_{p,n_2}(t)}{l_2!n_2!}\frac{\partial_t^{l_3} v_{p,n_3}(t)}{l_3!n_3!}\Big] + \frac{\partial_t^l u_{p,n}(t)}{n!} \tag{104}$$

for all $n \geq 0$, all $l \geq 0$.

We set $\mathcal{W} \subset \mathcal{T}_p$ as a proper subsector centered at 0. We introduce the next set of sequences

$$\mathbb{V}_{p,l,n} = \sup_{t\in\mathcal{W}} |\partial_t^l v_{p,n}(t)| \tag{105}$$

where by definition,

$$\mathbb{V}_{p,l,j} = \sup_{t \in \mathcal{W}} |\partial_t^l \check{\varphi}_j(t)| \tag{106}$$

for $0 \leq j \leq \kappa - 1$, along with

$$\mathbb{E}_{\underline{k},l,n} = \sup_{t \in \mathcal{W}} |\partial_t^l e_{\underline{k},n}(t)|, \quad \mathbb{A}_{l,n} = \sup_{t \in \mathcal{W}} |\partial_t^l a_n(t)|, \quad \mathbb{U}_{p,l,n} = \sup_{t \in \mathcal{W}} |\partial_t^l u_{p,n}(t)| \tag{107}$$

for all $l, n \geq 0$. Since the map $\sigma_{q;t}^{-l_3} : t \mapsto q^{-l_3} t$ leaves $\mathcal{W}$ stable (i.e $\sigma_{q;t}^{-l_3}(\mathcal{W}) \subset \mathcal{W}$) for any integer $l_3 \geq 0$, we deduce from the recursion (104) a sequence of inequalities

$$\frac{\mathbb{V}_{p,l,n+\kappa}}{n!} \leq \Big[\sum_{\underline{k}=(k_1,k_2,k_3) \in \mathcal{B}} \sum_{n_1+n_2=n} \sum_{l_1+l_2=l} l! \frac{\mathbb{E}_{\underline{k},l_1,n_1}}{l_1! n_1!} \frac{\mathbb{V}_{p,l_2+k_1,n_2+k_2}}{l_2! n_2!} q^{-k_3 l_2} \Big] + \Big[\sum_{n_1+n_2+n_3=n} \sum_{l_1+l_2+l_3=l} l! \frac{\mathbb{A}_{l_1,n_1}}{l_1! n_1!} \frac{\mathbb{V}_{p,l_2,n_2}}{l_2! n_2!} \frac{\mathbb{V}_{p,l_3,n_3}}{l_3! n_3!} \Big] + \frac{\mathbb{U}_{p,l,n}}{n!} \tag{108}$$

for all $l, n \geq 0$.

We introduce a sequence denoted $\mathbf{v}_{p,l,n}$ which fulfills the next recursion relation

$$\frac{\mathbf{v}_{p,l,n+\kappa}}{n!} = \Big[\sum_{\underline{k}=(k_1,k_2,k_3) \in \mathcal{B}} \sum_{n_1+n_2=n} \sum_{l_1+l_2=l} l! \frac{\mathbb{E}_{\underline{k},l_1,n_1}}{l_1! n_1!} \frac{\mathbf{v}_{p,l_2+k_1,n_2+k_2}}{l_2! n_2!} q^{-k_3 l_2} \Big] + \Big[\sum_{n_1+n_2+n_3=n} \sum_{l_1+l_2+l_3=l} l! \frac{\mathbb{A}_{l_1,n_1}}{l_1! n_1!} \frac{\mathbf{v}_{p,l_2,n_2}}{l_2! n_2!} \frac{\mathbf{v}_{p,l_3,n_3}}{l_3! n_3!} \Big] + \frac{\mathbb{U}_{p,l,n}}{n!} \tag{109}$$

with given initial data

$$\mathbf{v}_{p,l,j} = \mathbb{V}_{p,l,j} \quad , \quad 0 \leq j \leq \kappa - 1 \tag{110}$$

By construction, in comparing (108) and (109), (110), we observe the crucial fact that

$$\mathbb{V}_{p,l,n} \leq \mathbf{v}_{p,l,n} \tag{111}$$

for all $l, n \geq 0$. Let us define the next formal *generating series*

$$\mathbf{v}_p(T, X) = \sum_{l,n \geq 0} \mathbf{v}_{p,l,n} \frac{T^l}{l!} \frac{X^n}{n!} \tag{112}$$

together with

$$\mathbb{E}_{\underline{k}}(T, X) = \sum_{l,n \geq 0} \mathbb{E}_{\underline{k},l,n} \frac{T^l}{l!} \frac{X^n}{n!}, \quad \mathbb{A}(T, X) = \sum_{l,n \geq 0} \mathbb{A}_{l,n} \frac{T^l}{l!} \frac{X^n}{n!}, \quad \mathbb{U}_p(T, X) = \sum_{l,n \geq 0} \mathbb{U}_{p,l,n} \frac{T^l}{l!} \frac{X^n}{n!} \tag{113}$$

A direct computation following from the recursion (109), (110) shows that the formal series $\mathbf{v}_p(T, X)$ solves the next nonlinear Cauchy problem

$$\partial_X^\kappa \mathbf{v}_p(T, X) = \sum_{\underline{k}=(k_1,k_2,k_3) \in \mathcal{B}} \mathbb{E}_{\underline{k}}(T, X) \Big(\partial_T^{k_1} \partial_X^{k_2} \mathbf{v}_p \Big)(q^{-k_3} T, X) + \mathbb{A}(T, X) \mathbf{v}_p^2(T, X) + \mathbb{U}_p(T, X) \tag{114}$$

for prescribed Cauchy data

$$(\partial_X^j \mathbf{v}_p)(T,0) = \boldsymbol{\varphi}_{j,p}(T) \quad , \quad 0 \leq j \leq \kappa - 1 \tag{115}$$

where

$$\boldsymbol{\varphi}_{j,p}(T) = \sum_{l \geq 0} \mathbb{V}_{p,l,j} \frac{T^l}{l!}$$

for $0 \leq j \leq \kappa - 1$, whose coefficients are defined by (106). Since $\check{\varphi}_j(t)$ are polynomials in t, we deduce that $\boldsymbol{\varphi}_{j,p}(T)$ are also polynomials in the variable T, for $0 \leq j \leq \kappa - 1$.

4.2 Asymptotic expansions and bounds for the $n-$th derivative of the holomorphic solutions to the linear Cauchy problem discussed in Section 2.1

In this short subsection, we draw attention to parts of the results obtained in our previous work [13] that will be applied in the next subsections 4.3 and 4.5. Namely, as a consequence of Theorem 2 from [13], we get the following statement on the asymptotic expansion in time t of the maps $u_p(t,z)$ (described in Theorem 1)

Theorem 3 *1. There exists a formal power series $\hat{u}(t,z) = \sum_{n \geq 0} u_n(z) t^n$ with bounded holomorphic coefficients $u_n(z)$ on some fixed disc $D_{\frac{1}{2C_6}}$ for a well chosen constant $C_6 > 0$, which represent the common asymptotic expansion of all the functions $t \mapsto u_p(t,z)$ on $\mathcal{T}_p$, for $0 \leq p \leq \varsigma - 1$, uniformly relatively to z on $D_{\frac{1}{2C_6}}$. It means that, for each $0 \leq p \leq \varsigma - 1$, for each proper subsector $\mathcal{W} \subset \mathcal{T}_p$, for each integer $N \geq 0$, one can single out a constant $c(N, \mathcal{W}) > 0$ with*

$$\sup_{z \in D_{\frac{1}{2C_6}}} |u_p(t,z) - \sum_{n=0}^{N} u_n(z) t^n| \leq c(N,\mathcal{W}) |t|^{N+1}$$

for all $t \in \mathcal{W}$.

2. The maps $t \mapsto u_p(t,z)$ are infinitely often differentiable at the origin and

$$(\partial_t^l u_p)(0,z) = l! u_l(z) \tag{116}$$

for all $l \geq 0$, all $z \in D_{\frac{1}{2C_6}}$, given that $0 \leq p \leq \varsigma - 1$.

The second point 2. of the above result is not mentioned in [13] but is a direct consequence of the first item 1. by application of a classical result in asymptotics mentioned in Proposition 8 p. 66 of [1].

Furthermore, in the corollary 1 of Theorem 2 from [13], we derive important asymptotic bounds for the $l-$th derivative of the partial maps $t \mapsto u_p(t,z)$ on bounded sectors. Indeed, the next result holds

Theorem 4 *For each $0 \leq p \leq \varsigma - 1$, for each proper subsector $\mathcal{W}' \subset \mathcal{T}_p$, one can select two constants $C', M' > 0$ for which*

$$\sup_{z \in D_{\frac{1}{2C_6}}} |\partial_t^l u_p(t,z)| \leq C'(M')^l l! \Gamma(\frac{l}{k}) q^{\frac{l^2}{2}} \tag{117}$$

for all integers $l \geq 1$, all $t \in \mathcal{W}'$, where $C_6 > 0$ is the well chosen constant fixed in Theorem 3.

4.3 Banach spaces of formal power series

In this subsection, we unveil the definition and useful features of the Banach spaces in which we plan to seek for solutions to the aforementioned Cauchy problem (114), (115).

Definition 4 *Let $T_0, X_0 > 0$ be real numbers. We set $s > 0$ and $q > 1$ as real numbers. We define the space $G_{s,q}(T_0, X_0)$ as the vector space of formal power series $\mathbf{V}(T, X) \in \mathbb{C}[[T, X]]$,*

$$\mathbf{V}(T, X) = \sum_{l,n\geq 0} \mathbf{v}_{l,n} \frac{T^l}{l!} \frac{X^n}{n!}$$

such that the norm

$$||\mathbf{V}(T, X)||_{(T_0,X_0,s,q)} := \sum_{l,n\geq 0} \frac{|\mathbf{v}_{l,n}|}{q^{l^2/2}} \frac{T_0^l X_0^n}{\Gamma(1 + (s + 1)l + n)}$$

is finite. One checks that the vector space $G_{s,q}(T_0, X_0)$ equipped with the norm $||.||_{(T_0,X_0,s,q)}$ represents a complex Banach space.

Remark: 1) In the case $q = 1$, similar norms have been introduced by M. Miyake in the work [20] in order to construct formal power series solutions of Gevrey type to *linear* PDEs with analytic coefficients.
2) In the case $q > 1$, related norms on formal series have been introduced by the author and colleagues in order to solve several *linear* $q-$difference differential problems, see for instance [12], [16].
3) It is worth noticing that this is the earliest occurrence of such norms in the study of nonlinear $q-$difference differential problems in the author's overall investigations.

The next proposition is central in order to deal with the nonlinearity of the problem (114), (115).

Proposition 10 *Let $\mathbf{V}_1(T, X), \mathbf{V}_2(T, X) \in G_{s,q}(T_0, X_0)$. Then, the product $\mathbf{V}_1(T, X)\mathbf{V}_2(T, X)$ belongs to $G_{s,q}(T_0, X_0)$ and one can find a constant $L_1 > 0$ (depending on s) such that*

$$||\mathbf{V}_1(T, X)\mathbf{V}_2(T, X)||_{(T_0,X_0,s,q)} \leq L_1||\mathbf{V}_1(T, X)||_{(T_0,X_0,s,q)}||\mathbf{V}_2(T, X)||_{(T_0,X_0,s,q)} \quad (118)$$

In other words, $(G_{s,q}(T_0, X_0), ||.||_{(T_0,X_0,s,q)})$ turns out to be a Banach algebra.

Proof Let

$$\mathbf{V}_j(T, X) = \sum_{l,n\geq 0} \mathbf{v}^j_{l,n} \frac{T^l}{l!} \frac{X^n}{n!}$$

for $j = 1, 2$ two elements of $G_{s,q}(T_0, X_0)$. Their product writes

$$\mathbf{V}_1(T, X)\mathbf{V}_2(T, X) = \sum_{l,n\geq 0} \left(\sum_{\substack{l_1+l_2=l\\ n_1+n_2=n}} l!n! \frac{\mathbf{v}^1_{l_1,n_1}}{l_1!n_1!} \frac{\mathbf{v}^2_{l_2,n_2}}{l_2!n_2!} \right) \frac{T^l}{l!} \frac{X^n}{n!}$$

By Definition, its norm fufills the bounds

$$||\mathbf{V}_1(T, X)\mathbf{V}_2(T, X)||_{(T_0,X_0,s,q)} \leq \sum_{l,n\geq 0} \left(\sum_{\substack{l_1+l_2=l\\ n_1+n_2=n}} l!n! \frac{|\mathbf{v}^1_{l_1,n_1}|}{l_1!n_1!} \frac{|\mathbf{v}^2_{l_2,n_2}|}{l_2!n_2!} \right) \frac{1}{q^{l^2/2}} \frac{1}{\Gamma(1 + (s + 1)l + n)} T_0^l X_0^n \quad (119)$$

Since $q > 1$, we first observe that

$$q^{l_1^2/2} q^{l_2^2/2} \leq q^{l^2/2} \tag{120}$$

for all integers $l_1, l_2 \geq 0$ with $l_1 + l_2 = l$.
The next lemma is essential.

Lemma 3 *There exists a constant $L_1 > 0$ (depending on s) such that*

$$\frac{l!n!}{l_1!n_1!l_2!n_2!} \frac{1}{\Gamma(1+(s+1)l+n)} \leq L_1 \frac{1}{\Gamma(1+(s+1)l_1+n_1)\Gamma(1+(s+1)l_2+n_2)} \tag{121}$$

for all integers $l_1, l_2, n_1, n_2 \geq 0$ with $l_1 + l_2 = l$ and $n_1 + n_2 = n$.

Proof We depart from the next inequality

$$\frac{l!n!}{l_1!n_1!l_2!n_2!} \leq \frac{(l+n)!}{(l_1+n_1)!(l_2+n_2)!} = \frac{\Gamma(1+l+n)}{\Gamma(1+l_1+n_1)\Gamma(1+l_2+n_2)} \tag{122}$$

for all $l_1, l_2, n_1, n_2 \geq 0$ with $l_1 + l_2 = l$ and $n_1 + n_2 = n$. These bounds straightly follow from the binomial expansion for each term of the identity $(1+x)^l(1+x)^n = (1+x)^{n+l}$, for $x \in \mathbb{R}$.

In the next step, we need to prove the next

Lemma 4 *The next inequality*

$$\frac{\Gamma(1+(s+1)l_1+n_1)\Gamma(1+(s+1)l_2+n_2)}{\Gamma(2+(s+1)(l_1+l_2)+n_1+n_2)} \leq \frac{\Gamma(1+l_1+n_1)\Gamma(1+l_2+n_2)}{\Gamma(2+l_1+l_2+n_1+n_2)} \tag{123}$$

holds for all $l_1, l_2, n_1, n_2 \geq 0$

Proof We recall the next identity defining the so-called *Beta function* (see [1], Appendix B)

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt \tag{124}$$

provided that $\alpha, \beta > 0$ are real numbers. As a result, observing that

$$f(t) = (1-t)^{sl_1} t^{sl_2} \leq 1$$

provided that $t \in [0,1]$, we get that

$$\frac{\Gamma(1+(s+1)l_1+n_1)\Gamma(1+(s+1)l_2+n_2)}{\Gamma(2+(s+1)(l_1+l_2)+n_1+n_2)} = \int_0^1 (1-t)^{(s+1)l_1+n_1} t^{(s+1)l_2+n_2} dt$$
$$\leq \int_0^1 (1-t)^{l_1+n_1} t^{l_2+n_2} dt = \frac{\Gamma(1+l_1+n_1)\Gamma(1+l_2+n_2)}{\Gamma(2+l_1+l_2+n_1+n_2)}$$

from which Lemma 4 follows. □

From the functional property $\Gamma(z+1) = z\Gamma(z)$ for any $z > 0$, we can factorize

$$\begin{aligned} &\Gamma(2+(s+1)(l_1+l_2)+n_1+n_2) \\ &= (1+(s+1)(l_1+l_2)+n_1+n_2)\Gamma(1+(s+1)(l_1+l_2)+n_1+n_2), \\ &\Gamma(2+l_1+l_2+n_1+n_2) = (1+l_1+l_2+n_1+n_2)\Gamma(1+l_1+l_2+n_1+n_2) \end{aligned} \tag{125}$$

Combining the bounds (123) and the expansions (125), we get that

$$\frac{\Gamma(1+(s+1)l_1+n_1)\Gamma(1+(s+1)l_2+n_2)}{\Gamma(1+(s+1)(l_1+l_2)+n_1+n_2)} \leq C(l,n)\frac{\Gamma(1+l_1+n_1)\Gamma(1+l_2+n_2)}{\Gamma(1+l_1+l_2+n_1+n_2)} \tag{126}$$

where

$$C(l,n) = \frac{1+(s+1)l+n}{1+l+n}$$

where $l = l_1 + l_2$, $n = n_1 + n_2$ with $l_1, l_2, n_1, n_2 \geq 0$. Besided a constant $L_1 > 0$ (relying on s) can be found such that

$$C_{l,n} \leq L_1 \tag{127}$$

for all $l, n \geq 0$. Finally, the collection of the bounds (122), (126) and (127) yields the forecast estimates (121). □

Owing to the upper bounds (120) and (121) applied to (119), we reach the awaited inequality

$$\begin{aligned} &||\mathbf{V}_1(T,X)\mathbf{V}_2(T,X)||_{(T_0,X_0,s,q)} \tag{128}\\ &\leq L_1 \sum_{l,n\geq 0}\Big(\sum_{\substack{l_1+l_2=l\\ n_1+n_2=n}} \frac{|\mathbf{v}^1_{l_1,n_1}|}{q^{l_1^2/2}\Gamma(1+(s+1)l_1+n_1)}\frac{|\mathbf{v}^2_{l_2,n_2}|}{q^{l_2^2/2}\Gamma(1+(s+1)l_2+n_2)}\Big)T_0^lX_0^n\\ &= L_1||\mathbf{V}_1(T,X)||_{(T_0,X_0,s,q)}||\mathbf{V}_2(T,X)||_{(T_0,X_0,s,q)} \end{aligned}$$

□

The next proposition helps us in addressing the linear part of the Cauchy problem (114), (115).

Proposition 11 *Let $k_1, k_2, k_3 \geq 0$ be integers under the requirement*

$$k_2 \geq (s+1)k_1 \quad , \quad k_3 \geq k_1 \tag{129}$$

Then, the linear operator $\sigma_{q^{-k_3};T} \circ \partial_T^{k_1}\partial_X^{-k_2}$ is bounded from the space $G_{s,q}(T_0,X_0)$ into itself. In other words, there exists a constant $L_2 > 0$ (depending on q, k_1, k_2, s) with

$$||\Big(\partial_T^{k_1}\partial_X^{-k_2}\mathbf{V}\Big)(q^{-k_3}T,X)||_{(T_0,X_0,s,q)} \leq L_2X_0^{k_2}T_0^{-k_1}||\mathbf{V}(T,X)||_{(T_0,X_0,s,q)} \tag{130}$$

for all $\mathbf{V}(T,X) \in G_{s,q}(T_0,X_0)$.

Proof Let $\mathbf{V}(T,X) = \sum_{l,n\geq 0}\mathbf{v}_{l,n}T^l/l!X^n/n!$ be in $G_{s,q}(T_0,X_0)$. The action of the $q-$difference differential operator is expressed through

$$\Big(\partial_T^{k_1}\partial_X^{-k_2}\mathbf{V}\Big)(q^{-k_3}T,X) = \sum_{l\geq 0,n\geq k_2}\mathbf{v}_{l+k_1,n-k_2}q^{-k_3l}\frac{T^l}{l!}\frac{X^n}{n!}$$

and enables us to rewrite the norm

$$\begin{aligned} &T_0^{k_1}X_0^{-k_2}||\Big(\partial_T^{k_1}\partial_X^{-k_2}\mathbf{V}\Big)(q^{-k_3}T,X)||_{(T_0,X_0,s,q)} \tag{131}\\ &= \sum_{l\geq 0,n\geq k_2}\frac{|\mathbf{v}_{l+k_1,n-k_2}|}{q^{l^2/2}}\frac{q^{-k_3l}}{\Gamma(1+(s+1)l+n)}T_0^{l+k_1}X_0^{n-k_2}\\ &= \sum_{l\geq 0,n\geq k_2}\mathfrak{C}_{l,n}\frac{|\mathbf{v}_{l+k_1,n-k_2}|}{q^{(l+k_1)^2/2}}\frac{1}{\Gamma(1+(s+1)(l+k_1)+n-k_2)}T_0^{l+k_1}X_0^{n-k_2} \end{aligned}$$

where

$$\mathfrak{C}_{l,n} = \frac{q^{(l+k_1)^2/2}}{q^{l^2/2}} q^{-k_3 l} \frac{\Gamma(1+(s+1)(l+k_1)+n-k_2)}{\Gamma(1+(s+1)l+n)}$$

for all $l \geq 0$, $n \geq k_2$. In the continuing part of the proof, we show that the sequence $\mathfrak{C}_{l,n}$ is actually bounded by some constant.

Indeed, recall from [1], Appendix B, that for a given $a \in \mathbb{R}$, one can find a constant $K_a > 0$ (relying on a) such that

$$\frac{\Gamma(x+a)}{\Gamma(x)} \leq K_a x^a \tag{132}$$

provided that $x \geq 1$. Consequently, we get two constants $K_{s,k_1,k_2} > 0$ and $L_{2.1} > 0$ (depending on s, k_1, k_2) with

$$\frac{\Gamma(1+(s+1)l+n+(s+1)k_1-k_2)}{\Gamma(1+(s+1)l+n)} \leq K_{s,k_1,k_2} \frac{1}{(1+(s+1)l+n)^{k_2-(s+1)k_1}} \leq L_{2.1}$$

for all $l \geq 0$, $n \geq k_2$, provided that (129) holds. On the other hand,

$$\frac{q^{(l+k_1)^2/2}}{q^{l^2/2}} q^{-k_3 l} = q^{l(k_1-k_3)} q^{k_1^2/2} \leq q^{k_1^2/2} \tag{133}$$

for all $l \geq 0$ under the assumption (129). As a result, we obtain

$$\mathfrak{C}_{l,n} \leq q^{k_1^2/2} L_{2.1} \tag{134}$$

for all $l \geq 0$, $n \geq k_2$. Finally, based on (131) and (134), we deduce

$$T_0^{k_1} X_0^{-k_2} || \left(\partial_T^{k_1} \partial_X^{-k_2} \mathbf{V} \right) (q^{-k_3} T, X) ||_{(T_0,X_0,s,q)} \leq q^{k_1^2/2} L_{2.1} || \mathbf{V}(T,X) ||_{(T_0,X_0,s,q)}$$

which is tantamount to (130). □

In the next proposition, we show that the coefficients, the forcing term and Cauchy data of the problem (114), (115) belongs to the Banach space $G_{s,q}(T_0, X_0)$ for well chosen parameters.

Proposition 12 *Let k, q chosen as in Section 2.*
1) The series $\mathbb{U}_p(T,X)$ appertains to $G_{1/k,q}(T_0,X_0)$ for $T_0, X_0 > 0$ small enough.
2) The series $\mathbb{E}_{\underline{k}}(T,X)$ and $\mathbb{A}(T,X)$ belong to $G_{1/k,q}(T_0,X_0)$ for $T_0, X_0 > 0$ small enough.
3) We set up

$$\mathbb{W}_p(T,X) := \sum_{j=0}^{\kappa-1} \varphi_{j,p}(T) \frac{X^j}{j!} \tag{135}$$

For any non negative integers $k_1, k_2, k_3 \geq 0$, the polynomials $\left(\partial_T^{k_1} \partial_X^{k_2} \mathbb{W}_p \right) (q^{-k_3} T, X)$ belong to $G_{1/k,q}(T_0,X_0)$ for any given $T_0, X_0 > 0$.

Proof We focus on the first point 1). According to Theorem 3 stated in Subsection 4.2, with the help of the Cauchy formula, we deduce that the sequence $\mathbb{U}_{p,l,n}$ defined in (107) is subjected to bounds of the form

$$\mathbb{U}_{p,l,n} \leq \mathbb{U}_{0,p} (\mathbb{U}_1)^l (\mathbb{U}_2)^n l! \Gamma(l/k) q^{l^2/2} n! \tag{136}$$

for some selected constants $\mathbb{U}_{0,p}, \mathbb{U}_1, \mathbb{U}_2 > 0$, whenever $l \geq 1$, $n \geq 0$. Besides, from the formula (124), we observe in particular that

$$\Gamma(\alpha)\Gamma(\beta) \leq \Gamma(\alpha+\beta) \tag{137}$$

provided that $\alpha, \beta \geq 1$. This last inequality combined with the functional relation $\Gamma(z+1) = z\Gamma(z)$ for any $z > 0$ begets

$$l!\Gamma(l/k)n! \leq \Gamma(2 + (\frac{1}{k}+1)l + n) = (1 + (\frac{1}{k}+1)l + n)\Gamma(1 + (\frac{1}{k}+1)l + n) \tag{138}$$

for all $l \geq k$, all $n \geq 0$. Thereby, we deduce constants $\check{\mathbb{U}}_{0,p}, \check{\mathbb{U}}_1, \check{\mathbb{U}}_2 > 0$ with

$$\mathbb{U}_{p,l,n} \leq \check{\mathbb{U}}_{0,p}(\check{\mathbb{U}}_1)^l(\check{\mathbb{U}}_2)^n\Gamma(1 + (\frac{1}{k}+1)l + n)q^{l^2/2} \tag{139}$$

for all $l, n \geq 0$. Bestowing the last expansion of (113) implies the next norm bounds

$$\begin{aligned} ||\mathbb{U}_p(T,X)||_{(T_0,X_0,1/k,q)} &= \sum_{l,n\geq 0} \frac{|\mathbb{U}_{p,l,n}|}{q^{l^2/2}} \frac{T_0^l X_0^n}{\Gamma(1+(\frac{1}{k}+1)l+n)} \\ &\leq \check{\mathbb{U}}_{0,p} \sum_{l,n\geq 0} (T_0\check{\mathbb{U}}_1)^l (X_0\check{\mathbb{U}}_2)^n \leq 4\check{\mathbb{U}}_{0,p} \end{aligned} \tag{140}$$

provided that $T_0, X_0 > 0$ are constrained to $T_0\check{\mathbb{U}}_1 \leq 1/2$ and $X_0\check{\mathbb{U}}_2 \leq 1/2$.

We address the second point 2). According to the assumptions of Section 2, the functions $e_{\underline{k}}(t,z)$ and $a(t,z)$ are polynomial in t and bounded holomorphic on some disc D_R relatively to z, for all $\underline{k} \in \mathcal{B}$. From the Cauchy formula, we deduce that the first two sequences in (107) satisfy the bounds

$$\mathbb{E}_{\underline{k},l,n} \leq \mathbb{E}_{0,\underline{k}}(\mathbb{E}_1)^l(\mathbb{E}_2)^n l!n! \quad , \quad \mathbb{A}_{l,n} \leq \mathbb{A}_0(\mathbb{A}_1)^l(\mathbb{A}_2)^n l!n! \tag{141}$$

for some well chosen constants $\mathbb{E}_{0,\underline{k}}, \mathbb{E}_1, \mathbb{E}_2 > 0$ and $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2 > 0$, for all $l, n \geq 0$. Owing to (137) and since the map $x \mapsto \Gamma(x)$ is increasing on $[2, +\infty)$ (see [1], Appendix B), we check that

$$l!n! \leq \Gamma(2+l+n) \leq \Gamma(2+(\frac{1}{k}+1)l+n)q^{l^2/2} = (1+(\frac{1}{k}+1)l+n)\Gamma(1+(\frac{1}{k}+1)l+n)q^{l^2/2} \tag{142}$$

for all $l, n \geq 0$. Whence, constants $\check{\mathbb{E}}_{0,\underline{k}}, \check{\mathbb{E}}_1, \check{\mathbb{E}}_2 > 0$ and $\check{\mathbb{A}}_0, \check{\mathbb{A}}_1, \check{\mathbb{A}}_2 > 0$ can be found with

$$\begin{aligned} \mathbb{E}_{\underline{k},l,n} &\leq \check{\mathbb{E}}_{0,\underline{k}}(\check{\mathbb{E}}_1)^l(\check{\mathbb{E}}_2)^n\Gamma(1 + (\frac{1}{k}+1)l + n)q^{l^2/2}, \\ \mathbb{A}_{l,n} &\leq \check{\mathbb{A}}_0(\check{\mathbb{A}}_1)^l(\check{\mathbb{A}}_2)^n\Gamma(1 + (\frac{1}{k}+1)l + n)q^{l^2/2} \end{aligned} \tag{143}$$

for all $l, n \geq 0$. Thanks to the first two expansions of (113), we deduce the control of the norms

$$\begin{aligned} ||\mathbb{E}_{\underline{k}}(T,X)||_{(T_0,X_0,1/k,q)} &= \sum_{l,n\geq 0} \frac{|\mathbb{E}_{\underline{k},l,n}|}{q^{l^2/2}} \frac{T_0^l X_0^n}{\Gamma(1+(\frac{1}{k}+1)l+n)} \\ &\leq \check{\mathbb{E}}_{0,\underline{k}} \sum_{l,n\geq 0} (T_0\check{\mathbb{E}}_1)^l (X_0\check{\mathbb{E}}_2)^n \leq 4\check{\mathbb{E}}_{0,\underline{k}} \end{aligned} \tag{144}$$

together with

$$||\mathbb{A}(T,X)||_{(T_0,X_0,1/k,q)} = \sum_{l,n\geq 0} \frac{|\mathbb{A}_{l,n}|}{q^{l^2/2}} \frac{T_0^l X_0^n}{\Gamma(1+(\frac{1}{k}+1)l+n)} \leq \check{\mathbb{A}}_0 \sum_{l,n\geq 0} (T_0\check{\mathbb{A}}_1)^l (X_0\check{\mathbb{A}}_2)^n \leq 4\check{\mathbb{A}}_0 \tag{145}$$

whenever $T_0, X_0 > 0$ are submitted to $T_0\check{\mathbb{E}}_1 \leq 1/2$, $T_0\check{\mathbb{A}}_1 \leq 1/2$ and $X_0\check{\mathbb{E}}_2 \leq 1/2$, $X_0\check{\mathbb{A}}_2 \leq 1/2$.

Finally, the last point 3) is straightforward since for any given integers $k_1, k_2, k_3 \geq 0$, the quantity

$$||\left(\partial_T^{k_1}\partial_X^{k_2}\mathbb{W}_p\right)(q^{-k_3}T,X)||_{(T_0,X_0,1/k,q)}$$

is a polynomial in the variables T_0, X_0 and is therefore finite for any fixed $T_0, X_0 > 0$. □

4.4 Solving the auxiliary Cauchy problem

In this subsection, we seek for a formal power series solution to the nonlinear Cauchy problem (114), (115) by means of a decomposition

$$\mathbf{v}_p(T,X) = \partial_X^{-\kappa}\mathbf{Y}_p(T,X) + \mathbb{W}_p(T,X) \tag{146}$$

where the polynomial $\mathbb{W}_p(T,X)$ is displayed in (135), for some formal series $\mathbf{Y}_p(T,X)$ to be determined.

We observe that $\mathbf{v}_p(T,X)$ fulfills the problem (114), (115) if the expression $\mathbf{Y}_p(T,X)$ solves the next *fixed point equation*

$$\mathbf{Y}_p(T,X) = \sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}} \mathbb{E}_{\underline{k}}(T,X)\left(\partial_T^{k_1}\partial_X^{k_2-\kappa}\mathbf{Y}_p\right)(q^{-k_3}T,X) + \mathbb{A}(T,X)(\partial_X^{-\kappa}\mathbf{Y}_p(T,X)+\mathbb{W}_p(T,X))^2 + \mathbb{U}_p(T,X) + \sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}} \mathbb{E}_{\underline{k}}(T,X)\left(\partial_T^{k_1}\partial_X^{k_2}\mathbb{W}_p\right)(q^{-k_3}T,X) \tag{147}$$

Our ensuing undertaking is the construction of a solution of this latter equation (147) within the Banach space of formal series discussed in the previous section 4.3. In order to fulfill this objective, we set up the next nonlinear mapping $\mathfrak{A} : \mathbb{C}[[T,X]] \rightarrow \mathbb{C}[[T,X]]$ defined as

$$\mathfrak{A}(\mathbf{Y}(T,X)) := \sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}} \mathbb{E}_{\underline{k}}(T,X)\left(\partial_T^{k_1}\partial_X^{-(\kappa-k_2)}\mathbf{Y}\right)(q^{-k_3}T,X) + \mathbb{A}(T,X)(\partial_X^{-\kappa}\mathbf{Y}(T,X)+\mathbb{W}_p(T,X))^2 + \mathbb{U}_p(T,X) + \sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}} \mathbb{E}_{\underline{k}}(T,X)\left(\partial_T^{k_1}\partial_X^{k_2}\mathbb{W}_p\right)(q^{-k_3}T,X) \tag{148}$$

In the next proposition, we discuss sufficient conditions under which $\mathfrak{A}$ represents a shrinking map acting on some small ball centered at the origin in the space $G_{1/k,q}(T_0,X_0)$

Proposition 13 *Taking for granted the condition (98), some small real number $\xi > 0$ can be singled out such that if $0 < X_0 < \xi$, one can choose a radius $\varpi > 0$ such that $\mathfrak{A}$ suffers the next features:*

Let B_{ϖ} be the ball centered at 0 with radius ϖ in the space $G_{1/k,q}(T_0,X_0)$.

1. *The next inclusion*

$$\mathfrak{A}(B_{\varpi}) \subset B_{\varpi} \tag{149}$$

holds.

2. *For any* $\mathbf{Y}_1, \mathbf{Y}_2 \in B_{\varpi}$*, we have*

$$||\mathfrak{A}(\mathbf{Y}_1) - \mathfrak{A}(\mathbf{Y}_2)||_{(T_0,X_0,1/k,q)} \leq \frac{1}{2}||\mathbf{Y}_1 - \mathbf{Y}_2||_{(T_0,X_0,1/k,q)} \tag{150}$$

Proof We behold the first property. Let $\mathbf{Y}(T,X)$ belong to the ball B_{ϖ}. Under the constraint (98), the propositions 10, 11 and 12 allow the next inequality

$$\begin{aligned} &||\mathbb{E}_{\underline{k}}(T,X)\left(\partial_T^{k_1}\partial_X^{-(\kappa-k_2)}\mathbf{Y}\right)(q^{-k_3}T,X)||_{(T_0,X_0,1/k,q)} \leq L_1||\mathbb{E}_{\underline{k}}(T,X)||_{(T_0,X_0,1/k,q)} \\ &\times ||\left(\partial_T^{k_1}\partial_X^{-(\kappa-k_2)}\mathbf{Y}\right)(q^{-k_3}T,X)||_{(T_0,X_0,1/k,q)} \leq 4L_1L_2\check{\mathbb{E}}_{0,\underline{k}}X_0^{\kappa-k_2}T_0^{-k_1}||\mathbf{Y}(T,X)||_{(T_0,X_0,1/k,q)} \end{aligned} \tag{151}$$

to hold true along with

$$\begin{aligned} &||\mathbb{A}(T,X)(\partial_X^{-\kappa}\mathbf{Y}(T,X) + \mathbb{W}_p(T,X))^2||_{(T_0,X_0,1/k,q)} \leq L_1^2||\mathbb{A}(T,X)||_{(T_0,X_0,1/k,q)} \\ &\qquad \times ||\partial_X^{-\kappa}\mathbf{Y}(T,X) + \mathbb{W}_p(T,X)||^2_{(T_0,X_0,1/k,q)} \\ &\qquad \leq 4L_1^2\check{\mathbb{A}}_0\left(L_2X_0^{\kappa}||\mathbf{Y}(T,X)||_{(T_0,X_0,1/k,q)} + ||\mathbb{W}_p(T,X)||_{(T_0,X_0,1/k,q)}\right)^2 \end{aligned} \tag{152}$$

for some constants $L_1, L_2 > 0$ and $\check{\mathbb{E}}_{0,\underline{k}}, \check{\mathbb{A}}_0 > 0$. Eventually, due to Propositions 10 and 12 we arrive at

$$\begin{aligned} &||\mathbb{E}_{\underline{k}}(T,X)\left(\partial_T^{k_1}\partial_X^{k_2}\mathbb{W}_p\right)(q^{-k_3}T,X)||_{(T_0,X_0,1/k,q)} \\ &\qquad \leq L_1||\mathbb{E}_{\underline{k}}(T,X)||_{(T_0,X_0,1/k,q)}||\left(\partial_T^{k_1}\partial_X^{k_2}\mathbb{W}_p\right)(q^{-k_3}T,X)||_{(T_0,X_0,1/k,q)} \\ &\qquad\qquad \leq 4L_1\check{\mathbb{E}}_{0,\underline{k}}||\left(\partial_T^{k_1}\partial_X^{k_2}\mathbb{W}_p\right)(q^{-k_3}T,X)||_{(T_0,X_0,1/k,q)} \end{aligned} \tag{153}$$

for some constants $L_1 > 0$ and $\check{\mathbb{E}}_{0,\underline{k}} > 0$.

From now on we choose a radius $\varpi > 0$ and a tiny constant $\xi > 0$ with $0 < X_0 < \xi$ aiming to the next constraint

$$\begin{aligned} &\sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}} 4L_1L_2\check{\mathbb{E}}_{0,\underline{k}}X_0^{\kappa-k_2}T_0^{-k_1}\varpi \\ &\qquad + 4L_1^2\check{\mathbb{A}}_0\left(L_2X_0^{\kappa}\varpi + ||\mathbb{W}_p(T,X)||_{(T_0,X_0,1/k,q)}\right)^2 + 4\check{\mathbb{U}}_{0,p} \\ &\qquad\qquad + \sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}} 4L_1\check{\mathbb{E}}_{0,\underline{k}}||\left(\partial_T^{k_1}\partial_X^{k_2}\mathbb{W}_p\right)(q^{-k_3}T,X)||_{(T_0,X_0,1/k,q)} \leq \varpi \end{aligned} \tag{154}$$

Piling up the above estimates (151), (152) and (153) under the limitation (154) hints at the due inclusion (149).

We address the second property. Let $\mathbf{Y}_1(T,X), \mathbf{Y}_2(T,X)$ be taken within the ball B_{ϖ}. The above inequality (151) prompts the next bounds for the linear piece

$$(155)\quad ||\mathbb{E}_{\underline{k}}(T,X)\Big(\partial_T^{k_1}\partial_X^{-(\kappa-k_2)}(\mathbf{Y}_1-\mathbf{Y}_2)\Big)(q^{-k_3}T,X)||_{(T_0,X_0,1/k,q)} \\ \leq 4L_1L_2\check{\mathbb{E}}_{0,\underline{k}}X_0^{\kappa-k_2}T_0^{-k_1}||\mathbf{Y}_1(T,X)-\mathbf{Y}_2(T,X)||_{(T_0,X_0,1/k,q)}$$

With an eye toward the nonlinear block, by means of the basic identity $a^2-b^2=(a-b)(a+b)$, we factorize the difference of squares as

$$\Delta_{1,2} = (\partial_X^{-\kappa}\mathbf{Y}_1(T,X)+\mathbb{W}_p(T,X))^2 - (\partial_X^{-\kappa}\mathbf{Y}_2(T,X)+\mathbb{W}_p(T,X))^2 \\ = \partial_X^{-\kappa}(\mathbf{Y}_1(T,X)-\mathbf{Y}_2(T,X)) \times \Big(\partial_X^{-\kappa}\mathbf{Y}_1(T,X)+2\mathbb{W}_p(T,X)+\partial_X^{-\kappa}\mathbf{Y}_2(T,X)\Big)$$

Consequently, on account of Propositions 10, 11 and 12 the next estimates

$$(156)\quad ||\mathbb{A}(T,X)\Delta_{1,2}||_{(T_0,X_0,1/k,q)} \leq L_1^2||\mathbb{A}(T,X)||_{(T_0,X_0,1/k,q)} \\ \times ||\partial_X^{-\kappa}(\mathbf{Y}_1(T,X)-\mathbf{Y}_2(T,X))||_{(T_0,X_0,1/k,q)} \\ \times ||\partial_X^{-\kappa}\mathbf{Y}_1(T,X)+2\mathbb{W}_p(T,X)+\partial_X^{-\kappa}\mathbf{Y}_2(T,X)||_{(T_0,X_0,1/k,q)} \\ \leq L_1^2 4\check{\mathbb{A}}_0 L_2 X_0^{\kappa}\Big(L_2X_0^{\kappa}||\mathbf{Y}_1(T,X)||_{(T_0,X_0,1/k,q)} \\ +2||\mathbb{W}_p(T,X)||_{(T_0,X_0,1/k,q)}+L_2X_0^{\kappa}||\mathbf{Y}_2(T,X)||_{(T_0,X_0,1/k,q)}\Big)||\mathbf{Y}_1(T,X)-\mathbf{Y}_2(T,X)||_{(T_0,X_0,1/k,q)}$$

follow. Then, we pick out a small constant $\xi > 0$ with $0 < X_0 < \xi$ intending to the next condition

$$(157)\quad \sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}} 4L_1L_2\check{\mathbb{E}}_{0,\underline{k}}X_0^{\kappa-k_2}T_0^{-k_1} \\ + L_1^2 4\check{\mathbb{A}}_0 L_2 X_0^{\kappa}\Big(2L_2X_0^{\kappa}\varpi + 2||\mathbb{W}_p(T,X)||_{(T_0,X_0,1/k,q)}\Big) \leq 1/2$$

Storing up the previous bounds (155) and (156) under the restriction (157) triggers the Lipschitz property (150).

At last, we compel the constants $\varpi > 0$ and $\xi > 0$ to fulfill both constraints (154) and (157) in order to guarantee each of the two foretold properties (149) and (150). □

In the upcoming proposition, we solve the auxiliary nonlinear Cauchy problem (114), (115) amidst the Banach spaces introduced in Section 4.3.

Proposition 14 *Let us assume that the condition (98) holds. Let the constants $\varpi, \xi > 0$ be determined by the proposition 13. Then, the nonlinear Cauchy problem (114), (115) owns a formal power series solution $\mathbf{v}_p(T,X)$ that belongs to the space $G_{1/k,q}(T_0,X_0)$ presuming that $0 < X_0 < \xi$. Along with it, one can find a constant $L_2 > 0$ (resting on k,q,κ) such that*

$$(158)\quad ||\mathbf{v}_p(T,X)||_{(T_0,X_0,1/k,q)} \leq L_2X_0^{\kappa}\varpi + ||\mathbb{W}_p(T,X)||_{(T_0,X_0,1/k,q)}$$

Proof On the basis of Proposition 13, the classical fixed point theorem for shrinking maps in complete metric spaces can be applied to the map $\mathfrak{A} : B_{\varpi} \rightarrow B_{\varpi}$ from the plain observation that (B_{ϖ}, d) represents a complete metric space for the distance $d(x,y) = ||x-y||_{(T_0,X_0,1/k,q)}$ since the vector space $G_{1/k,q}(T_0,X_0)$ endowed with the norm $||.||_{(T_0,X_0,1/k,q)}$ is a Banach space.

Indeed, under the conditions asked in Proposition 13, the map $\mathfrak{A} : B_{\varpi} \rightarrow B_{\varpi}$ appears to be of Lipschitz type 1/2 and hence shrinking. Whence, $\mathfrak{A} : B_{\varpi} \rightarrow B_{\varpi}$ has a unique fixed point, denoted $\mathbf{Y}_p(T, X) \in B_{\varpi}$, signifying that

$$\mathfrak{A}(\mathbf{Y}_p) = \mathbf{Y}_p$$

In particular, a unique solution $\mathbf{Y}_p$ for the equation (147) is found in the ball B_{ϖ}. Along with it, due to Proposition 11, norms estimates can be achieved,

$$||\partial_X^{-\kappa}\mathbf{Y}_p||_{(T_0,X_0,1/k,q)} \leq L_2 X_0^{\kappa} ||\mathbf{Y}_p||_{(T_0,X_0,1/k,q)} \leq L_2 X_0^{\kappa} \varpi$$

Consequently, the splitting (146) attests that the map $\mathbf{v}_p(T, X) = \partial_X^{-\kappa}\mathbf{Y}_p(T, X) + \mathbb{W}_p(T, X)$ appertains to $G_{1/k,q}(T_0, X_0)$ and solves the problem (114), (115) under the restriction (158). □

4.5 Asymptotic expansions of mixed order

In order to describe the type of asymptotic behaviour which arises in our settings, we need the following definition issued from our recent work [14].

Definition 5 *Let $(\mathbb{F}, ||.||_{\mathbb{F}})$ be a complex Banach space. We set $k \geq 1$ an integer and $q > 1$ a real number. Let $f : \mathcal{T} \rightarrow \mathbb{F}$ be a holomorphic map, where $\mathcal{T}$ stands for a bounded sector in $\mathbb{C}^*$ centered at 0. Then, the map f enjoys the property of having the formal series*

$$\hat{f}(t) = \sum_{n \geq 0} a_n t^n \in \mathbb{F}[[t]]$$

as Gevrey asymptotic expansion of mixed order $(1/k; (q, 1))$ on $\mathcal{T}$ if for each closed proper subsector $\mathcal{W}$ of $\mathcal{T}$ centered at 0, two constants $C, M > 0$ can be distinguished with

$$||f(t) - \sum_{n=0}^{N} a_n t^n||_{\mathbb{F}} \leq CM^{N+1}\Gamma(\frac{N+1}{k})q^{\frac{(N+1)^2}{2}}|t|^{N+1}$$

for all integers $N \geq 0$ and any $t \in \mathcal{W}$.

We are ready to enunciate the second main outcome of the work.

Theorem 5 *We consider the set $\{v_p\}_{0 \leq p \leq \varsigma - 1}$ of solutions to the main Cauchy problem (21), (22) established in Theorem 2. If one sets $\mathbb{F}_{Z_1}$ as the Banach space of bounded holomorphic functions on the disc D_{Z_1} equipped with the sup norm, then each partial map $t \mapsto v_p(t, z)$ can be viewed as a holomorphic map from the sector $\mathcal{T}_p$ into $\mathbb{F}_{Z_1}$, for $0 \leq p \leq \varsigma - 1$, as long as $0 < Z_1 < Z_0/2$, where Z_0 is set up in Theorem 2.*

Hence, provided that $Z_1 > 0$ is taken small enough, for all $0 \leq p \leq \varsigma - 1$, the maps $t \mapsto v_p(t, z)$ share a common formal power series

$$\hat{v}(t, z) = \sum_{l \geq 0} h_l(z) t^l \in \mathbb{F}_{Z_1}[[t]] \quad (159)$$

as Gevrey asymptotic expansion of mixed order $(1/k; (q, 1))$ on $\mathcal{T}_p$. To rephrase it, for each $0 \leq p \leq \varsigma - 1$, for each proper subsector $\mathcal{W} \subset \mathcal{T}_p$, two constants $C, M > 0$ can be singled out with

$$\sup_{z \in D_{Z_1}} |v_p(t, z) - \sum_{l=0}^{N} h_l(z) t^l| \leq CM^{N+1}\Gamma(\frac{N+1}{k})q^{\frac{(N+1)^2}{2}}|t|^{N+1} \quad (160)$$

for all $t \in \mathcal{W}$, all integers $N \geq 0$.

Proof According to the bounds (158) from Proposition 14, the *unique* formal series $\mathbf{v}_p(T,X) \in \mathbb{C}[[T,X]]$ solution of the Cauchy problem (114), (115) given by the expansion (112) is compelled to satisfy the next bounds

$$\mathbf{v}_{p,l,n} \leq (L_2 X_0^{\kappa}\varpi + ||\mathbb{W}_p(T,X)||_{(T_0,X_0,1/k,q)})(\frac{1}{T_0})^l(\frac{1}{X_0})^n\Gamma(1+(\frac{1}{k}+1)l+n)q^{l^2/2} \tag{161}$$

for all $l,n \geq 0$. On the other hand, the next lemma will be required.

Lemma 5 *There exist constants $K_j > 0$, $j=1,2,3$ such that*

$$\Gamma(\alpha+\beta) \leq K_1(K_2)^{\alpha}(K_3)^{\beta}\Gamma(\alpha)\Gamma(\beta) \tag{162}$$

for all real numbers $\alpha,\beta \geq 1$.

Proof We provide a complete proof of this classical result since it is not contained in our reference [1] on special functions. We depart from the Stirling formula (see [1] Appendix B), which ensures the existence of two constants $A_1, A_2 > 0$ for which

$$A_1 x^x x^{-1/2}e^{-x} \leq \Gamma(x) \leq A_2 x^x x^{-1/2} e^{-x}$$

for all $x \geq 1$. Therefrom,

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \leq \frac{A_2}{A_1^2}\Big(\frac{\alpha\beta}{\alpha+\beta}\Big)^{1/2}\Big(1+\frac{\beta}{\alpha}\Big)^{\alpha}\Big(1+\frac{\alpha}{\beta}\Big)^{\beta} \tag{163}$$

Besides, one can pinpoint two constants $B_1 > 0$ and $B_2 > 1$ such that

$$\Big(\frac{\alpha\beta}{\alpha+\beta}\Big)^{1/2} \leq \beta^{1/2} \leq B_1(B_2)^{\beta} \tag{164}$$

for all $\beta \geq 1$ and from the classical estimates $\log(1+x) \leq x$ for $x > 0$, we deduce that

$$\Big(1+\frac{\beta}{\alpha}\Big)^{\alpha} = \exp(\alpha\log(1+\frac{\beta}{\alpha})) \leq e^{\beta} \quad , \quad \Big(1+\frac{\alpha}{\beta}\Big)^{\beta} \leq e^{\alpha} \tag{165}$$

for all real numbers $\alpha,\beta \geq 1$. Lastly, piling up the above bounds (163), (164) and (165) yields the awaited estimates (162). □

In view of the inequality (111), we deduce from (161) together with (162) the next decisive bounds for the higher order time derivatives

$$\sup_{t\in\mathcal{W}}|v_{p,n}(t)| \leq (L_2X_0^{\kappa}\varpi + ||\mathbb{W}_p(T,X)||_{(T_0,X_0,1/k,q)})(\frac{1}{X_0})^n n! \leq \tilde{\mathbb{V}}_0(\frac{1}{X_0})^n n! \tag{166}$$

for all $n \geq 0$, together with

$$\begin{aligned}(167)\quad &\sup_{t\in\mathcal{W}}|\partial_t^l v_{p,n}(t)| \\ &\leq (L_2X_0^{\kappa}\varpi + ||\mathbb{W}_p(T,X)||_{(T_0,X_0,1/k,q)})(\frac{1}{T_0})^l(\frac{1}{X_0})^n\Gamma(1+(\frac{1}{k}+1)l+n)q^{l^2/2} \\ &\leq \hat{\mathbb{V}}_0(\hat{\mathbb{V}}_1)^l(\hat{\mathbb{V}}_2)^n\Gamma((\frac{1}{k}+1)l)\Gamma(1+n)q^{l^2/2} = \hat{\mathbb{V}}_0(\hat{\mathbb{V}}_1)^l(\hat{\mathbb{V}}_2)^n\frac{1}{(\frac{1}{k}+1)l}\Gamma(1+(\frac{1}{k}+1)l)n!q^{l^2/2} \\ &\leq \check{\mathbb{V}}_0(\check{\mathbb{V}}_1)^l(\check{\mathbb{V}}_2)^n l!n!\Gamma(\frac{1}{k}l)q^{l^2/2}\end{aligned}$$

for all $l \geq 1$, $n \geq 0$, for some constants $\tilde{\mathbb{V}}_0 > 0$, $\hat{\mathbb{V}}_0, \hat{\mathbb{V}}_1, \hat{\mathbb{V}}_2 > 0$ and $\check{\mathbb{V}}_0, \check{\mathbb{V}}_1, \check{\mathbb{V}}_2 > 0$ (relying $k, q, \kappa, T_0, X_0, \varpi$ and $\mathbb{V}_{p,l,j}$ for $0 \leq j \leq \kappa - 1$), where $\mathcal{W}$ stands for a fixed subsector of $\mathcal{T}_p$.

From the Taylor formula with integral remainder, for each $0 \leq p \leq \varsigma - 1$, $n \geq 0$, we can expand the function $v_{p,n}(t)$ as follows

$$v_{p,n}(t) = \sum_{l=0}^{N} \frac{(\partial_t^l v_{p,n})(0)}{l!} t^l + R_{N+1,p,n}(t) \tag{168}$$

where

$$R_{N+1,p,n}(t) = \int_0^t \partial_h^{N+1} v_{p,n}(h) \frac{(t-h)^N}{N!} dh$$

for all $t \in \mathcal{W}$, all integers $N \geq 0$. The remainder $R_{N+1,p,n}(t)$ can be estimated from above by means of the latter key bounds (167),

$$\begin{aligned} |R_{N+1,p,n}(t)| &\leq \Big(\int_0^{|t|} (|t|-u)^N du \Big) \check{\mathbb{V}}_0 (\check{\mathbb{V}}_1)^{N+1} (N+1) (\check{\mathbb{V}}_2)^n n! \Gamma(\frac{1}{k}(N+1)) q^{(N+1)^2/2} \\ &\leq (\check{\mathbb{V}}_0 (\check{\mathbb{V}}_2)^n n!) (\check{\mathbb{V}}_1)^{N+1} \Gamma(\frac{1}{k}(N+1)) q^{(N+1)^2/2} |t|^{N+1} \end{aligned} \tag{169}$$

for all $t \in \mathcal{W}$.

For each $0 \leq p \leq \varsigma - 1$, we define the formal power series

$$\hat{v}_p(t,z) = \sum_{l,n \geq 0} (\partial_t^l v_{p,n})(0) \frac{t^l}{l!} \frac{z^n}{n!} \in \mathbb{C}[[t,z]] \tag{170}$$

It turns out that $\hat{v}_p(t,z)$ can be rewritten in the form

$$\hat{v}_p(t,z) = \sum_{l \geq 0} h_{p,l}(z) t^l \tag{171}$$

where the expressions $h_{p,l}(z)$ are holomorphic on a disc D_{Z_1} provided that its radius $Z_1 > 0$ is taken small enough. Indeed, the first term $h_{p,0}(z)$ is expressed through the series

$$h_{p,0}(z) = \sum_{n \geq 0} v_{p,n}(0) \frac{z^n}{n!}$$

which represents a holomorphic function on D_{Z_1} for $Z_1 < X_0/2$ since

$$|h_{p,0}(z)| \leq \tilde{\mathbb{V}}_0 \sum_{n \geq 0} (\frac{|z|}{X_0})^n \leq 2\tilde{\mathbb{V}}_0$$

owing to the bounds (166), whenever $|z| < X_0/2$. Besides, for all $l \geq 1$, the coefficients $h_{p,l}(z)$ are given by the expansions

$$h_{p,l}(z) = \frac{1}{l!} \Big(\sum_{n \geq 0} (\partial_t^l v_{p,n})(0) \frac{z^n}{n!} \Big)$$

which define holomophic maps on D_{Z_1}, when $Z_1 < \frac{1}{2\check{\mathbb{V}}_2}$ since

$$|h_{p,l}(z)| \leq \check{\mathbb{V}}_0 (\check{\mathbb{V}}_1)^l \Gamma(\frac{l}{k}) q^{l^2/2} \sum_{n \geq 0} (\check{\mathbb{V}}_2)^n |z|^n \leq 2\check{\mathbb{V}}_0 (\check{\mathbb{V}}_1)^l \Gamma(\frac{l}{k}) q^{l^2/2}$$

according to the bounds (167), given that $|z| < \frac{1}{2\check{\mathbb{V}}_2}$.

Consequently, in view of both expansions (168) and (171) along with the remainder estimates (169), we deduce the next error bounds

$$(172)\quad |v_p(t,z) - \sum_{l=0}^{N} h_{p,l}(z)t^l| = \Big|\sum_{n\geq 0} v_{p,n}(t)\frac{z^n}{n!} - \sum_{n\geq 0}\Big(\sum_{l=0}^{N}(\partial_t^l v_{p,n})(0)\frac{t^l}{l!}\Big)\frac{z^n}{n!}\Big|$$
$$= \Big|\sum_{n\geq 0} R_{N+1,p,n}(t)\frac{z^n}{n!}\Big| \leq \check{\mathbb{V}}_0(\check{\mathbb{V}}_1)^{N+1}\Gamma(\frac{1}{k}(N+1))q^{\frac{(N+1)^2}{2}}|t|^{N+1}\sum_{n\geq 0}(|z|\check{\mathbb{V}}_2)^n$$
$$\leq 2\check{\mathbb{V}}_0(\check{\mathbb{V}}_1)^{N+1}\Gamma(\frac{1}{k}(N+1))q^{\frac{(N+1)^2}{2}}|t|^{N+1}$$

for all $t \in \mathcal{W}$, all integers $N \geq 0$, given that $|z| < Z_1$, for $Z_1 > 0$ chosen as above.

In the last part of the proof disclosed within the next lemma, we show that the coefficients $h_{p,l}(z)$ do not depend on p for all $l \geq 0$.

Lemma 6 *The coefficients*

$$\hat{v}_{p,l,n} = (\partial_t^l v_{p,n})(0)$$

of the formal series $\hat{v}_p(t,z)$ given by (170) do not depend on p, for all $l,n \geq 0$. As a result, the coefficients $h_{p,l}(z)$ of the formal expansion (171) do not depend on p for all $l \geq 0$ and thereby each formal series $\hat{v}_p(t,z)$ turns out to be written as a single formal series

$$\hat{v}(t,z) = \sum_{l\geq 0} h_l(z)t^l$$

with holomorphic coefficients $h_l(z)$ on D_{Z_1}, for $l \geq 0$.

Proof Paying regard to the relation (104), we get in particular the next recursion for the sequence $\hat{v}_{p,l,n}$,

$$(173)\quad \frac{\hat{v}_{p,l,n+\kappa}}{n!} = \Big[\sum_{\underline{k}=(k_1,k_2,k_3)\in\mathcal{B}}\ \sum_{n_1+n_2=n}\ \sum_{l_1+l_2=l} l!\frac{\partial_t^{l_1}e_{\underline{k},n_1}(0)}{l_1!n_1!}\frac{\hat{v}_{p,l_2+k_1,n_2+k_2}}{l_2!n_2!}q^{-k_3l_2}\Big]$$
$$+\Big[\sum_{n_1+n_2+n_3=n}\ \sum_{l_1+l_2+l_3=l} l!\frac{\partial_t^{l_1}a_{n_1}(0)}{l_1!n_1!}\frac{\hat{v}_{p,l_2,n_2}}{l_2!n_2!}\frac{\hat{v}_{p,l_3,n_3}}{l_3!n_3!}\Big] + \frac{\partial_t^l u_{p,n}(0)}{n!}$$

with prescribed set of data

$$(174)\quad \hat{v}_{p,l,j} = (\partial_t^l\check{\varphi}_j)(0)\quad,\quad 0 \leq j \leq \kappa-1,\quad l \geq 0$$

Now, we observe that the above prescribed quantities $\hat{v}_{p,l,j}$ for $0 \leq j \leq \kappa-1$ and $l \geq 0$ along with the forcing terms $\frac{\partial_t^l u_{p,n}(0)}{n!}$ for all $l \geq 0$, all $n \geq 0$, *do not depend* on p, according to our assumption (22) and Theorem 3, 2. . As a result, we deduce by induction from the above recursion (173) that the whose sequence $\hat{v}_{p,l,n}$ do not depend on p for all $l,n \geq 0$. □

At last, the statement of Theorem 5 issues from the above lemma 6 and the error bounds (172). □

5 Confluence as q tends to 1

Throughout this section, the notations introduced in the earlier sections of the work are lightly modified. Our objective is now to keep track of the dependence of the family of solutions $\{v_p(t,z)\}_{0 \leq p \leq \varsigma-1}$ to the initial problem (21), (22) relatively to the parameter $q > 1$, constructed in Theorem 2. On that account, we denote $v_{p;q}(t,z)$ the function $v_p(t,z)$. We also attach a second index q to the Borel map $\Theta_p(u,z)$ by setting $\Theta_p(u,z) = \Theta_{p;q}(u,z)$ within the integral representation (86). From now on, the real parameter q is chosen inside an interval $(1, q_0]$ for some fixed real number $q_0 > 1$.

5.1 A limit singular linear Cauchy problem

In this subsection, we call attention to parts of the results reached in our previous work [13] that will be applied within the next subsection. We keep the same notations as the ones introduced in Section 2.1.

We consider the next singular Cauchy problem

$$P(t^{k+1}\partial_t)\partial_z^S u_{;1}(t,z) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}} c_{\underline{l}}(z) t^{l_0} (t^{k+1}\partial_t)^{l_1} \partial_z^{l_2} u_{;1}(t,z) \quad (175)$$

for given Cauchy data

$$(\partial_z^j u_{;1})(t,0) = \varphi_j(t) \quad , \quad 0 \leq j \leq S-1. \quad (176)$$

where all the data $k, S, P, \mathcal{A}$ and the coefficients $c_{\underline{l}}(z)$ with $\underline{l} \in \mathcal{A}$ along with the initial data $\varphi_j(t)$ for $0 \leq j \leq S-1$ are already declared in Section 2.1.

In Section 8.1 of the paper [13], the next statement is outlined

Proposition 15 *Let $\underline{\mathcal{D}} = \{\underline{\mathcal{T}}, \underline{\mathcal{U}}\}$ be an admissible set of sectors as chosen in Definition 2. Let $\mathcal{U}$ be one sector belonging to the family of unbounded sectors $\underline{\mathcal{U}}$.*

One can build up a solution $u_{;1}(t,z)$ to the Cauchy problem (175), (176) which is bounded holomorphic on a domain $\mathcal{T} \times D_{\frac{1}{2C_8}}$ where the bounded sector $\mathcal{T}$ belongs to the family $\underline{\mathcal{T}}$ of bounded sectors from $\underline{\mathcal{D}}$ and corresponds to $\mathcal{U}$ under the requirement 2) of Definition 2 and where $C_8 > 0$ is some well chosen constant. The map $u_{;1}(t,z)$ can be expressed by means of a Laplace transform of order k,

$$u_{;1}(t,z) = k \int_{L_\gamma} w_{;1}(u,z) \exp(-(u/t)^k) du/u \quad (177)$$

along a halfline $L_\gamma \subset \mathcal{U} \cup \{0\}$ described in Definition 2 2), for all $(t,z) \in \mathcal{T} \times D_{\frac{1}{2C_8}}$.

The Borel map $w_{;1}(u,z)$ represents a holomorphic function on $\mathcal{U} \times D_{\frac{1}{2C_8}}$. Its Taylor expansion

$$w_{;1}(u,z) = \sum_{n\geq 0} w_{n;1}(u) z^n/n! \quad (178)$$

suffers the next bounds

$$|w_{n;1}(u)| \leq C_7 (C_8)^n n! |u| \exp\left(k_1 \log^2(|u| + u_0) + \alpha \log(|u| + u_0)\right) \quad (179)$$

for all $n \geq 0$, all $u \in \mathcal{U} \cup \{0\}$, for fittingly chosen constants $C_7, k_1 > 0$ and $u_0 > 1, \alpha \geq 0$.

5.2 A limit nonlinear Cauchy problem

In this subsection, a novel Cauchy problem is introduced that we call the limit problem as $q > 1$ tends to 1. Its shape is displayed as follows

$$\partial_z^{\kappa} v_{;1}(t,z) = \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C}} d_{\underline{h}}(z) t^{h_0}((t^{k+1}\partial_t)^{h_1}\partial_z^{h_2} v_{;1})(t,z) + a(t,z)v_{;1}^2(t,z) + u_{;1}(t,z) \tag{180}$$

for given Cauchy data

$$(\partial_z^j v_{;1})(t,0) = \check{\varphi}_j(t) \quad , \quad 0 \leq j \leq \kappa - 1 \tag{181}$$

where the forcing term $u_{;1}(t,z)$ is the holomorphic solution of the singular linear Cauchy problem (175), (176) unveiled in the proposition 15 of the former subsection. Besides, all the items $\kappa, k, \mathcal{C}$ along with the coefficients $d_{\underline{h}}(z)$, $a(t,z)$ for $\underline{h} \in \mathcal{C}$ and the Cauchy data $\check{\varphi}_j(t)$ for $0 \leq j \leq \kappa - 1$ are those already introduced in Section 2.2.

We aim for a solution to (180), (181) having the profile of a Laplace transform of order k,

$$v_{;1}(t,z) = k\int_{L_\gamma} \Theta_{;1}(u,z)\exp(-(u/t)^k)du/u \tag{182}$$

along the halfline L_γ given in (177), where the Borel map $\Theta_{;1}(u,z)$ is holomorphic with respect to u on $\mathcal{U}$ and analytic w.r.t z on some small disc D_r centered at 0 with radius $r > 0$.

The same computations as the ones of Section 2.3 by means of Proposition 1 shows that the Borel map $\Theta_{;1}(u,z)$ solves that next auxiliary Cauchy problem

$$\begin{aligned} \partial_z^{\kappa}\Theta_{;1}(u,z) = &\sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)(ku^k)^{h_1}(\partial_z^{h_2}\Theta_{;1})(u,z) \\ &+ \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{u^k}{\Gamma(h_0/k)}\int_0^{u^k}(u^k-s)^{\frac{h_0}{k}-1}(ks)^{h_1}(\partial_z^{h_2}\Theta_{;1})(s^{1/k},z)\frac{ds}{s} \\ &+ a_0(z)u^k\int_0^{u^k}\Theta_{;1}((u^k-s)^{1/k},z)\Theta_{;1}(s^{1/k},z)\frac{1}{(u^k-s)s}ds \\ &+ \sum_{h=1}^{A} a_h(z)\frac{u^k}{\Gamma(h/k)}\int_0^{u^k}(u^k-s)^{\frac{h}{k}-1}\Big(s\int_0^s \Theta_{;1}((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)\frac{1}{(s-s_1)s_1}ds_1\Big)\frac{ds}{s} \\ &+ w_{;1}(u,z) \end{aligned} \tag{183}$$

for given Cauchy data

$$(\partial_z^j\Theta_{;1})(u,0) = \check{P}_j(u) \quad , \quad 0 \leq j \leq \kappa - 1 \tag{184}$$

For later need, we make the change of variables $s = u^k x$ and $s_1 = sx_1$ for $0 \leq x, x_1 \leq 1$ in the integrals involved in (183). As a result, $\Theta_{;1}(u,z)$ solves that next auxiliary Cauchy problem

$$(185)\quad \partial_z^{\kappa}\Theta_{;1}(u,z) = \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)(ku^k)^{h_1}(\partial_z^{h_2}\Theta_{;1})(u,z)$$
$$+ \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{u^{h_0+kh_1}}{\Gamma(h_0/k)}\int_0^1 (1-x)^{\frac{h_0}{k}-1}k^{h_1}x^{h_1}\partial_z^{h_2}\Theta_{;1}(ux^{1/k},z)\frac{dx}{x}$$
$$+ a_0(z)\int_0^1 \Theta_{;1}(u(1-x)^{1/k},z)\Theta_{;1}(ux^{1/k},z)\frac{dx}{(1-x)x}$$
$$+ \sum_{h=1}^{A} a_h(z)\frac{u^h}{\Gamma(h/k)}\int_0^1 (1-x)^{\frac{h}{k}-1}\Big(\int_0^1 \Theta_{;1}(ux^{1/k}(1-x_1)^{1/k},z)\Theta_{;1}(ux^{1/k}x_1^{1/k},z)\frac{dx_1}{(1-x_1)x_1}\Big)\frac{dx}{x}$$
$$+ w_{;1}(u,z)$$

for given Cauchy data (184).

The next proposition comes along with the same steps as in the proofs of Proposition 9 and Theorem 2.

Proposition 16 *Let $\underline{\mathcal{D}} = (\underline{\mathcal{T}},\underline{\mathcal{U}})$ be the admissible set of sectors distinguished in Proposition 15. Let $\mathcal{U}$ be the infinite sector belonging to $\underline{\mathcal{U}}$ selected in Proposition 15.*

A solution $v_{;1}(t,z)$ to the Cauchy problem (180), (181) can be reached, that is bounded and holomorphic on a domain $\mathcal{T}\times D_{Z_{0;1}/2}$ for some small radius $Z_{0;1}>0$, where $\mathcal{T}$ is the bounded sector from $\underline{\mathcal{T}}$ singled out in Proposition 15. Moreover, $v_{;1}(t,z)$ is described using a Laplace transform of order k,

$$(186)\qquad v_{;1}(t,z) = k\int_{L_\gamma}\Theta_{;1}(u,z)\exp(-(u/t)^k)du/u$$

for all $(t,z)\in\mathcal{T}\times D_{Z_{0;1}/2}$, along the halfline L_γ that appears in the representation (177). The Borel map $\Theta_{;1}(u,z)$ stands for a holomorphic map on the domain $\mathcal{U}\times D_{Z_{0;1}/2}$ which solves the above auxiliary Cauchy problem (183), (185) and (184). Besides, the map $\Theta_{;1}(u,z)$ belongs to the space $G^k_{(\sigma,Z_{0;1},\mathcal{U})}$ and enjoys a decomposition of the form

$$(187)\qquad \Theta_{;1}(u,z) = \partial_z^{-\kappa}\Xi_{;1}(u,z) + \check{\Psi}(u,z)$$

where $\check{\Psi}(u,z)$ is defined in (61) and $\Xi_{;1}(u,z)$ belongs to $G^k_{(\sigma,Z_{0;1},\mathcal{U})}$ and satisfies

$$(188)\qquad ||\Xi_{;1}(u,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \leq \upsilon_{;1}$$

for some suitable constant $\upsilon_{;1}>0$.

In the ensuing corollary, we observe that $v_{;1}(t,z)$ actually solves a limit nonlinear Cauchy problem with analytic coefficients in space z in the vicinity of the origin and polynomial in time t.

Corollary 2 *The analytic map $v_{;1}(t,z)$ solves on the product $\mathcal{T}\times D_{Z_{0;1}/2}$ a particular Cauchy problem relying polynomially on time t with the shape*

$$(189)\qquad P(t^{k+1}\partial_t)\partial_z^{S+\kappa}v_{;1}(t,z) = \check{G}(t,z,\{\partial_t^{r_1}\partial_z^{r_2}v_{;1}(t,z)\}_{\underline{r}=(r_1,r_2)\in\check{\mathcal{R}}})$$

for given Cauchy data of the form

$$(\partial_z^j v_{;1})(t,0) = \check{\varphi}_j(t) \quad , \quad 0 \leq j \leq \kappa - 1 \tag{190}$$

together with

$$(\partial_z^j v_{;1})(t,0) = \check{\phi}_{j;1}(t) \quad , \quad \kappa \leq j \leq \kappa + S - 1 \tag{191}$$

for well selected polynomials $\check{\phi}_{j;1}(t)$, $\kappa \leq j \leq \kappa + S - 1$ (that are independent from q). The set $\check{\mathcal{R}}$ is a finite subset of $\mathbb{N}^2$ for $(r_1, r_2) \in \check{\mathcal{R}}$ with the property that $r_2 < S + \kappa$ for any $(r_1, r_2) \in \check{\mathcal{R}}$. The map $\check{G}(t, z, (u_{\underline{r}})_{\underline{r} \in \check{\mathcal{R}}})$ is polynomial in time t and in its arguments $u_{\underline{r}}$ with $\underline{r} \in \check{\mathcal{R}}$ and analytic w.r.t z on $D_{Z_{0;1}/2}$.

Proof We set up the differential operators

$$\mathcal{D}_1(t, z, \partial_t, \partial_z) := P(t^{k+1}\partial_t)\partial_z^S - \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}} c_{\underline{l}}(z) t^{l_0} (t^{k+1}\partial_t)^{l_1} \partial_z^{l_2}$$

and

$$\mathcal{D}_2(t, z, \partial_t, \partial_z)v := \partial_z^{\kappa} - \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C}} d_{\underline{h}}(z) t^{h_0} ((t^{k+1}\partial_t)^{h_1} \partial_z^{h_2})v - a(t,z)v^2$$

According to Proposition 16, $v_{;1}(t, z)$ satisfies

$$\mathcal{D}_2(t, z, \partial_t, \partial_z) v_{;1}(t, z) = u_{;1}(t, z) \tag{192}$$

and based on Proposition 15, the map $u_{;1}(t, z)$ solves

$$\mathcal{D}_1(t, z, \partial_t, \partial_z) u_{;1}(t, z) = 0 \tag{193}$$

provided that $(t, z) \in \mathcal{T} \times D_{Z_{0;1}/2}$. By pairing (192) together with (193), we obtain

$$\mathcal{D}_1(t, z, \partial_t, \partial_z) \circ \mathcal{D}_2(t, z, \partial_t, \partial_z) v_{;1}(t, z) = 0 \tag{194}$$

that is represented by the equation (189). Regarding the Cauchy data, the map $v_{;1}$ suffers the conditions (181) which are replicated in (190). Moreover, the constraints (176) put on $u_{;1}(t, z)$ can be reworded as

$$(\partial_z^j \circ \mathcal{D}_2(t, z, \partial_t, \partial_z) v_{;1})(t, 0) = \varphi_j(t)$$

for $0 \leq j \leq S - 1$ which begets the assumptions (191) for properly chosen polynomials $\check{\phi}_{l;1}(t)$ with complex coefficients for $\kappa \leq l \leq \kappa + S - 1$. □

5.3 Analytic solutions to the limit singular linear Cauchy problem under the action of a $q-$difference operator

In this subsection, we take heed of a technical result achieved in our foregoing work [13] that will be called upon in the next subsection. We keep the notations of Subsection 5.1.

In Section 8.2 of [13], the next result is established.

Proposition 17 *Let $\beta \geq 1$ be an integer and let $q \in (1, q_0]$. Two constants $C_9, C_{10} > 0$ (that are unrelated to q) can be found such that*

$$|w_{n;1}(u) - w_{n;1}(q^{-\beta}u)| \leq |q^{-\beta} - 1| C_9 (C_{10})^n n! |u| \exp(k_1 \log^2(|u| + u_0) + \alpha \log(|u| + u_0)) \tag{195}$$

for all $u \in \mathcal{U} \cup \{0\}$, for all integers $n \geq 0$, where the constants $k_1 > 0$, $u_0 > 1$, $\alpha \geq 0$ and the unbounded sector $\mathcal{U}$ appear in Proposition 15.

5.4 Analytic solutions to the limit nonlinear Cauchy problem under the action of a $q-$difference operator

This subsection is dedicated to the proof of the next technical proposition.

Proposition 18 *Let $\beta \geq 1$ be an integer and set $q \in (1, q_0]$. Then, one can find some constants $\upsilon_{;1} > 0$ and $\tilde{M}_{1;1} > 0$ (unrelated to q) such that for any given $\sigma > 0$, if $\tilde{Z}_0 > 0$ is taken small enough, the next bounds*

$$||\Theta_{;1}(u,z) - \Theta_{;1}(q^{-\beta}u,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq \tilde{M}_{1;1}(1-q^{-\beta}) \tag{196}$$

hold.

Proof The proof is rather lengthy and is made up with several steps.

In the **first step** of the proof, we formulate a Cauchy problem, stated below in (206), (207), that the difference

$$\Delta^{\beta}\Theta_{;1}(u,z) := \Theta_{;1}(u,z) - \Theta_{;1}(q^{-\beta}u,z) \tag{197}$$

is compelled to solve. We first state a Cauchy problem satisfied by the quantity $\Theta_{;1}(q^{-\beta}u,z)$ by substituting u by $q^{-\beta}u$ in the equations (185), (184).

$$\begin{aligned} \partial_z^{\kappa}\Theta_{;1}(q^{-\beta}u,z) &= \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)(k(q^{-\beta}u)^k)^{h_1}(\partial_z^{h_2}\Theta_{;1})(q^{-\beta}u,z) \\ &+ \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{(q^{-\beta}u)^{h_0+kh_1}}{\Gamma(h_0/k)}\int_0^1 (1-x)^{\frac{h_0}{k}-1}k^{h_1}x^{h_1}\partial_z^{h_2}\Theta_{;1}(q^{-\beta}ux^{1/k},z)\frac{dx}{x} \\ &+ a_0(z)\int_0^1 \Theta_{;1}(q^{-\beta}u(1-x)^{1/k},z)\Theta_{;1}(q^{-\beta}ux^{1/k},z)\frac{dx}{(1-x)x} \\ &+\sum_{h=1}^{A} a_h(z)\frac{(q^{-\beta}u)^h}{\Gamma(h/k)}\int_0^1 (1-x)^{\frac{h}{k}-1}\left(\int_0^1 \Theta_{;1}(q^{-\beta}ux^{1/k}(1-x_1)^{1/k},z)\Theta_{;1}(q^{-\beta}ux^{1/k}x_1^{1/k},z)\frac{dx_1}{(1-x_1)x_1}\right)\frac{dx}{x} \\ &+ w_{;1}(q^{-\beta}u,z) \end{aligned} \tag{198}$$

$$(\partial_z^j\Theta_{;1})(q^{-\beta}u,0) = \check{P}_j(q^{-\beta}u) \quad , \quad 0 \leq j \leq \kappa-1 \tag{199}$$

On the way, we need to perform some practical computations. Namely, using the basic identity $ab - cd = (a-c)b + c(b-d)$, we recast the next list of pieces in a suitable manner :

$$\begin{aligned} &(ku^k)^{h_1}\partial_z^{h_2}\Theta_{;1}(u,z) - (k(q^{-\beta}u)^k)^{h_1}(\partial_z^{h_2}\Theta_{;1})(q^{-\beta}u,z) \\ &= \left[(ku^k)^{h_1} - (k(q^{-\beta}u)^k)^{h_1}\right]\partial_z^{h_2}\Theta_{;1}(u,z) + (k(q^{-\beta}u)^k)^{h_1}\left[\partial_z^{h_2}\Theta_{;1}(u,z) - \partial_z^{h_2}\Theta_{;1}(q^{-\beta}u,z)\right] \end{aligned} \tag{200}$$

and

$$\begin{aligned} &u^{h_0+kh_1}\int_0^1 (1-x)^{\frac{h_0}{k}-1}x^{h_1}\partial_z^{h_2}\Theta_{;1}(ux^{1/k},z)\frac{dx}{x} \\ &\quad - (q^{-\beta}u)^{h_0+kh_1}\int_0^1 (1-x)^{\frac{h_0}{k}-1}x^{h_1}\partial_z^{h_2}\Theta_{;1}(q^{-\beta}ux^{1/k},z)\frac{dx}{x} \\ &= \left[u^{h_0+kh_1} - (q^{-\beta}u)^{h_0+kh_1}\right]\int_0^1 (1-x)^{\frac{h_0}{k}-1}x^{h_1}\partial_z^{h_2}\Theta_{;1}(ux^{1/k},z)\frac{dx}{x} \\ &+ (q^{-\beta}u)^{h_0+kh_1}\int_0^1 (1-x)^{\frac{h_0}{k}-1}x^{h_1}\left[\partial_z^{h_2}\Theta_{;1}(ux^{1/k},z) - \partial_z^{h_2}\Theta_{;1}(q^{-\beta}ux^{1/k},z)\right]\frac{dx}{x} \end{aligned} \tag{201}$$

along with

$$
(202) \quad \int_0^1 \Theta_{;1}(u(1-x)^{1/k},z)\Theta_{;1}(ux^{1/k},z)\frac{dx}{(1-x)x} - \int_0^1 \Theta_{;1}(q^{-\beta}u(1-x)^{1/k},z)\Theta_{;1}(q^{-\beta}ux^{1/k},z)\frac{dx}{(1-x)x}
$$

$$
= \int_0^1 \Big(\big[\Theta_{;1}(u(1-x)^{1/k},z) - \Theta_{;1}(q^{-\beta}u(1-x)^{1/k},z)\big]\Theta_{;1}(ux^{1/k},z) + \Theta_{;1}(q^{-\beta}u(1-x)^{1/k},z)\big[\Theta_{;1}(ux^{1/k},z) - \Theta_{;1}(q^{-\beta}ux^{1/k},z)\big]\Big)\frac{dx}{(1-x)x}
$$

and

$$
(203) \quad u^h \int_0^1 (1-x)^{\frac{h}{k}-1}\Big(\int_0^1 \Theta_{;1}(ux^{1/k}(1-x_1)^{1/k},z)\Theta_{;1}(ux^{1/k}x_1^{1/k},z)\frac{dx_1}{(1-x_1)x_1}\Big)\frac{dx}{x}
$$

$$
- (q^{-\beta}u)^h \int_0^1 (1-x)^{\frac{h}{k}-1}\Big(\int_0^1 \Theta_{;1}(q^{-\beta}ux^{1/k}(1-x_1)^{1/k},z)\Theta_{;1}(q^{-\beta}ux^{1/k}x_1^{1/k},z)\frac{dx_1}{(1-x_1)x_1}\Big)\frac{dx}{x}
$$

$$
= \Big[u^h - (q^{-\beta}u)^h\Big] \int_0^1 (1-x)^{\frac{h}{k}-1}\Big(\int_0^1 \Theta_{;1}(ux^{1/k}(1-x_1)^{1/k},z)\Theta_{;1}(ux^{1/k}x_1^{1/k},z)\frac{dx_1}{(1-x_1)x_1}\Big)\frac{dx}{x}
$$

$$
+ (q^{-\beta}u)^h \int_0^1 (1-x)^{\frac{h}{k}-1}\Big[\int_0^1 \Big\{ \Big(\Theta_{;1}(ux^{1/k}(1-x_1)^{1/k},z) - \Theta_{;1}(q^{-\beta}ux^{1/k}(1-x_1)^{1/k},z)\Big) \times \Theta_{;1}(ux^{1/k}x_1^{1/k},z) + \Theta_{;1}(q^{-\beta}ux^{1/k}(1-x_1)^{1/k},z) \times \Big(\Theta_{;1}(ux^{1/k}x_1^{1/k},z) - \Theta_{;1}(q^{-\beta}ux^{1/k}x_1^{1/k},z)\Big)\Big\}\frac{dx_1}{(1-x_1)x_1}\Big]\frac{dx}{x}
$$

Based on the two equations (185), (198) together with the prescribed data (184), (199) and thanks to the above computations (200), (201), (202), (203), we can now exhibit a Cauchy problem fulfilled by the difference $\Delta^{\beta}\Theta_{;1}(u,z)$ given in (197) owning the following shape :

$$
\begin{aligned}
(204)\quad \partial_z^{\kappa}\Delta^{\beta}\Theta_{;1}(u,z) = &\sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)\Big\{\big[(ku^k)^{h_1} - (k(q^{-\beta}u)^k)^{h_1}\big]\partial_z^{h_2}\Theta_{;1}(u,z)\\
&+ (k(q^{-\beta}u)^k)^{h_1}\partial_z^{h_2}\Delta^{\beta}\Theta_{;1}(u,z)\Big\} + \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{k^{h_1}}{\Gamma(h_0/k)}\Big\{\big[u^{h_0+kh_1} - (q^{-\beta}u)^{h_0+kh_1}\big]\\
&\times\int_0^1 (1-x)^{\frac{h_0}{k}-1}x^{h_1}\partial_z^{h_2}\Theta_{;1}(ux^{1/k},z)\frac{dx}{x} + (q^{-\beta}u)^{h_0+kh_1}\int_0^1 (1-x)^{\frac{h_0}{k}-1}x^{h_1}(\partial_z^{h_2}\Delta^{\beta}\Theta_{;1})(ux^{1/k},z)\frac{dx}{x}\Big\}\\
&+a_0(z)\int_0^1 \Big\{\Delta^{\beta}\Theta_{;1}(u(1-x)^{1/k},z)\Theta_{;1}(ux^{1/k},z) + \Theta_{;1}(q^{-\beta}u(1-x)^{1/k},z)\Delta^{\beta}\Theta_{;1}(ux^{1/k},z)\Big\}\frac{dx}{(1-x)x}\\
&+\sum_{h=1}^{A} a_h(z)\frac{1}{\Gamma(h/k)}\Big\{(u^h - (q^{-\beta}u)^h)\Big[\int_0^1 (1-x)^{\frac{h}{k}-1}\\
&\times\Big(\int_0^1 \Theta_{;1}(ux^{1/k}(1-x_1)^{1/k},z)\Theta_{;1}(ux^{1/k}x_1^{1/k},z)\frac{dx_1}{(1-x_1)x_1}\Big)\frac{dx}{x}\Big]\\
&+ (q^{-\beta}u)^h\int_0^1 (1-x)^{\frac{h}{k}-1}\Big[\int_0^1 \{\Delta^{\beta}\Theta_{;1}(ux^{1/k}(1-x_1)^{1/k},z)\Theta_{;1}(ux^{1/k}x_1^{1/k},z)\\
&+ \Theta_{;1}(q^{-\beta}ux^{1/k}(1-x_1)^{1/k},z)\Delta^{\beta}\Theta_{;1}(ux^{1/k}x_1^{1/k},z)\}\frac{dx_1}{(1-x_1)x_1}\Big]\frac{dx}{x}\Big\}\\
&+ w_{;1}(u,z) - w_{;1}(q^{-\beta}u,z)
\end{aligned}
$$

for given Cauchy data

$$
(205)\qquad (\partial_z^j\Delta^{\beta}\Theta_{;1})(u,0) = \check{P}_j(u) - \check{P}_j(q^{-\beta}u)\quad,\quad 0\leq j\leq \kappa-1.
$$

In the upcoming step, we plan to solve the problem (204), (205) within the Banach space described in Subsection 3.1. In the first instance, we need to rephrase the integral operators involved in (204) in terms of those appearing in Subsection 3.1 by means of the parametrization $s = u^k x$, $s_1 = sx_1$ for $0\leq x, x_1\leq 1$. Indeed, we deduce that $\Delta^{\beta}\Theta_{;1}(u,z)$ turns out to solve the next *linear* Cauchy problem

$$
\begin{aligned}
(206)\quad \partial_z^{\kappa}\Delta^{\beta}\Theta_{;1}(u,z) = &\sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)\Big\{k^{h_1}\left[1-q^{-\beta k h_1}\right]u^{kh_1}\partial_z^{h_2}\Theta_{;1}(u,z)\\
&+(k(q^{-\beta}u)^k)^{h_1}\partial_z^{h_2}\Delta^{\beta}\Theta_{;1}(u,z)\Big\} + \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{k^{h_1}}{\Gamma(h_0/k)}\Big\{\left[1-q^{-\beta(h_0+kh_1)}\right]\\
&\times u^k\int_0^{u^k}(u^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)\frac{ds}{s}\\
&+q^{-\beta(h_0+kh_1)}u^k\int_0^{u^k}(u^k-s)^{\frac{h_0}{k}-1}s^{h_1}(\partial_z^{h_2}\Delta^{\beta}\Theta_{;1})(s^{1/k},z)\frac{ds}{s}\Big\}\\
&+a_0(z)u^k\int_0^{u^k}\Big\{\Delta^{\beta}\Theta_{;1}((u^k-s)^{1/k},z)\Theta_{;1}(s^{1/k},z)+\Theta_{;1}(q^{-\beta}(u^k-s)^{1/k},z)\Delta^{\beta}\Theta_{;1}(s^{1/k},z)\Big\}\frac{ds}{(u^k-s)s}\\
&+\sum_{h=1}^{A}a_h(z)\frac{1}{\Gamma(h/k)}\Big\{\left[1-q^{-\beta h}\right]u^k\int_0^{u^k}(u^k-s)^{\frac{h}{k}-1}\Big(s\int_0^s\Theta_{;1}((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)\frac{ds_1}{(s-s_1)s_1}\Big)\frac{ds}{s}\\
&+q^{-\beta h}u^k\int_0^{u^k}(u^k-s)^{\frac{h}{k}-1}\Big[s\int_0^s\big\{\Delta^{\beta}\Theta_{;1}((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)\\
&+\Theta_{;1}(q^{-\beta}(s-s_1)^{1/k},z)\Delta^{\beta}\Theta_{;1}(s_1^{1/k},z)\big\}\frac{ds_1}{(s-s_1)s_1}\Big]\frac{ds}{s}\Big\}\\
&+w_{;1}(u,z)-w_{;1}(q^{-\beta}u,z)
\end{aligned}
$$

for assigned Cauchy data

$$
(207)\qquad (\partial_z^j\Delta^{\beta}\Theta_{;1})(u,0) = \check{P}_j(u)-\check{P}_j(q^{-\beta}u)\quad ,\quad 0\leq j\leq \kappa-1.
$$

In the **second step** of the proof, we seek for a solution to the above problem (206), (207) expressed by means of the next splitting

$$
(208)\qquad \Delta^{\beta}\Theta_{;1}(u,z) = \partial_z^{-\kappa}\Delta^{\beta}\Xi_{;1}(u,z)+\Delta^{\beta}\check{\Psi}(u,z)
$$

for some expression $\Delta^{\beta}\Xi_{;1}(u,z)$ to be determined, where

$$
(209)\qquad \Delta^{\beta}\check{\Psi}(u,z) = \sum_{j=0}^{\kappa-1}\Delta^{\beta}\check{P}_j(u)\frac{z^j}{j!}
$$

for $\Delta^{\beta}\check{P}_j(u) = \check{P}_j(u)-\check{P}_j(q^{-\beta}u)$, with $0\leq j\leq \kappa-1$.

We pinpoint the crucial fact that $\Delta^{\beta}\Theta_{;1}(u,z)$ solves the problem (206), (207) if the quantity

$\Delta^{\beta}\Xi_{;1}(u,z)$ satisfies the next *fixed point equation*

$$
\begin{aligned}
(210)\quad \Delta^{\beta}\Xi_{;1}(u,z) = &\sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)\Big\{ k^{h_1}\left[1-q^{-\beta k h_1}\right] u^{kh_1}\partial_z^{h_2}\Theta_{;1}(u,z) \\
&+ (k(q^{-\beta}u)^k)^{h_1}\left(\partial_z^{-(\kappa-h_2)}\Delta^{\beta}\Xi_{;1}(u,z) + \partial_z^{h_2}\Delta^{\beta}\check{\Psi}(u,z)\right)\Big\} \\
&+ \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{k^{h_1}}{\Gamma(h_0/k)}\Big\{\left[1-q^{-\beta(h_0+kh_1)}\right] \\
&\times u^k \int_0^{u^k} (u^k-s)^{\frac{h_0}{k}-1} s^{h_1}\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)\frac{ds}{s} \\
&+ q^{-\beta(h_0+kh_1)}u^k\int_0^{u^k}(u^k-s)^{\frac{h_0}{k}-1}s^{h_1}\left(\partial_z^{-(\kappa-h_2)}\Delta^{\beta}\Xi_{;1}(s^{1/k},z) + \partial_z^{h_2}\Delta^{\beta}\check{\Psi}(s^{1/k},z)\right)\frac{ds}{s}\Big\} \\
&+ a_0(z)u^k\int_0^{u^k}\Big\{\left(\partial_z^{-\kappa}\Delta^{\beta}\Xi_{;1}((u^k-s)^{1/k},z) + \Delta^{\beta}\check{\Psi}((u^k-s)^{1/k},z)\right)\Theta_{;1}(s^{1/k},z) \\
&+ \Theta_{;1}(q^{-\beta}(u^k-s)^{1/k},z)\left(\partial_z^{-\kappa}\Delta^{\beta}\Xi_{;1}(s^{1/k},z) + \Delta^{\beta}\check{\Psi}(s^{1/k},z)\right)\Big\}\frac{ds}{(u^k-s)s} \\
&+\sum_{h=1}^{A} a_h(z)\frac{1}{\Gamma(h/k)}\Big\{\left[1-q^{-\beta h}\right]u^k\int_0^{u^k}(u^k-s)^{\frac{h}{k}-1}\Big(s\int_0^s \Theta_{;1}((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)\frac{ds_1}{(s-s_1)s_1}\Big)\frac{ds}{s} \\
&+ q^{-\beta h}u^k\int_0^{u^k}(u^k-s)^{\frac{h}{k}-1}\Big[s\int_0^s\Big\{\left(\partial_z^{-\kappa}\Delta^{\beta}\Xi_{;1}((s-s_1)^{1/k},z) + \Delta^{\beta}\check{\Psi}((s-s_1)^{1/k},z)\right)\Theta_{;1}(s_1^{1/k},z) \\
&+ \Theta_{;1}(q^{-\beta}(s-s_1)^{1/k},z)\left(\partial_z^{-\kappa}\Delta^{\beta}\Xi_{;1}(s_1^{1/k},z) + \Delta^{\beta}\check{\Psi}(s_1^{1/k},z)\right)\Big\}\frac{ds_1}{(s-s_1)s_1}\Big]\frac{ds}{s}\Big\} \\
&+ w_{;1}(u,z) - w_{;1}(q^{-\beta}u,z)
\end{aligned}
$$

Our next duty will be to search for a solution and *provide bounds relatively to q* for a solution to (210) in the Banach space $G^k_{(\sigma,\tilde{Z}_0,\mathcal{U})}$, for any given $\sigma > 0$, provided that $\tilde{Z}_0 > 0$ is chosen small enough. With that in mind, we introduce the linear map

$$
\begin{aligned}
(211)\quad \mathfrak{D}(\Delta^{\beta}\Xi(\tau,z)) = & \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)\Big\{ k^{h_1}\big[1-q^{-\beta k h_1}\big]\tau^{kh_1}\partial_z^{h_2}\Theta_{;1}(\tau,z)\\
& + (k(q^{-\beta}\tau)^k)^{h_1}\Big(\partial_z^{-(\kappa-h_2)}\Delta^{\beta}\Xi(\tau,z) + \partial_z^{h_2}\Delta^{\beta}\check{\Psi}(\tau,z)\Big)\Big\}\\
& + \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{k^{h_1}}{\Gamma(h_0/k)}\Big\{\big[1-q^{-\beta(h_0+kh_1)}\big]\\
& \times \tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)\frac{ds}{s}\\
& + q^{-\beta(h_0+kh_1)}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\Big(\partial_z^{-(\kappa-h_2)}\Delta^{\beta}\Xi(s^{1/k},z)+\partial_z^{h_2}\Delta^{\beta}\check{\Psi}(s^{1/k},z)\Big)\frac{ds}{s}\Big\}\\
& + a_0(z)\tau^k\int_0^{\tau^k}\Big\{\Big(\partial_z^{-\kappa}\Delta^{\beta}\Xi((\tau^k-s)^{1/k},z)+\Delta^{\beta}\check{\Psi}((\tau^k-s)^{1/k},z)\Big)\Theta_{;1}(s^{1/k},z)\\
& + \Theta_{;1}(q^{-\beta}(\tau^k-s)^{1/k},z)\Big(\partial_z^{-\kappa}\Delta^{\beta}\Xi(s^{1/k},z)+\Delta^{\beta}\check{\Psi}(s^{1/k},z)\Big)\Big\}\frac{ds}{(\tau^k-s)s}\\
& + \sum_{h=1}^{A} a_h(z)\frac{1}{\Gamma(h/k)}\Big\{\big[1-q^{-\beta h}\big]\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h}{k}-1}\Big(s\int_0^s\Theta_{;1}((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)\frac{ds_1}{(s-s_1)s_1}\Big)\frac{ds}{s}\\
& + q^{-\beta h}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h}{k}-1}\Big[s\int_0^s\Big\{\big(\partial_z^{-\kappa}\Delta^{\beta}\Xi((s-s_1)^{1/k},z)+\Delta^{\beta}\check{\Psi}((s-s_1)^{1/k},z)\big)\Theta_{;1}(s_1^{1/k},z)\\
& + \Theta_{;1}(q^{-\beta}(s-s_1)^{1/k},z)\big(\partial_z^{-\kappa}\Delta^{\beta}\Xi(s_1^{1/k},z)+\Delta^{\beta}\check{\Psi}(s_1^{1/k},z)\big)\Big\}\frac{ds_1}{(s-s_1)s_1}\Big]\frac{ds}{s}\Big\}\\
& + w_{;1}(\tau,z) - w_{;1}(q^{-\beta}\tau,z)
\end{aligned}
$$

In the next lemma, we discuss sufficient conditions under which $\mathfrak{D}$ acts as a shrinking map on a small ball centered at 0, *whose radius depends on* q, in the space $G^k_{(\sigma,\tilde{Z}_0,\mathcal{U})}$.

Lemma 7 *We take for granted that the conditions (19) hold. Then, one can single out a small real number* $\tilde{\chi} > 0$ *in a way that if* $0 < \tilde{Z}_0 < \tilde{\chi}$*, one can select a radius* $\tilde{\upsilon} > 0$ *which is independent of* q *(but relies on* q_0 *such that* $1 < q \leq q_0$*), such that* $\mathfrak{D}$ *possesses the next two qualities: Let* $B_{\tilde{\upsilon}(1-q^{-\beta})}$ *be a ball centered at 0 with radius* $\tilde{\upsilon}(1-q^{-\beta})$

1. $\mathfrak{D}$ *maps* $B_{\tilde{\upsilon}(1-q^{-\beta})}$ *into itself, meaning that*

$$(212)\qquad \mathfrak{D}(B_{\tilde{\upsilon}(1-q^{-\beta})}) \subset B_{\tilde{\upsilon}(1-q^{-\beta})}$$

2. *For any* $\Delta^{\beta}\Xi_1, \Delta^{\beta}\Xi_2 \in B_{\tilde{\upsilon}(1-q^{-\beta})}$*, we have*

$$(213)\qquad ||\mathfrak{D}(\Delta^{\beta}\Xi_1) - \mathfrak{D}(\Delta^{\beta}\Xi_2)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq \frac{1}{2}||\Delta^{\beta}\Xi_1 - \Delta^{\beta}\Xi_2||_{(\sigma,\tilde{Z}_0,\mathcal{U})}$$

Proof As a prefatory material, norm bounds estimates are required for some pieces of the map $\mathfrak{D}$.

a) Indeed, at first we need estimates for the norm $||w_{;1}(\tau,z)-w_{;1}(q^{-\beta}\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}$. According to the expansion (178) and the definition of the norm, we get

$$||w_{;1}(\tau,z)-w_{;1}(q^{-\beta}\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} = \sum_{n\geq 0} ||w_{n;1}(\tau)-w_{n;1}(q^{-\beta}\tau)||_{(n,\sigma,\mathcal{U})}\frac{\tilde{Z}_0^n}{n!}$$

and owing to the bounds (195) brought to mind in Proposition 17, we can upper bound the coefficients of this latter series by

$$||w_{n;1}(\tau)-w_{n;1}(q^{-\beta}\tau)||_{(n,\sigma,\mathcal{U})} = \sup_{\tau\in\mathcal{U}} \frac{1+|\tau|^{2k}}{|\tau|}\exp(-\sigma r_b(n)|\tau|^k)|w_{n;1}(\tau)-w_{n;1}(q^{-\beta}\tau)|$$
$$\leq (1-q^{-\beta})C_9(C_{10})^n n! \sup_{x\geq 0}(1+x^{2k})\exp(-\sigma x^k)\exp(k_1\log^2(x+u_0)+\alpha\log(x+u_0))$$
$$\leq (1-q^{-\beta})\tilde{C}_9(C_{10})^n n!$$

for some constant $\tilde{C}_9$ (unrelated to q), for all integers $n\geq 0$. Thereupon, it leads to

$$||w_{;1}(\tau,z)-w_{;1}(q^{-\beta}\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq (1-q^{-\beta})\tilde{C}_9\sum_{n\geq 0}(C_{10}\tilde{Z}_0)^n \leq 2(1-q^{-\beta})\tilde{C}_9 \tag{214}$$

provided that $\tilde{Z}_0 < \frac{1}{2C_{10}}$.

b) We focus on the quantities $||\tau^{kh_1}\partial_z^{h_2}\Theta_{;1}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}$ for integers $h_1,h_2\geq 0$ under the constraint (19) and on $||\Theta_{;1}(q^{-\beta}\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}$.

According to the decomposition (187) with the bounds (188) and owing to Propositions 2 and 6, we obtain

$$||\tau^{kh_1}\partial_z^{h_2}\Theta_{;1}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq ||\tau^{kh_1}\partial_z^{h_2}\Theta_{;1}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \tag{215}$$
$$\leq ||\tau^{kh_1}\partial_z^{-(\kappa-h_2)}\Xi_{;1}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} + ||\tau^{kh_1}\partial_z^{h_2}\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}$$
$$\leq M_1 Z_{0;1}^{\kappa-h_2}\upsilon_{;1} + ||\tau^{kh_1}\partial_z^{h_2}\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}$$

along with

$$||\Theta_{;1}(q^{-\beta}\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq ||\Theta_{;1}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \tag{216}$$
$$\leq ||(\partial_z^{-\kappa}\Xi_{;1})(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} + ||\check{\Psi}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}$$
$$\leq M_1 Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}$$

for some constant $M_1>0$ (independent of q but relying on q_0 such that $1<q\leq q_0$) provided that $\tilde{Z}_0 < Z_{0;1}$.

c) We ask for sharp bounds for the norms $||\tau^{kh_1}\partial_z^{h_2}\Delta^{\beta}\check{\Psi}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}$, for integers $h_1,h_2\geq 0$ under the constraint (19). Departing from the expansion (209), we observe that

$$\tau^{kh_1}\partial_z^{h_2}\Delta^{\beta}\check{\Psi}(\tau,z) = \sum_{j=0}^{\kappa-1-h_2}\tau^{kh_1}\Delta^{\beta}\check{P}_{j+h_2}(\tau)\frac{z^j}{j!}$$

and its norm writes

$$||\tau^{kh_1}\partial_z^{h_2}\Delta^{\beta}\check{\Psi}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} = \sum_{j=0}^{\kappa-1-h_2}||\tau^{kh_1}\Delta^{\beta}\check{P}_{j+h_2}(\tau)||_{(j,\sigma,\mathcal{U})}\frac{\tilde{Z}_0^j}{j!} \tag{217}$$

where

$$(218)\quad ||\tau^{kh_1}\Delta^{\beta}\check{P}_{j+h_2}(\tau)||_{(j,\sigma,\mathcal{U})} = \sup_{\tau\in\mathcal{U}} \frac{1+|\tau|^{2k}}{|\tau|} \exp(-\sigma r_b(j)|\tau|^k)|\tau|^{kh_1}|\Delta^{\beta}\check{P}_{j+h_2}(\tau)| \leq \sup_{\tau\in\mathcal{U}} \frac{1+|\tau|^{2k}}{|\tau|} \exp(-\sigma|\tau|^k)|\tau|^{kh_1}|\Delta^{\beta}\check{P}_{j+h_2}(\tau)|$$

since $r_b(j) \geq 1$, for all $j \geq 0$. Furthermore, we can recast the difference as an integral

$$(219)\quad \Delta^{\beta}\check{P}_{j+h_2}(\tau) = \int_{q^{-\beta}\tau}^{\tau} \check{P}'_{j+h_2}(s)ds = (1-q^{-\beta})\tau \int_0^1 \check{P}'_{j+h_2}(\tau h + q^{-\beta}\tau(1-h))dh$$

by means of the parametrization $s = \tau h + q^{-\beta}\tau(1-h)$ with $0 \leq h \leq 1$. Since $\check{P}_{j+h_2}$ is a polynomial with complex coefficients, its derivative can be writen in the form

$$\check{P}'_{j+h_2}(s) = \sum_{h\in\check{J}'_{j+h_2}} \check{P}'_{j+h_2,h}s^h$$

for some finite subset $\check{J}'_{j+h_2}$ of $\mathbb{N}$ and complex coefficients $\check{P}'_{j+h_2,h}$. We set

$$|\check{P}'_{j+h_2}|(x) = \sum_{h\in\check{J}'_{j+h_2}} |\check{P}'_{j+h_2,h}|x^h$$

and since

$$|\tau h + q^{-\beta}\tau(1-h)| \leq |\tau|h + |q^{-\beta}||\tau|(1-h) \leq |\tau|$$

for all $\tau \in \mathbb{C}$, all $0 \leq h \leq 1$, we get in particular that

$$(220)\quad |\check{P}'_{j+h_2}(\tau h + q^{-\beta}\tau(1-h))| \leq |\check{P}'_{j+h_2}|(|\tau|)$$

whenever $\tau \in \mathbb{C}$ and $0 \leq h \leq 1$. We deduce from (219) and (220) that

$$(221)\quad |\Delta^{\beta}\check{P}_{j+h_2}(\tau)| \leq (1-q^{-\beta})|\tau||\check{P}'_{j+h_2}|(|\tau|)$$

for all $\tau \in \mathbb{C}$. Thereupon, gathering (218) and (221) yields a constant $M_{k,h_1,j,h_2} > 0$ for which

$$(222)\quad ||\tau^{kh_1}\Delta^{\beta}\check{P}_{j+h_2}(\tau)||_{(j,\sigma,\mathcal{U})} \leq (1-q^{-\beta}) \sup_{x\geq 0}(1+x^{2k})\exp(-\sigma x^k)x^{kh_1}|\check{P}'_{j+h_2}|(x) \leq (1-q^{-\beta})M_{k,h_1,j,h_2}$$

Keeping in mind (217), we finally get a constant $N_{k,h_1,h_2,\kappa,\tilde{Z}_0} > 0$ such that

$$(223)\quad ||\tau^{kh_1}\partial_z^{h_2}\Delta^{\beta}\check{\Psi}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq (1-q^{-\beta})N_{k,h_1,h_2,\kappa,\tilde{Z}_0}$$

d) Bounds for the quantity $||\tau^k \int_0^{\tau^k}(\tau^k - s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})}$ are also required, for integers $h_0,h_1,h_2 \geq 0$ under the constraint (19). Based on the decomposition (187) with the bounds (188), Propositions 4 and 6 allow to get a constant $M_3 > 0$ with

$$(224)\quad ||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1} \partial_z^{h_2} \Theta_{;1}(s^{1/k}, z) \frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})}$$
$$\leq ||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1} \partial_z^{h_2} \Theta_{;1}(s^{1/k}, z) \frac{ds}{s}||_{(\sigma,Z_{0;1},\mathcal{U})}$$
$$\leq ||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1} \partial_z^{-(\kappa-h_2)} \Xi_{;1}(s^{1/k}, z) \frac{ds}{s}||_{(\sigma,Z_{0;1},\mathcal{U})}$$
$$+ ||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1} \partial_z^{h_2} \check{\Psi}(s^{1/k}, z) \frac{ds}{s}||_{(\sigma,Z_{0;1},\mathcal{U})}$$
$$\leq M_3 Z_{0;1}^{\kappa-h_2} \upsilon_{;1} + ||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1} \partial_z^{h_2} \check{\Psi}(s^{1/k}, z) \frac{ds}{s}||_{(\sigma,Z_{0;1},\mathcal{U})}$$

presuming that $\tilde{Z}_0 < Z_{0;1}$.

e) We demande also accurate estimates for the norms

$$\Delta_I := ||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1} \partial_z^{h_2} \Delta^{\beta} \check{\Psi}(s^{1/k}, z) \frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})}$$

for integers $h_0, h_1, h_2 \geq 0$ under the constraint (19). From the expansion (209), we check that

$$\partial_z^{h_2} \Delta^{\beta} \check{\Psi}(s^{1/k}, z) = \sum_{j=0}^{\kappa-1-h_2} \Delta^{\beta} \check{P}_{j+h_2}(s^{1/k}) \frac{z^j}{j!}$$

and therefore

$$(225)\qquad \Delta_I = \sum_{j=0}^{\kappa-1-h_2} ||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1-1} \Delta^{\beta} \check{P}_{j+h_2}(s^{1/k}) ds||_{(j,\sigma,\mathcal{U})} \frac{\tilde{Z}_0^j}{j!}$$

On the basis of the upper bounds (221), by means of the change of variable $h = |\tau|^k x$ with $0 \leq x \leq 1$ and keeping in mind the lower bounds $r_b(j) \geq 1$, we deduce a constant $\bar{M}_{k,h_0,h_1,h_2,j} > 0$ for which

$$(226)\quad ||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1-1} \Delta^{\beta} \check{P}_{j+h_2}(s^{1/k}) ds||_{(j,\sigma,\mathcal{U})}$$
$$\leq \sup_{\tau \in \mathcal{U}} \frac{1 + |\tau|^{2k}}{|\tau|} \exp(-\sigma r_b(j)|\tau|^k)|\tau|^k \int_0^{|\tau|^k} (|\tau|^k - h)^{\frac{h_0}{k}-1} h^{h_1-1} (1 - q^{-\beta}) h^{1/k} |\check{P}'_{j+h_2}|(h^{1/k}) dh$$
$$\leq (1 - q^{-\beta}) \sup_{y \geq 0} (1 + y^{2k}) \exp(-\sigma y^k) y^{h_0+kh_1} |\check{P}'_{j+h_2}|(y) \int_0^1 (1 - x)^{\frac{h_0}{k}-1} x^{h_1-1+\frac{1}{k}} dx$$
$$\leq (1 - q^{-\beta}) \bar{M}_{k,h_0,h_1,h_2,j}$$

At last, pairing (225) with (226) gives rise to a constant $\bar{N}_{k,h_0,h_1,h_2,\kappa,\tilde{Z}_0} > 0$ such that

$$(227)\qquad ||\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1} s^{h_1} \partial_z^{h_2} \Delta^{\beta} \check{\Psi}(s^{1/k}, z) \frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq (1 - q^{-\beta}) \bar{N}_{k,h_0,h_1,h_2,\kappa,\tilde{Z}_0}.$$

We are now ready to come to the core of the proof. We fix our attention on the first item 1.

Let $\Delta^{\beta}\Xi$ belong to $G^{k}_{(\sigma,\tilde{Z}_0,\mathcal{U})}$ such that

$$||\Delta^{\beta}\Xi(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq \tilde{\upsilon}(1-q^{-\beta}) \tag{228}$$

We provide explicit bounds for each block of the map $\mathfrak{D}(\Delta^{\beta}\Xi)$.
Proposition 5 and (215) return

$$\begin{aligned} &||d_{\underline{h}}(z)k^{h_1}\big[1-q^{-\beta k h_1}\big]\tau^{kh_1}\partial_z^{h_2}\Theta_{;1}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\ &\leq k^{h_1}\big[1-q^{-\beta k h_1}\big]|d_{\underline{h}}|(\tilde{Z}_0)\Big(M_1 Z_{0;1}^{\kappa-h_2}\upsilon_{;1} + ||\tau^{kh_1}\partial_z^{h_2}\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\Big) \end{aligned} \tag{229}$$

Propositions 2 and 5 together with (223) trigger

$$\begin{aligned} &||d_{\underline{h}}(z)k^{h_1}q^{-\beta k h_1}\tau^{kh_1}\Big(\partial_z^{-(\kappa-h_2)}\Delta^{\beta}\Xi(\tau,z)+\partial_z^{h_2}\Delta^{\beta}\check{\Psi}(\tau,z)\Big)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\ &\leq |d_{\underline{h}}|(\tilde{Z}_0)k^{h_1}q^{-\beta k h_1}\Big(M_1\tilde{Z}_0^{\kappa-h_2}||\Delta^{\beta}\Xi(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} + (1-q^{-\beta})N_{k,h_1,h_2,\kappa,\tilde{Z}_0}\Big) \end{aligned} \tag{230}$$

Proposition 5 and (224) afford

$$\begin{aligned} &||d_{\underline{h}}(z)\frac{k^{h_1}}{\Gamma(h_0/k)}[1-q^{-\beta(h_0+kh_1)}]\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\ &\leq |d_{\underline{h}}|(\tilde{Z}_0)\frac{k^{h_1}}{\Gamma(h_0/k)}[1-q^{-\beta(h_0+kh_1)}] \\ &\times\Big(M_3 Z_{0;1}^{\kappa-h_2}\upsilon_{;1} + ||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\check{\Psi}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,Z_{0;1},\mathcal{U})}\Big) \end{aligned} \tag{231}$$

Propositions 4 and 5 along with (227) prompt

$$\begin{aligned} &||d_{\underline{h}}(z)\frac{k^{h_1}}{\Gamma(h_0/k)}q^{-\beta(h_0+kh_1)}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1} \\ &\times\Big(\partial_z^{-(\kappa-h_2)}\Delta^{\beta}\Xi(s^{1/k},z)+\partial_z^{h_2}\Delta^{\beta}\check{\Psi}(s^{1/k},z)\Big)\frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\ &\leq |d_{\underline{h}}|(\tilde{Z}_0)\frac{k^{h_1}}{\Gamma(h_0/k)}q^{-\beta(h_0+kh_1)}\Big(M_3\tilde{Z}_0^{\kappa-h_2}||\Delta^{\beta}\Xi(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} + (1-q^{-\beta})\bar{N}_{k,h_0,h_1,h_2,\kappa,\tilde{Z}_0}\Big) \end{aligned} \tag{232}$$

Propositions 2, 3 and 5 with the help of (215), (216) and (223) furnish

$$\begin{aligned} &||a_0(z)\tau^k\int_0^{\tau^k}\Big\{\Big(\partial_z^{-\kappa}\Delta^{\beta}\Xi((\tau^k-s)^{1/k},z)+\Delta^{\beta}\check{\Psi}((\tau^k-s)^{1/k},z)\Big)\Theta_{;1}(s^{1/k},z) \\ &+\Theta_{;1}(q^{-\beta}(\tau^k-s)^{1/k},z)\Big(\partial_z^{-\kappa}\Delta^{\beta}\Xi(s^{1/k},z)+\Delta^{\beta}\check{\Psi}(s^{1/k},z)\Big)\Big\}\frac{ds}{(\tau^k-s)s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\ &\leq |a_0|(\tilde{Z}_0)M_2\Big(||\partial_z^{-\kappa}\Delta^{\beta}\Xi(\tau,z)+\Delta^{\beta}\check{\Psi}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}||\Theta_{;1}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\ &+||\Theta_{;1}(q^{-\beta}\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}||\partial_z^{-\kappa}\Delta^{\beta}\Xi(\tau,z)+\Delta^{\beta}\check{\Psi}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}\Big) \\ &\leq |a_0|(\tilde{Z}_0)M_2\Big[\big(M_1\tilde{Z}_0^{\kappa}||\Delta^{\beta}\Xi(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}+(1-q^{-\beta})N_{k,0,0,\kappa,\tilde{Z}_0}\big) \\ &\times\big(M_1 Z_{0;1}^{\kappa}\upsilon_{;1}+||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big) \\ &+\big(M_1 Z_{0;1}^{\kappa}\upsilon_{;1}+||\check{\Psi}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big)\big(M_1\tilde{Z}_0^{\kappa}||\Delta^{\beta}\Xi(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}+(1-q^{-\beta})N_{k,0,0,\kappa,\tilde{Z}_0}\big)\Big] \end{aligned} \tag{233}$$

Propositions 3, 4 and 5 together with (215) beget

$$
\begin{aligned}
(234)\quad & ||a_h(z)\frac{1}{\Gamma(h/k)}\left[1-q^{-\beta h}\right]\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h}{k}-1} \\
&\times\left(s\int_0^s \Theta_{;1}((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)\frac{ds_1}{(s-s_1)s_1}\right)\frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\
&\leq |a_h|(\tilde{Z}_0)\frac{1}{\Gamma(h/k)}\left[1-q^{-\beta h}\right]M_3||\tau^k\int_0^{\tau^k}\Theta_{;1}((\tau^k-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)\frac{ds_1}{(\tau^k-s_1)s_1}||_{\sigma,\tilde{Z}_0,\mathcal{U})} \\
&\leq |a_h|(\tilde{Z}_0)\frac{1}{\Gamma(h/k)}\left[1-q^{-\beta h}\right]M_3M_2||\Theta_{;1}(\tau,z)||^2_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\
&\leq |a_h|(\tilde{Z}_0)\frac{1}{\Gamma(h/k)}\left[1-q^{-\beta h}\right]M_3M_2\left(M_1Z_{0;1}^{\kappa}\upsilon_{;1}+||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\right)^2
\end{aligned}
$$

Proposition 2, 3, 4 and 5 coupled with (215), (216) and (223) breed

$$
\begin{aligned}
(235)\quad & ||a_h(z)\frac{1}{\Gamma(h/k)}q^{-\beta h}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h}{k}-1} \\
&\times\Big[s\int_0^s\Big\{\left(\partial_z^{-\kappa}\Delta^\beta\Xi((s-s_1)^{1/k},z)+\Delta^\beta\check{\Psi}((s-s_1)^{1/k},z)\right)\Theta_{;1}(s_1^{1/k},z) \\
&+\Theta_{;1}(q^{-\beta}(s-s_1)^{1/k},z)\left(\partial_z^{-\kappa}\Delta^\beta\Xi(s_1^{1/k},z)+\Delta^\beta\check{\Psi}(s_1^{1/k},z)\right)\Big\}\frac{ds_1}{(s-s_1)s_1}\Big]\frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\
&\leq |a_h|(\tilde{Z}_0)\frac{1}{\Gamma(h/k)}q^{-\beta h}M_3||\tau^k\int_0^{\tau^k}\Big\{\left(\partial_z^{-\kappa}\Delta^\beta\Xi((\tau^k-s_1)^{1/k},z)+\Delta^\beta\check{\Psi}((\tau^k-s_1)^{1/k},z)\right)\Theta_{;1}(s_1^{1/k},z) \\
&+\Theta_{;1}(q^{-\beta}(\tau^k-s_1)^{1/k},z)\left(\partial_z^{-\kappa}\Delta^\beta\Xi(s_1^{1/k},z)+\Delta^\beta\check{\Psi}(s_1^{1/k},z)\right)\Big\}\frac{ds_1}{(\tau^k-s_1)s_1}||_{\sigma,\tilde{Z}_0,\mathcal{U})} \\
&\leq |a_h|(\tilde{Z}_0)\frac{1}{\Gamma(h/k)}q^{-\beta h}M_3M_2\Big[||\partial_z^{-\kappa}\Delta^\beta\Xi(\tau,z)+\Delta^\beta\check{\Psi}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}||\Theta_{;1}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\
&+||\Theta_{;1}(q^{-\beta}\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}||\partial_z^{-\kappa}\Delta^\beta\Xi(\tau,z)+\Delta^\beta\check{\Psi}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}\Big] \\
&\leq |a_h|(\tilde{Z}_0)\frac{1}{\Gamma(h/k)}q^{-\beta h}M_3M_2\Big[\left(M_1\tilde{Z}_0^{\kappa}||\Delta^\beta\Xi(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}+(1-q^{-\beta})N_{k,0,0,\kappa,\tilde{Z}_0}\right) \\
&\times\left(M_1Z_{0;1}^{\kappa}\upsilon_{;1}+||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\right) \\
&+\left(M_1Z_{0;1}^{\kappa}\upsilon_{;1}+||\check{\Psi}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\right)\left(M_1\tilde{Z}_0^{\kappa}||\Delta^\beta\Xi(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}+(1-q^{-\beta})N_{k,0,0,\kappa,\tilde{Z}_0}\right)\Big]
\end{aligned}
$$

From now on, we select a small sized quantity $\tilde{Z}_0>0$ and suitable $\tilde{\upsilon}>0$ (taken independently of $q \in (1,q_{0}])$ in a way that the next inequality holds

$$
\begin{aligned}
(236) \quad & \sum_{\underline{h}=(h_{0},h_{1},h_{2},h_{3}) \in \mathcal{C};h_{0}=0} |d_{\underline{h}}|(\tilde{Z}_{0}) \Big\{ k^{h_1}[1-q^{-\beta k h_{1}}] \big(M_{1}Z_{0;1}^{\kappa - h_{2}} \upsilon_{;1} + ||\tau^{kh_1}\partial_{z}^{h_2}\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \big) \\
& + k^{h_1}q^{-\beta k h_{1}} \big(M_{1}\tilde{Z}_{0}^{\kappa-h_{2}}\tilde{\upsilon}(1-q^{-\beta}) + (1-q^{-\beta})N_{k,h_{1},h_{2},\kappa,\tilde{Z}_{0}} \big) \Big\} \\
& + \sum_{\underline{h}=(h_{0},h_{1},h_{2},h_{3}) \in \mathcal{C};h_{0} \geq 1} |d_{\underline{h}}|(\tilde{Z}_{0}) \frac{k^{h_1}}{\Gamma(h_{0}/k)} \Big\{ [1-q^{-\beta(h_{0}+kh_{1})}] \\
& \times \big(M_{3}Z_{0;1}^{\kappa-h_{2}}\upsilon_{;1} + ||\tau^{k} \int_{0}^{\tau^k} (\tau^{k}-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_{z}^{h_2}\check{\Psi}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,Z_{0;1},\mathcal{U})} \big) \\
& + q^{-\beta(h_{0}+kh_{1})} \big(M_{3}\tilde{Z}_{0}^{\kappa-h_{2}}\tilde{\upsilon}(1-q^{-\beta}) + (1-q^{-\beta})\bar{N}_{k,h_{0},h_{1},h_{2},\kappa,\tilde{Z}_{0}} \big) \Big\} \\
& + |a_{0}|(\tilde{Z}_{0})M_{2} \Big[\big(M_{1}\tilde{Z}_{0}^{\kappa}\tilde{\upsilon}(1-q^{-\beta}) + (1-q^{-\beta})N_{k,0,0,\kappa,\tilde{Z}_{0}} \big) \\
& \times \big(M_{1}Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \big) \\
& + \big(M_{1}Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \big) \big(M_{1}\tilde{Z}_{0}^{\kappa}\tilde{\upsilon}(1-q^{-\beta}) + (1-q^{-\beta})N_{k,0,0,\kappa,\tilde{Z}_{0}} \big) \Big] \\
& + \sum_{h=1}^{A} |a_{h}|(\tilde{Z}_{0}) \frac{1}{\Gamma(h/k)} \Big\{ [1-q^{-\beta h}]M_{3}M_{2} \big(M_{1}Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \big)^{2} \\
& + q^{-\beta h}M_{3}M_{2} \Big[\big(M_{1}\tilde{Z}_{0}^{\kappa}\tilde{\upsilon}(1-q^{-\beta}) + (1-q^{-\beta})N_{k,0,0,\kappa,\tilde{Z}_{0}} \big) \\
& \times \big(M_{1}Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \big) \\
& + \big(M_{1}Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \big) \big(M_{1}\tilde{Z}_{0}^{\kappa}\tilde{\upsilon}(1-q^{-\beta}) + (1-q^{-\beta})N_{k,0,0,\kappa,\tilde{Z}_{0}} \big) \Big] \Big\} \\
& + 2(1-q^{-\beta})\tilde{C}_{9} \leq \tilde{\upsilon}(1-q^{-\beta})
\end{aligned}
$$

Notice that the above constraint (236) can be achieved since all the quantities

$$
\frac{1-q^{-\beta k h_{1}}}{1-q^{-\beta}} = \sum_{j=0}^{kh_{1}-1} (q^{-\beta})^{j}, \quad \frac{1-q^{-\beta(h_{0}+kh_{1})}}{1-q^{-\beta}} = \sum_{j=0}^{h_{0}+kh_{1}-1} (q^{-\beta})^{j}, \quad \frac{1-q^{-\beta k h}}{1-q^{-\beta}} = \sum_{j=0}^{kh-1} (q^{-\beta})^{j}
$$

remain bounded for $q \in (1,q_{0}]$ provided that $kh_{1} \geq 1$, $h_{0}+kh_{1} \geq 1$ and $h \geq 1$.

At last, the collection of all the inequalities (214), (229), (230), (231), (232), (233), (234) and (235) listed overhead under the condition (236) yields the first expected item (212).

We concentrate on the second feature 2. of the map $\mathfrak{D}$. Let $\Delta^{\beta}\Xi_{1},\Delta^{\beta}\Xi_{2}$ be elements of $G_{(\sigma,\tilde{Z}_{0},\mathcal{U})}^{k}$ with

$$
||\Delta^{\beta}\Xi_{j}(\tau,z)||_{(\sigma,\tilde{Z}_{0},\mathcal{U})} \leq \tilde{\upsilon}(1-q^{-\beta})
$$

for $j=1,2$.

According to the computations made to treat the first property 1. of $\mathfrak{D}$, we deduce forthrightly the next list of inequalities

$$
\begin{aligned}
(237) \quad & ||d_{\underline{h}}(z)k^{h_1}q^{-\beta k h_{1}}\tau^{kh_1}\partial_{z}^{-(\kappa-h_{2})} \big(\Delta^{\beta}\Xi_{1}(\tau,z) - \Delta^{\beta}\Xi_{2}(\tau,z) \big) ||_{(\sigma,\tilde{Z}_{0},\mathcal{U})} \\
& \leq |d_{\underline{h}}|(\tilde{Z}_{0})k^{h_1}q^{-\beta k h_{1}}M_{1}\tilde{Z}_{0}^{\kappa-h_{2}}||\Delta^{\beta}\Xi_{1}(\tau,z) - \Delta^{\beta}\Xi_{2}(\tau,z)||_{(\sigma,\tilde{Z}_{0},\mathcal{U})}
\end{aligned}
$$

and

$$(238)\quad ||d_{\underline{h}}(z)\frac{k^{h_1}}{\Gamma(h_0/k)}q^{-\beta(h_0+kh_1)}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1} \times \partial_z^{-(\kappa-h_2)}\big(\Delta^\beta\Xi_1(s^{1/k},z)-\Delta^\beta\Xi_2(s^{1/k},z)\big)\frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq |d_{\underline{h}}|(\tilde{Z}_0)\frac{k^{h_1}}{\Gamma(h_0/k)}q^{-\beta(h_0+kh_1)}M_3\tilde{Z}_0^{\kappa-h_2}||\Delta^\beta\Xi_1(\tau,z)-\Delta^\beta\Xi_2(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}$$

along with

$$(239)\quad ||a_0(z)\tau^k\int_0^{\tau^k}\Big\{\Big(\partial_z^{-\kappa}(\Delta^\beta\Xi_1((\tau^k-s)^{1/k},z)-\Delta^\beta\Xi_2((\tau^k-s)^{1/k},z))\Big)\Theta_{;1}(s^{1/k},z) + \Theta_{;1}(q^{-\beta}(\tau^k-s)^{1/k},z)\Big(\partial_z^{-\kappa}(\Delta^\beta\Xi_1(s^{1/k},z)-\Delta^\beta\Xi_2(s^{1/k},z))\Big)\Big\}\frac{ds}{(\tau^k-s)s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq |a_0|(\tilde{Z}_0)M_2\Big[\big(M_1\tilde{Z}_0^\kappa||\Delta^\beta\Xi_1(\tau,z)-\Delta^\beta\Xi_2(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}\big) \times\big(M_1Z_{0;1}^\kappa\upsilon_{;1}+||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big) + \big(M_1Z_{0;1}^\kappa\upsilon_{;1}+||\check{\Psi}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big)\big(M_1\tilde{Z}_0^\kappa||\Delta^\beta\Xi_1(\tau,z)-\Delta^\beta\Xi_2(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}\big)\Big]$$

and

$$(240)\quad ||a_h(z)\frac{1}{\Gamma(h/k)}q^{-\beta h}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h}{k}-1} \times\Big[s\int_0^s\Big\{\big(\partial_z^{-\kappa}(\Delta^\beta\Xi_1((s-s_1)^{1/k},z)-\Delta^\beta\Xi_2((s-s_1)^{1/k},z))\big)\Theta_{;1}(s_1^{1/k},z) + \Theta_{;1}(q^{-\beta}(s-s_1)^{1/k},z)\big(\partial_z^{-\kappa}(\Delta^\beta\Xi_1(s_1^{1/k},z)-\Delta^\beta\Xi_2(s_1^{1/k},z))\big)\Big\}\frac{ds_1}{(s-s_1)s_1}\Big]\frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq |a_h|(\tilde{Z}_0)\frac{1}{\Gamma(h/k)}q^{-\beta h}M_3M_2\Big[\big(M_1\tilde{Z}_0^\kappa||\Delta^\beta\Xi_1(\tau,z)-\Delta^\beta\Xi_2(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}\big) \times\big(M_1Z_{0;1}^\kappa\upsilon_{;1}+||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big) + \big(M_1Z_{0;1}^\kappa\upsilon_{;1}+||\check{\Psi}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big)\big(M_1\tilde{Z}_0^\kappa||\Delta^\beta\Xi_1(\tau,z)-\Delta^\beta\Xi_2(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})}\big)\Big]$$

Hereafter, we choose $\tilde{Z}_0 > 0$ small enough in a way that

$$(241)\quad \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0}|d_{\underline{h}}|(\tilde{Z}_0)k^{h_1}q^{-\beta h_1 k}M_1\tilde{Z}_0^{\kappa-h_2} + \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1}|d_{\underline{h}}|(\tilde{Z}_0)\frac{k^{h_1}}{\Gamma(h_0/k)}q^{-\beta(h_0+kh_1)}M_3\tilde{Z}_0^{\kappa-h_2} + |a_0|(\tilde{Z}_0)M_2\Big[M_1\tilde{Z}_0^\kappa\big(M_1Z_{0;1}^\kappa\upsilon_{;1}+||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big) + \big(M_1Z_{0;1}^\kappa\upsilon_{;1}+||\check{\Psi}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big)M_1\tilde{Z}_0^\kappa\Big] + \sum_{h=1}^{A}|a_h|(\tilde{Z}_0)\frac{1}{\Gamma(h/k)}q^{-\beta h}M_3M_2\Big[M_1\tilde{Z}_0^\kappa\big(M_1Z_{0;1}^\kappa\upsilon_{;1}+||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big) + \big(M_1Z_{0;1}^\kappa\upsilon_{;1}+||\check{\Psi}(q^{-\beta}\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big)M_1\tilde{Z}_0^\kappa\Big] \leq 1/2$$

and gathering (237), (238), (239) and (240) gives rise to the shrinking property (213).

Eventually, we select the constants $\tilde{Z}_0 > 0$ and $\tilde{\upsilon} > 0$ in a way that both constraints (236) and (241) hold concomitantly. Lemma 7 follows. □

In the upcoming lemma, we solve the linear Cauchy problem (206), (207) within the Banach space $G^k_{(\sigma,\tilde{Z}_0,\mathcal{U})}$.

Lemma 8 *Let us presume that the condition (19) hold. We single out the constants $\tilde{Z}_0 > 0$ and $\tilde{\upsilon} > 0$ (independently of q in $(1, q_0]$) as in Lemma 7. Then, the linear Cauchy problem (206), (207) owns a solution $\Delta^{\beta}\Theta_{;1}(u, z)$ that belong to the Banach space $G^k_{(\sigma,\tilde{Z}_0,\mathcal{U})}$ for any given $\sigma > 0$. Along with it, a constant $\tilde{M}_{1;1} > 0$ (unrelated to $q \in (1, q_0]$) can be found with the next feature*

$$||\Delta^{\beta}\Theta_{;1}(u, z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq \tilde{M}_{1;1}(1 - q^{-\beta}) \tag{242}$$

Proof Derived from Lemma 7, the classical fixed point theorem for shrinking maps on metric spaces can be used for the map $\mathfrak{D} : B_{\tilde{\upsilon}(1-q^{-\beta})} \to B_{\tilde{\upsilon}(1-q^{-\beta})}$ according to the fact $(B_{\tilde{\upsilon}(1-q^{-\beta})}, d)$ stands for a complete metric space for the distance $d(x, y) = ||x - y||_{(\sigma,\tilde{Z}_0,\mathcal{U})}$. Whence, $\mathfrak{D}$: $B_{\tilde{\upsilon}(1-q^{-\beta})} \to B_{\tilde{\upsilon}(1-q^{-\beta})}$ carries a unique fixed point denoted $\Delta^{\beta}\Xi_{;1}(u, z)$ inside the ball $B_{\tilde{\upsilon}(1-q^{-\beta})}$, meaning that

$$\mathfrak{D}(\Delta^{\beta}\Xi_{;1}) = \Delta^{\beta}\Xi_{;1}$$

As a result, a unique solution $\Delta^{\beta}\Xi_{;1}$ for the equation (210) is established in the ball $B_{\tilde{\upsilon}(1-q^{-\beta})}$. Furthermore, the proposition 2 conjointly with the decomposition (208) and the bounds (223) certify that the map

$$\Delta^{\beta}\Theta_{;1}(u, z) = \partial_z^{-\kappa}\Delta^{\beta}\Xi_{;1}(u, z) + \Delta^{\beta}\check{\Psi}(u, z)$$

belongs to $G^k_{(\sigma,\tilde{Z}_0,\mathcal{U})}$, solves the problem (206), (207) and suffers the next upper estimates

$$\begin{aligned} ||\Delta^{\beta}\Theta_{;1}(u, z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} &\leq M_1\tilde{Z}_0^{\kappa}||\Delta^{\beta}\Xi_{;1}(u, z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} + ||\Delta^{\beta}\check{\Psi}(u, z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \\ &\leq M_1\tilde{Z}_0^{\kappa}\tilde{\upsilon}(1 - q^{-\beta}) + (1 - q^{-\beta})N_{k,0,0,\kappa,\tilde{Z}_0} \leq \tilde{M}_{1;1}(1 - q^{-\beta}) \end{aligned} \tag{243}$$

for some constant $\tilde{M}_{1;1} > 0$ which is unattached to q in the range $(1, q_0]$. □

According to Lemma 8, it turns out that the *unique* formal series in z with holomorphic coefficients on $\mathcal{U}$ solution of (206), (207) is subjected to the bounds (242). Since the difference $\Theta_{;1}(u, z) - \Theta_{;1}(q^{-\beta}u, z)$, which represents in particular a formal power series in z with holomorphic coefficients on $\mathcal{U}$, is shown to solve (206), (207) in the first step of Proposition 18, it must coincide with the solution $\Delta^{\beta}\Theta_{;1}(u, z)$ constructed above in the second step of the proof and we deduce conclusively that the forsought estimates (196) hold for it. This completes the proof of the proposition. □

5.5 Error bounds between the Borel maps of the analytic solutions to the linear Cauchy problems (11), (12) and (175), (176).

In this subsection, we remind the reader of a result obtained in our previous work [13] related to the dependence of the family of solutions $\{u_p(t, z)\}_{0\leq p\leq\varsigma-1}$ to the linear Cauchy problem (11), (12) relatively to the parameter $q > 1$ set up in Theorem 1. This result will be applied in the ensuing subsection.

We consider the admissible set $\underline{\mathcal{D}} = \{\underline{\mathcal{T}}, \underline{\mathcal{U}}\}$ of sectors chosen in Proposition 15, where an unbounded sector $\mathcal{U}$ and corresponding bounded sector $\mathcal{T}$ are distinguished. We denote

$$u(t,z) = k\int_{L_{\gamma}} w_{;q}(u,z)\exp(-(u/t)^{k})\frac{du}{u}$$

for $(t,z) \in \mathcal{T} \times D_{\frac{1}{2C_4}}$, the solution of the problem (11), (12) displayed in Theorem 1, along the halfline L_{γ} taken in Proposition 15, where the Borel map $w_{;q}(u,z)$ has now an attached index q in order to keep track of the dependence in $q > 1$. According to Theorem 1, $w_{;q}(u,z)$ represents a holomorphic function on the domain $\mathcal{U} \times D_{\frac{1}{2C_4}}$ with a Taylor expansion of the form

$$w_{;q}(u,z) = \sum_{n \geq 0} w_{n;q}(u)\frac{z^n}{n!} \tag{244}$$

whenever $u \in \mathcal{U}$, $z \in D_{\frac{1}{2C_4}}$. We keep the notations of Section 5.1. In Section 8.3 of [13], the next result is stated.

Proposition 19 *Let $q \in (1, q_0]$. Two constants $C_{11}, C_{12} > 0$ (that are unconnected to q) can be singled out such that*

$$|w_{n;q}(u) - w_{n;1}(u)| \leq (q-1)C_{11}(C_{12})^{n} n! |u| \exp\left(k_1 \log^2(|u| + u_0) + \alpha \log(|u| + u_0)\right) \tag{245}$$

for all $u \in \mathcal{U} \cup \{0\}$, all integers $n \geq 0$, where the constants $k_1 > 0$, $u_0 > 1$, $\alpha \geq 0$ are fixed in Proposition 15.

5.6 Error bounds between the analytic solutions of the nonlinear auxiliary Cauchy problems (30), (31) and (183), (184).

This subsection is devoted to the expounding of the next proposition which plays a central role in the upcoming third main result of this work discussed in Theorem 6.

Proposition 20 *Let $q \in (1, q_0]$. Then, one can select two constants $\check{Z}_0 > 0$ and $\check{M}_1 > 0$, that are unrelated to q, such that for any given $\sigma > 0$, the next bounds*

$$||\Theta_{;1}(u,z) - \Theta_{;q}(u,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq \check{M}_1 \check{Z}_0^{\kappa}(q-1) \tag{246}$$

hold, where $\Theta_{;q}$ stands for the solution to the Cauchy problem (30), (31) that belongs to the space $G^{k}_{(\sigma,Z_0,\mathcal{U})}$ for some $Z_0 > 0$ (relying on q_0 but independent of q) built up in Proposition 9 and where $\Theta_{;1}$ solves the problem (183), (184) and is exhibited in Proposition 16.

Proof The proof is split up is two main parts.

In the **first part**, we frame a Cauchy problem specified later on in (251), (252) which is fulfilled by the difference

$$\Delta^{1;q}\Theta(u,z) := \Theta_{;1}(u,z) - \Theta_{;q}(u,z)$$

for which we seek upper bounds.

To that end, we need some prelusive computations. The basic identity $ab - cd = (a-c)b + c(b-d)$ will help us in recasting in an appropriate manner the next list of differences appearing as building blocks of the pending equation (251). The differences dealing with the linear terms can be reorganized by inserting some auxiliary terms as

$$
\begin{aligned}
(247)\quad & (ku^k)^{h_1}\partial_z^{h_2}\Theta_{;1}(u,z) - (k(q^{-h_3}u)^k)^{h_1}\partial_z^{h_2}\Theta_{;q}(q^{-h_3}u,z)\\
&= \left[(ku^k)^{h_1} - (k(q^{-h_3}u)^k)^{h_1}\right]\partial_z^{h_2}\Theta_{;1}(u,z) + (k(q^{-h_3}u)^k)^{h_1}\left[\partial_z^{h_2}\Theta_{;1}(u,z) - \partial_z^{h_2}\Theta_{;q}(q^{-h_3}u,z)\right]\\
&= \left[(ku^k)^{h_1} - (k(q^{-h_3}u)^k)^{h_1}\right]\partial_z^{h_2}\Theta_{;1}(u,z) + (k(q^{-h_3}u)^k)^{h_1}\Big[\partial_z^{h_2}\Theta_{;1}(u,z) - \partial_z^{h_2}\Theta_{;1}(q^{-h_3}u,z)\\
&\qquad + \partial_z^{h_2}\Theta_{;1}(q^{-h_3}u,z) - \partial_z^{h_2}\Theta_{;q}(q^{-h_3}u,z)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
(248)\quad & u^k\int_0^{u^k}(u^k-s)^{\frac{h_0}{k}-1}(ks)^{h_1}\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)\frac{ds}{s}\\
&\quad - u^k\int_0^{u^k}(u^k-s)^{\frac{h_0}{k}-1}(k(q^{-h_3}s^{1/k})^k)^{h_1}\partial_z^{h_2}\Theta_{;q}(q^{-h_3}s^{1/k},z)\frac{ds}{s}\\
&= u^k\int_0^{u^k}(u^k-s)^{\frac{h_0}{k}-1}\Big\{\left[(ks)^{h_1} - (k(q^{-h_3}s^{1/k})^k)^{h_1}\right]\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)\\
&\quad + (k(q^{-h_3}s^{1/k})^k)^{h_1}\left[\partial_z^{h_2}\Theta_{;1}(s^{1/k},z) - \partial_z^{h_2}\Theta_{;q}(q^{-h_3}s^{1/k},z)\right]\Big\}\frac{ds}{s}\\
&= u^k\int_0^{u^k}(u^k-s)^{\frac{h_0}{k}-1}\Big\{\left[(ks)^{h_1} - (k(q^{-h_3}s^{1/k})^k)^{h_1}\right]\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)\\
&\quad + (k(q^{-h_3}s^{1/k})^k)^{h_1}\Big[\partial_z^{h_2}\Theta_{;1}(s^{1/k},z) - \partial_z^{h_2}\Theta_{;1}(q^{-h_3}s^{1/k},z)\\
&\qquad + \partial_z^{h_2}\Theta_{;1}(q^{-h_3}s^{1/k},z) - \partial_z^{h_2}\Theta_{;q}(q^{-h_3}s^{1/k},z)\Big]\Big\}\frac{ds}{s}
\end{aligned}
$$

along with the differences for the nonlinear terms that can be reshaped as

$$
\begin{aligned}
(249)\quad & u^k\int_0^{u^k}\Theta_{;1}((u^k-s)^{1/k},z)\Theta_{;1}(s^{1/k},z)\frac{ds}{(u^k-s)s}\\
&\quad - u^k\int_0^{u^k}\Theta_{;q}((u^k-s)^{1/k},z)\Theta_{;q}(s^{1/k},z)\frac{ds}{(u^k-s)s}\\
&= u^k\int_0^{u^k}\Big\{\left[\Theta_{;1}((u^k-s)^{1/k},z) - \Theta_{;q}((u^k-s)^{1/k},z)\right]\Theta_{;1}(s^{1/k},z)\\
&\quad + \Theta_{;q}((u^k-s)^{1/k},z)\left[\Theta_{;1}(s^{1/k},z) - \Theta_{;q}(s^{1/k},z)\right]\Big\}\frac{ds}{(u^k-s)s}
\end{aligned}
$$

and

$$(250)\quad u^{k}\int_{0}^{u^k}(u^{k}-s)^{\frac{h}{k}-1}\Big(s\int_{0}^{s}\Theta_{;1}((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)\frac{ds_1}{(s-s_1)s_1}\Big)\frac{ds}{s}$$
$$-u^{k}\int_{0}^{u^k}(u^{k}-s)^{\frac{h}{k}-1}\Big(s\int_{0}^{s}\Theta_{;q}((s-s_1)^{1/k},z)\Theta_{;q}(s_1^{1/k},z)\frac{ds_1}{(s-s_1)s_1}\Big)\frac{ds}{s}$$
$$=u^{k}\int_{0}^{u^k}(u^{k}-s)^{\frac{h}{k}-1}\Bigg(s\int_{0}^{s}\Big\{\big[\Theta_{;1}((s-s_1)^{1/k},z)-\Theta_{;q}((s-s_1)^{1/k},z)\big]\Theta_{;1}(s_1^{1/k},z)$$
$$+\Theta_{;q}((s-s_1)^{1/k},z)\big[\Theta_{;1}(s_1^{1/k},z)-\Theta_{;q}(s_1^{1/k},z)\big]\Big\}\frac{ds_1}{(s-s_1)s_1}\Bigg)\frac{ds}{s}$$

Owing to the fact that $\Theta_{;1}$ (resp. $\Theta_{;q}$) solves the linear convolution Cauchy problem (183), (184) (resp. (30), (31)), paying regard to the redrawn expressions (247), (248), (249) and (250) and keeping in mind the notation (197) of the previous section 5.4, we can state the linear convolution Cauchy problem fulfilled by the difference $\Delta^{1;q}\Theta(u,z)$ as follows

$$(251)\quad \partial_z^{\kappa}\Delta^{1;q}\Theta(u,z)=\sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0}d_{\underline{h}}(z)\Big(\big[(ku^k)^{h_1}-(k(q^{-h_3}u)^k)^{h_1}\big]\partial_z^{h_2}\Theta_{;1}(u,z)$$
$$+(k(q^{-h_3}u)^k)^{h_1}\big[\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(u,z)+\partial_z^{h_2}\Delta^{1;q}\Theta(q^{-h_3}u,z)\big]\Big)$$
$$+\sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1}d_{\underline{h}}(z)\frac{1}{\Gamma(h_0/k)}\Bigg(u^{k}\int_{0}^{u^k}(u^k-s)^{\frac{h_0}{k}-1}\Big[\big[(ks)^{h_1}-(k(q^{-h_3}s^{1/k})^k)^{h_1}\big]$$
$$\times\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)+(k(q^{-h_3}s^{1/k})^k)^{h_1}\big[\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(s^{1/k},z)+\partial_z^{h_2}\Delta^{1;q}\Theta(q^{-h_3}s^{1/k},z)\big]\Big]\frac{ds}{s}\Bigg)$$
$$+a_0(z)u^{k}\int_{0}^{u^k}\Big(\Delta^{1;q}\Theta((u^k-s)^{1/k},z)\Theta_{;1}(s^{1/k},z)+\Theta_{;q}((u^k-s)^{1/k},z)\Delta^{1;q}\Theta(s^{1/k},z)\Big)\frac{ds}{(u^k-s)s}$$
$$+\sum_{h=1}^{A}a_h(z)\frac{1}{\Gamma(h/k)}u^{k}\int_{0}^{u^k}(u^k-s)^{\frac{h}{k}-1}\Big(s\int_{0}^{s}\big[\Delta^{1;q}\Theta((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)$$
$$+\Theta_{;q}((s-s_1)^{1/k},z)\Delta^{1;q}\Theta(s_1^{1/k},z)\big]\frac{ds_1}{(s-s_1)s_1}\Big)\frac{ds}{s}+w_{;1}(u,z)-w_{;q}(u,z)$$

for prescribed vanishing Cauchy data

$$(252)\qquad (\partial_z^{j}\Delta^{1;q}\Theta)(u,0)\equiv 0\quad,\quad 0\leq j\leq\kappa-1$$

In the **ensuing part**, we intend to solve the above problem (251), (252) by way of the Banach spaces introduced in Subsection 3.1. Namely, we search for a solution to (251), (252) shaped as

$$(253)\qquad \Delta^{1;q}\Theta(u,z)=\partial_z^{-\kappa}\Delta^{1;q}\Xi(u,z)$$

for some expression $\Delta^{1;q}\Xi(u,z)$ to be specified. One checks that $\Delta^{1;q}\Theta(u,z)$ matches the problem

(251), (252) if the map $\Delta^{1;q}\Xi(u,z)$ is subjected to the next *fixed point equation*

$$(254)\quad \Delta^{1;q}\Xi(u,z) = \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)\Big(\big[(ku^k)^{h_1} - (k(q^{-h_3}u)^k)^{h_1}\big]\partial_z^{h_2}\Theta_{;1}(u,z)$$
$$+ (k(q^{-h_3}u)^k)^{h_1}\big[\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(u,z) + \partial_z^{-(\kappa-h_2)}\Delta^{1;q}\Xi(q^{-h_3}u,z)\big]\Big)$$
$$+ \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{1}{\Gamma(h_0/k)}\Big(u^k\int_0^{u^k}(u^k-s)^{\frac{h_0}{k}-1}\Big[\big[(ks)^{h_1} - (k(q^{-h_3}s^{1/k})^k)^{h_1}\big]$$
$$\times\partial_z^{h_2}\Theta_{;1}(s^{1/k},z) + (k(q^{-h_3}s^{1/k})^k)^{h_1}\big[\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(s^{1/k},z) + \partial_z^{-(\kappa-h_2)}\Delta^{1;q}\Xi(q^{-h_3}s^{1/k},z)\big]\Big]\frac{ds}{s}\Big)$$
$$+ a_0(z)u^k\int_0^{u^k}\Big(\partial_z^{-\kappa}\Delta^{1;q}\Xi((u^k-s)^{1/k},z)\Theta_{;1}(s^{1/k},z)$$
$$+ \Theta_{;q}((u^k-s)^{1/k},z)\partial_z^{-\kappa}\Delta^{1;q}\Xi(s^{1/k},z)\Big)\frac{ds}{(u^k-s)s}$$
$$+ \sum_{h=1}^{A} a_h(z)\frac{1}{\Gamma(h/k)}u^k\int_0^{u^k}(u^k-s)^{\frac{h}{k}-1}\Big(s\int_0^s\big[\partial_z^{-\kappa}\Delta^{1;q}\Xi((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)$$
$$+ \Theta_{;q}((s-s_1)^{1/k},z)\partial_z^{-\kappa}\Delta^{1;q}\Xi(s_1^{1/k},z)\big]\frac{ds_1}{(s-s_1)s_1}\Big)\frac{ds}{s} + w_{;1}(u,z) - w_{;q}(u,z)$$

In the sequel, we seek for a solution, *for which sharp bounds relatively to q are exhibited*, to (254) inside the Banach space $G^k_{(\sigma,\check{Z}_0,\mathcal{U})}$, for any prescribed $\sigma > 0$ in the condition that $\check{Z}_0 > 0$ remains small enough. In order to meet this objective, we set up the next linear mapping

$$(255)\quad \mathfrak{E}(\Delta^{1;q}\Xi(\tau,z)) := \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} d_{\underline{h}}(z)\Big(\big[(k\tau^k)^{h_1} - (k(q^{-h_3}\tau)^k)^{h_1}\big]\partial_z^{h_2}\Theta_{;1}(\tau,z)$$
$$+ (k(q^{-h_3}\tau)^k)^{h_1}\big[\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(\tau,z) + \partial_z^{-(\kappa-h_2)}\Delta^{1;q}\Xi(q^{-h_3}\tau,z)\big]\Big)$$
$$+ \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} d_{\underline{h}}(z)\frac{1}{\Gamma(h_0/k)}\Big(\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}\Big[\big[(ks)^{h_1} - (k(q^{-h_3}s^{1/k})^k)^{h_1}\big]$$
$$\times\partial_z^{h_2}\Theta_{;1}(s^{1/k},z) + (k(q^{-h_3}s^{1/k})^k)^{h_1}\big[\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(s^{1/k},z) + \partial_z^{-(\kappa-h_2)}\Delta^{1;q}\Xi(q^{-h_3}s^{1/k},z)\big]\Big]\frac{ds}{s}\Big)$$
$$+ a_0(z)\tau^k\int_0^{\tau^k}\Big(\partial_z^{-\kappa}\Delta^{1;q}\Xi((\tau^k-s)^{1/k},z)\Theta_{;1}(s^{1/k},z)$$
$$+ \Theta_{;q}((\tau^k-s)^{1/k},z)\partial_z^{-\kappa}\Delta^{1;q}\Xi(s^{1/k},z)\Big)\frac{ds}{(\tau^k-s)s}$$
$$+ \sum_{h=1}^{A} a_h(z)\frac{1}{\Gamma(h/k)}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h}{k}-1}\Big(s\int_0^s\big[\partial_z^{-\kappa}\Delta^{1;q}\Xi((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z)$$
$$+ \Theta_{;q}((s-s_1)^{1/k},z)\partial_z^{-\kappa}\Delta^{1;q}\Xi(s_1^{1/k},z)\big]\frac{ds_1}{(s-s_1)s_1}\Big)\frac{ds}{s} + w_{;1}(\tau,z) - w_{;q}(\tau,z)$$

In the following lemma, sufficient conditions are enunciated under which $\mathfrak{E}$ becomes a shrinking map on a tiny ball *whose radius hinges on q* centered at the origin in the space $G^k_{(\sigma,\check{Z}_0,\mathcal{U})}$.

Lemma 9 *Assume that the condition (19) holds. Then, a small sized real number* $\check{\chi} > 0$ *can be singled out in a way that if* $0 < \check{Z}_0 < \check{\chi}$*, a radius* $\check{\upsilon} > 0$ *can be distinguished (in a unrelated manner to* q*, but depending on* q_0 *for which* $1 < q \leq q_0$*), such that* $\mathfrak{E}$ *acquires the next two hallmarks: Let us denote* $B_{\check{\upsilon}(q-1)}$ *the ball centered at 0 with radius* $\check{\upsilon}(q-1)$ *in* $G^k_{(\sigma,\check{Z}_0,\mathcal{U})}$*,*

1. $\mathfrak{E}$ *maps* $B_{\check{\upsilon}(q-1)}$ *into itself, signifying that*

$$\mathfrak{E}(B_{\check{\upsilon}(q-1)}) \subset B_{\check{\upsilon}(q-1)} \tag{256}$$

2. *The inequality*

$$||\mathfrak{E}(\Delta^{1;q}\Xi_1) - \mathfrak{E}(\Delta^{1;q}\Xi_2)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq \frac{1}{2}||\Delta^{1;q}\Xi_1 - \Delta^{1;q}\Xi_2||_{(\sigma,\check{Z}_0,\mathcal{U})} \tag{257}$$

holds as long as $\Delta^{1;q}\Xi_1, \Delta^{1;q}\Xi_2 \in B_{\check{\upsilon}(q-1)}$*.*

Proof We first supply norm upper estimates for some parts of the map $\mathfrak{E}$.

a) Upper bounds are established for the norm $||w_{;1}(\tau,z) - w_{;q}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}$. Departing from the expressions (178) and (244), we arrive at

$$||w_{;1}(\tau,z) - w_{;q}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} = \sum_{n\geq 0} ||w_{n;1}(\tau) - w_{n;q}(\tau)||_{(n,\sigma,\mathcal{U})} \frac{\check{Z}_0^n}{n!}$$

and owing to the bounds (245) stirred up in Proposition 19, the coefficients of the above series can be upper bounded by

$$||w_{n;1}(\tau) - w_{n;q}(\tau)||_{(n,\sigma,\mathcal{U})} = \sup_{\tau\in\mathcal{U}} \frac{1+|\tau|^{2k}}{|\tau|} \exp(-\sigma r_b(n)|\tau|^k)|w_{n;1}(\tau) - w_{n;q}(u)|$$
$$\leq (q-1)C_{11}(C_{12})^n n! \sup_{x\geq 0}(1+x^{2k})\exp(-\sigma x^k)\exp\left(k_1\log^2(x+u_0) + \alpha\log(x+u_0)\right)$$
$$\leq (q-1)\check{C}_{11}(C_{12})^n n!$$

for some constant $\check{C}_{11} > 0$ (unrelated to q), for any integers $n \geq 0$. Thereby, it brings in

$$||w_{;1}(\tau,z) - w_{;q}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq (q-1)\check{C}_{11}\sum_{n\geq 0}(C_{12}\check{Z}_0)^n \leq 2(q-1)\check{C}_{11} \tag{258}$$

on the condition that $\check{Z}_0 < \frac{1}{2C_{12}}$.

b) Bearing in mind (215), we already know that

$$||\tau^{kh_1}\partial_z^{h_2}\Theta_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq M_1 Z_{0;1}^{\kappa-h_2}\upsilon_{;1} + ||\tau^{kh_1}\partial_z^{h_2}\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \tag{259}$$

for some constant $M_1 > 0$ (independent of q but relying on q_0), where $Z_{0;1}, \upsilon_{;1} > 0$ are defined in Proposition 16, as long as $\check{Z}_0 < Z_{0;1}$.

c) We need accurate bounds for the norms $||\tau^{kh_1}\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}$, for non negative integers h_1, h_2, h_3 submitted to the condition (19). Calling to mind Lemma 8, the next splitting

$$\Delta^{h_3}\Theta_{;1}(\tau,z) = \partial_z^{-\kappa}\Delta^{h_3}\Xi_{;1}(\tau,z) + \Delta^{h_3}\check{\Psi}(\tau,z) \tag{260}$$

holds where $\Delta^{h_3}\Xi_{;1}(\tau,z)$ belongs to the ball $B_{\tilde{\upsilon}(1-q^{-h_3})}$ in $G^{k}_{(\sigma,\tilde{Z}_0,\mathcal{U})}$ for some constants $\tilde{Z}_0,\tilde{\upsilon}>0$ and where $\Delta^{h_3}\check{\Psi}(\tau,z)$ is defined in (209). Owing to (223), we can set a constant $N_{k,h_1,h_2,\kappa,\tilde{Z}_0}>0$ with

(261) $$||\tau^{kh_1}\partial_z^{h_2}\Delta^{h_3}\check{\Psi}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq (1-q^{-h_3})N_{k,h_1,h_2,\kappa,\tilde{Z}_0}$$

Besides, according to Proposition 2, we get a constant $M_1>0$ (independent of q) such that

(262) $$||\tau^{kh_1}\partial_z^{-(\kappa-h_2)}\Delta^{h_3}\Xi_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq M_1\tilde{Z}_0^{\kappa-h_2}||\Delta^{h_3}\Xi_{;1}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq M_1\tilde{Z}_0^{\kappa-h_2}\tilde{\upsilon}(1-q^{-h_3})$$

Thereupon, we deduce from (261) and (262) the awaited bounds

(263) $$||\tau^{kh_1}\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq ||\tau^{kh_1}\partial_z^{-(\kappa-h_2)}\Delta^{h_3}\Xi_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} + ||\tau^{kh_1}\partial_z^{h_2}\Delta^{h_3}\check{\Psi}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq M_1\tilde{Z}_0^{\kappa-h_2}\tilde{\upsilon}(1-q^{-h_3}) + (1-q^{-h_3})N_{k,h_1,h_2,\kappa,\tilde{Z}_0}$$

provided that $\check{Z}_0<\tilde{Z}_0$.

d) Keeping in mind (224), we observe that

(264) $$||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq M_3Z_{0;1}^{\kappa-h_2}\upsilon_{;1} + ||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\check{\Psi}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,Z_{0;1},\mathcal{U})}$$

as long as that $\check{Z}_0<\tilde{Z}_0$.

e) Precise bounds for the norms

$$||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})}$$

are required. Owing to the bounds (227), we can exhibit a constant $\bar{N}_{k,h_0,h_1,h_2,\kappa,\tilde{Z}_0}>0$ with

(265) $$||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\Delta^{h_3}\check{\Psi}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq (1-q^{-h_3})\bar{N}_{k,h_0,h_1,h_2,\kappa,\tilde{Z}_0}$$

and according to Proposition 4, we get a constant $M_3>0$ (unrelated to q) with

(266) $$||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{-(\kappa-h_2)}\Delta^{h_3}\Xi_{;1}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq M_3\tilde{Z}_0^{\kappa-h_2}||\Delta^{h_3}\Xi_{;1}(\tau,z)||_{(\sigma,\tilde{Z}_0,\mathcal{U})} \leq M_3\tilde{Z}_0^{\kappa-h_2}\tilde{\upsilon}(1-q^{-h_3})$$

Thus, on grounds of (265) and (266), we reach the due bounds

(267) $$||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq ||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{-(\kappa-h_2)}\Delta^{h_3}\Xi_{;1}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})} + ||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\Delta^{h_3}\check{\Psi}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq M_3\tilde{Z}_0^{\kappa-h_2}\tilde{\upsilon}(1-q^{-h_3}) + (1-q^{-h_3})\bar{N}_{k,h_0,h_1,h_2,\kappa,\tilde{Z}_0}$$

whenever $\check{Z}_0 < \tilde{Z}_0$.

f) We also need the bounds

$$||\Theta_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq M_1 Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} \tag{268}$$

which is a particular case of (259) and according to Proposition 9, one can select constants $M_1, Z_0, \upsilon > 0$ (unrelated to q but depending on q_0) with

$$||\Theta_{;q}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq ||\Theta_{;q}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})} \leq M_1 Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})} \tag{269}$$

whenever $\check{Z}_0 < Z_0$.

We are now in position to reach the heart of the proof. We focus on the first attribute 1. We set $\Delta^{1;q}\Xi(\tau,z)$ in $G^k_{(\sigma,\check{Z}_0,\mathcal{U})}$ with

$$||\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq \check{\upsilon}(q-1) \tag{270}$$

We display explicit bounds for each piece of the map $\mathfrak{E}(\Delta^{1;q}\Xi)$.
Proposition 5 and (259) beget

$$\begin{aligned} ||d_{\underline{h}}(z)\left[(k\tau^k)^{h_1} - (k(q^{-h_3}\tau)^k)^{h_1}\right]\partial_z^{h_2}\Theta_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} &\leq k^{h_1}\left[1 - q^{-h_3kh_1}\right]|d_{\underline{h}}|(\check{Z}_0) \\ &\times \left(M_1 Z_{0;1}^{\kappa-h_2}\upsilon_{;1} + ||\tau^{kh_1}\partial_z^{h_2}\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\right) \end{aligned} \tag{271}$$

Proposition 5 and (263) yield

$$\begin{aligned} ||d_{\underline{h}}(z)(k(q^{-h_3}\tau)^k)^{h_1}\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} &\leq |d_{\underline{h}}|(\check{Z}_0)k^{h_1}q^{-h_3kh_1} \\ &\times \left(M_1\tilde{Z}_0^{\kappa-h_2}\tilde{\upsilon}(1-q^{-h_3}) + (1-q^{-h_3})N_{k,h_1,h_2,\kappa,\tilde{Z}_0}\right) \end{aligned} \tag{272}$$

Propositions 2 and 5 trigger

$$\begin{aligned} &||d_{\underline{h}}(z)(k(q^{-h_3}\tau)^k)^{h_1}\partial_z^{-(\kappa-h_2)}\Delta^{1;q}\Xi(q^{-h_3}\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \\ &\leq |d_{\underline{h}}|(\check{Z}_0)k^{h_1}q^{-h_3kh_1}\left(M_1\check{Z}_0^{\kappa-h_2}||\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\right) \\ &\leq |d_{\underline{h}}|(\check{Z}_0)k^{h_1}q^{-h_3kh_1}\left(M_1\check{Z}_0^{\kappa-h_2}\check{\upsilon}(q-1)\right) \end{aligned} \tag{273}$$

Proposition 5 and (264) prompt

$$\begin{aligned} &||d_{\underline{h}}(z)\frac{1}{\Gamma(h_0/k)}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}\left[(ks)^{h_1} - (k(q^{-h_3}s^{1/k})^k)^{h_1}\right]\partial_z^{h_2}\Theta_{;1}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})} \\ &\leq |d_{\underline{h}}|(\check{Z}_0)\frac{1}{\Gamma(h_0/k)}k^{h_1}\left[1-q^{-h_3kh_1}\right]\Big(M_3 Z_{0;1}^{\kappa-h_2}\upsilon_{;1} \\ &+ ||\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\check{\Psi}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,Z_{0;1},\mathcal{U})}\Big) \end{aligned} \tag{274}$$

Proposition 5 and (267) return

$$\begin{aligned} &||d_{\underline{h}}(z)\frac{1}{\Gamma(h_0/k)}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}(k(q^{-h_3}s^{1/k})^k)^{h_1}\partial_z^{h_2}\Delta^{h_3}\Theta_{;1}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})} \\ &\leq |d_{\underline{h}}|(\check{Z}_0)\frac{1}{\Gamma(h_0/k)}k^{h_1}q^{-h_3kh_1}\left(M_3\tilde{Z}_0^{\kappa-h_2}\tilde{\upsilon}(1-q^{-h_3}) + (1-q^{-h_3})\bar{N}_{k,h_0,h_1,h_2,\kappa,\tilde{Z}_0}\right) \end{aligned} \tag{275}$$

Propositions 4 and 5 furnish

$$
\begin{aligned}
(276)\quad & ||d_{\underline{h}}(z)\frac{1}{\Gamma(h_0/k)}\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h_0}{k}-1}(k(q^{-h_3}s^{1/k})^k)^{h_1} \\
& \times \partial_z^{-(\kappa-h_2)}\Delta^{1;q}\Xi(q^{-h_3}s^{1/k},z)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})} \\
& \leq |d_{\underline{h}}|(\check{Z}_0)\frac{1}{\Gamma(h_0/k)}k^{h_1}q^{-h_3kh_1}\big(M_3(\check{Z}_0)^{\kappa-h_2}||\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\big) \\
& \leq |d_{\underline{h}}|(\check{Z}_0)\frac{1}{\Gamma(h_0/k)}k^{h_1}q^{-h_3kh_1}\big(M_3(\check{Z}_0)^{\kappa-h_2}\check{\upsilon}(q-1)\big)
\end{aligned}
$$

Propositions 2, 3 and 5 with the help of (268) and (269) promote

$$
\begin{aligned}
(277)\quad & ||a_0(z)\tau^k \int_0^{\tau^k} \Big(\partial_z^{-\kappa}\Delta^{1;q}\Xi((\tau^k - s)^{1/k},z)\Theta_{;1}(s^{1/k},z) \\
& + \Theta_{;q}((\tau^k - s)^{1/k},z)\partial_z^{-\kappa}\Delta^{1;q}\Xi(s^{1/k},z)\Big)\frac{ds}{(\tau^k - s)s}||_{(\sigma,\check{Z}_0,\mathcal{U})} \\
& \leq |a_0|(\check{Z}_0)M_2\Big(||\partial_z^{-\kappa}\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}||\Theta_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \\
& + ||\Theta_{;q}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}||\partial_z^{-\kappa}\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\Big) \\
& \leq |a_0|(\check{Z}_0)M_2\Big[M_1(\check{Z}_0)^{\kappa}||\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\big(M_1Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big) \\
& + \big(M_1Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})}\big)M_1(\check{Z}_0)^{\kappa}||\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\Big] \\
& \leq |a_0|(\check{Z}_0)M_2\Big[M_1(\check{Z}_0)^{\kappa}\check{\upsilon}(q-1)\big(M_1Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big) \\
& + \big(M_1Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})}\big)M_1(\check{Z}_0)^{\kappa}\check{\upsilon}(q-1)\Big]
\end{aligned}
$$

Propositions 2, 3, 4 and 5 together with (268), (269) spark off

$$
\begin{aligned}
(278)\quad & ||a_h(z)\frac{1}{\Gamma(h/k)}\tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{h}{k}-1}\Big(s\int_0^s \left[\partial_z^{-\kappa}\Delta^{1;q}\Xi((s-s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z) \right. \\
& \left. + \Theta_{;q}((s-s_1)^{1/k},z)\partial_z^{-\kappa}\Delta^{1;q}\Xi(s_1^{1/k},z)\right] \frac{ds_1}{(s-s_1)s_1}\Big)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})} \\
& \leq |a_h|(\check{Z}_0)\frac{1}{\Gamma(h/k)}M_3||\tau^k \int_0^{\tau^k} \Big[\partial_z^{-\kappa}\Delta^{1;q}\Xi((\tau^k - s_1)^{1/k},z)\Theta_{;1}(s_1^{1/k},z) \\
& + \Theta_{;q}((\tau^k - s_1)^{1/k},z)\partial_z^{-\kappa}\Delta^{1;q}\Xi(s_1^{1/k},z)\Big]\frac{ds_1}{(\tau^k - s_1)s_1}||_{(\sigma,\check{Z}_0,\mathcal{U})} \\
& \leq |a_h|(\check{Z}_0)\frac{1}{\Gamma(h/k)}M_3M_2\Big[||\partial_z^{-\kappa}\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}||\Theta_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \\
& + ||\Theta_{;q}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}||\partial_z^{-\kappa}\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\Big] \\
& \leq |a_h|(\check{Z}_0)\frac{1}{\Gamma(h/k)}M_3M_2\Big[M_1(\check{Z}_0)^{\kappa}||\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\big(M_1Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big) \\
& + \big(M_1Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})}\big)M_1(\check{Z}_0)^{\kappa}||\Delta^{1;q}\Xi(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\Big] \\
& \leq |a_h|(\check{Z}_0)\frac{1}{\Gamma(h/k)}M_3M_2\Big[M_1(\check{Z}_0)^{\kappa}\check{\upsilon}(q-1)\big(M_1Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big) \\
& + \big(M_1Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})}\big)M_1(\check{Z}_0)^{\kappa}\check{\upsilon}(q-1)\Big]
\end{aligned}
$$

From now onwards, we choose a small sized quantity $\check{Z}_0 > 0$ and proper $\check{\upsilon} > 0$ (taken freely from $q \in (1, q_0])$ in a way that the next inequality holds

$$(279) \quad \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} \Big[k^{h_1}[1 - q^{-h_3kh_1}]|d_{\underline{h}}|(\check{Z}_0) \times \big(M_1 Z_{0;1}^{\kappa-h_2}\upsilon_{;1} + ||\tau^{kh_1}\partial_z^{h_2}\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big) + |d_{\underline{h}}|(\check{Z}_0)k^{h_1}q^{-h_3kh_1} \times \big(M_1 \tilde{Z}_0^{\kappa-h_2}\tilde{\upsilon}(1-q^{-h_3}) + (1-q^{-h_3})N_{k,h_1,h_2,\kappa,\tilde{Z}_0}\big) + |d_{\underline{h}}|(\check{Z}_0)k^{h_1}q^{-h_3kh_1}\big(M_1\check{Z}_0^{\kappa-h_2}\check{\upsilon}(q-1)\big)\Big]$$
$$+ \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} \Big[|d_{\underline{h}}|(\check{Z}_0)\frac{1}{\Gamma(h_0/k)}k^{h_1}[1-q^{-h_3kh_1}]\Big(M_3 Z_{0;1}^{\kappa-h_2}\upsilon_{;1} + ||\tau^k \int_0^{\tau^k}(\tau^k - s)^{\frac{h_0}{k}-1}s^{h_1}\partial_z^{h_2}\check{\Psi}(s^{1/k},z)\frac{ds}{s}||_{(\sigma,Z_{0;1},\mathcal{U})}\Big)$$
$$+ |d_{\underline{h}}|(\check{Z}_0)\frac{1}{\Gamma(h_0/k)}k^{h_1}q^{-h_3kh_1}\big(M_3\tilde{Z}_0^{\kappa-h_2}\tilde{\upsilon}(1-q^{-h_3}) + (1-q^{-h_3})\bar{N}_{k,h_0,h_1,h_2,\kappa,\tilde{Z}_0}\big)$$
$$+ |d_{\underline{h}}|(\check{Z}_0)\frac{1}{\Gamma(h_0/k)}k^{h_1}q^{-h_3kh_1}\big(M_3(\check{Z}_0)^{\kappa-h_2}\check{\upsilon}(q-1)\big)\Big]$$
$$+ |a_0|(\check{Z}_0)M_2M_1(\check{Z}_0)^{\kappa}\check{\upsilon}(q-1)\Big[M_1 Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} + M_1 Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})}\Big]$$
$$+ \sum_{h=1}^{A}|a_h|(\check{Z}_0)\frac{1}{\Gamma(h/k)}M_3M_2M_1(\check{Z}_0)^{\kappa}\check{\upsilon}(q-1)$$
$$\times \Big[M_1 Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} + M_1 Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})}\Big] + 2(q-1)\check{C}_{11} \leq \check{\upsilon}(q-1)$$

Observe that the above constraint (279) is achievable since all the quantities

$$\frac{1-q^{-h_3}}{q-1} = \frac{1}{q}\frac{1-(1/q)^{h_3}}{1-(1/q)} = \frac{1}{q}\sum_{j=0}^{h_3-1}(1/q)^j \quad , \quad \frac{1-q^{-h_3kh_1}}{q-1} = \frac{1}{q}\frac{1-(1/q)^{h_3kh_1}}{1-(1/q)} = \frac{1}{q}\sum_{j=0}^{h_3kh_1-1}(1/q)^j$$

remain bounded whenever $q \in (1, q_0]$, for given integers $h_3 \geq 1$, $h_3kh_1 \geq 1$.

Eventually, piling up the inequalities (258), (271), (272), (273), (274), (275), (276), (277) and (278) reached above under the condition (279) produces the awaited item (256).

We focalize on the second aspect 2. of the map $\mathfrak{E}$. Let $\Delta^{1;q}\Xi_1, \Delta^{1;q}\Xi_2$ be elements of $G^k_{(\sigma,\check{Z}_0,\mathcal{U})}$ such that

$$||\Delta^{1;q}\Xi_j(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq \check{\upsilon}(q-1)$$

for $j = 1, 2$.

The estimates involved in the first property 1. of $\mathfrak{E}$ enable us to write down the next set of inequalities

$$(280) \quad ||d_{\underline{h}}(z)(k(q^{-h_3}\tau)^k)^{h_1}\partial_z^{-(\kappa-h_2)}\big(\Delta^{1;q}\Xi_1(q^{-h_3}\tau,z) - \Delta^{1;q}\Xi_2(q^{-h_3}\tau,z)\big)||_{(\sigma,\check{Z}_0,\mathcal{U})}$$
$$\leq |d_{\underline{h}}|(\check{Z}_0)k^{h_1}q^{-h_3kh_1}\big(M_1\check{Z}_0^{\kappa-h_2}||\Delta^{1;q}\Xi_1(\tau,z) - \Delta^{1;q}\Xi_2(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\big)$$

and

$$
\begin{aligned}
(281)\quad & ||d_{\underline{h}}(z)\frac{1}{\Gamma(h_0/k)}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h_0}{k}-1}(k(q^{-h_3}s^{1/k})^k)^{h_1}\\
&\times\partial_z^{-(\kappa-h_2)}\big(\Delta^{1;q}\Xi_1(q^{-h_3}s^{1/k},z)-\Delta^{1;q}\Xi_2(q^{-h_3}s^{1/k},z)\big)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})}\\
&\leq |d_{\underline{h}}|(\check{Z}_0)\frac{1}{\Gamma(h_0/k)}k^{h_1}q^{-h_3kh_1}\big(M_3(\check{Z}_0)^{\kappa-h_2}||\Delta^{1;q}\Xi_1(\tau,z)-\Delta^{1;q}\Xi_2(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\big)
\end{aligned}
$$

along with

$$
\begin{aligned}
(282)\quad & ||a_0(z)\tau^k\int_0^{\tau^k}\Big(\partial_z^{-\kappa}\big(\Delta^{1;q}\Xi_1((\tau^k-s)^{1/k},z)-\Delta^{1;q}\Xi_2((\tau^k-s)^{1/k},z)\big)\Theta_{;1}(s^{1/k},z)\\
&+\Theta_{;q}((\tau^k-s)^{1/k},z)\partial_z^{-\kappa}\big(\Delta^{1;q}\Xi_1(s^{1/k},z)-\Delta^{1;q}\Xi_2(s^{1/k},z)\big)\Big)\frac{ds}{(\tau^k-s)s}||_{(\sigma,\check{Z}_0,\mathcal{U})}\\
&\leq |a_0|(\check{Z}_0)M_2\Big(||\partial_z^{-\kappa}(\Delta^{1;q}\Xi_1(\tau,z)-\Delta^{1;q}\Xi_2(\tau,z))||_{(\sigma,\check{Z}_0,\mathcal{U})}||\Theta_{;1}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\\
&+||\Theta_{;q}(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}||\partial_z^{-\kappa}(\Delta^{1;q}\Xi_1(\tau,z)-\Delta^{1;q}\Xi_2(\tau,z))||_{(\sigma,\check{Z}_0,\mathcal{U})}\Big)\\
&\leq |a_0|(\check{Z}_0)M_2\Big[M_1(\check{Z}_0)^{\kappa}||\Delta^{1;q}\Xi_1(\tau,z)-\Delta^{1;q}\Xi_2(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\big(M_1Z_{0;1}^{\kappa}\upsilon_{;1}+||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big)\\
&+\big(M_1Z_0^{\kappa}\upsilon+||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})}\big)M_1(\check{Z}_0)^{\kappa}||\Delta^{1;q}\Xi_1(\tau,z)-\Delta^{1;q}\Xi_2(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
(283)\quad & ||a_h(z)\frac{1}{\Gamma(h/k)}\tau^k\int_0^{\tau^k}(\tau^k-s)^{\frac{h}{k}-1}\\
&\times\Big(s\int_0^s\Big[\partial_z^{-\kappa}\big(\Delta^{1;q}\Xi_1((s-s_1)^{1/k},z)-\Delta^{1;q}\Xi_2((s-s_1)^{1/k},z)\big)\Theta_{;1}(s_1^{1/k},z)\\
&+\Theta_{;q}((s-s_1)^{1/k},z)\partial_z^{-\kappa}\big(\Delta^{1;q}\Xi_1(s_1^{1/k},z)-\Delta^{1;q}\Xi_2(s_1^{1/k},z)\big)\Big]\frac{ds_1}{(s-s_1)s_1}\Big)\frac{ds}{s}||_{(\sigma,\check{Z}_0,\mathcal{U})}\\
&\leq |a_h|(\check{Z}_0)\frac{1}{\Gamma(h/k)}M_3M_2\Big[M_1(\check{Z}_0)^{\kappa}||\Delta^{1;q}\Xi_1(\tau,z)-\Delta^{1;q}\Xi_2(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\\
&\times\big(M_1Z_{0;1}^{\kappa}\upsilon_{;1}+||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})}\big)\\
&+\big(M_1Z_0^{\kappa}\upsilon+||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})}\big)M_1(\check{Z}_0)^{\kappa}||\Delta^{1;q}\Xi_1(\tau,z)-\Delta^{1;q}\Xi_2(\tau,z)||_{(\sigma,\check{Z}_0,\mathcal{U})}\Big]
\end{aligned}
$$

From this point forward, we pick up $\check{Z}_0 > 0$ close enough to 0 in a way that

$$(284) \quad \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0=0} |d_{\underline{h}}|(\check{Z}_0)k^{h_1}q^{-h_3kh_1}M_1\check{Z}_0^{\kappa-h_2} + \sum_{\underline{h}=(h_0,h_1,h_2,h_3)\in\mathcal{C};h_0\geq 1} |d_{\underline{h}}|(\check{Z}_0)\frac{1}{\Gamma(h_0/k)}k^{h_1}q^{-h_3kh_1}M_3(\check{Z}_0)^{\kappa-h_2}$$
$$+ |a_0|(\check{Z}_0)M_2M_1(\check{Z}_0)^{\kappa}\Big[M_1Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} + M_1Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})}\Big]$$
$$+ \sum_{h=1}^{A}|a_h|(\check{Z}_0)\frac{1}{\Gamma(h/k)}M_3M_2M_1(\check{Z}_0)^{\kappa}\Big[M_1Z_{0;1}^{\kappa}\upsilon_{;1} + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_{0;1},\mathcal{U})} + M_1Z_0^{\kappa}\upsilon + ||\check{\Psi}(\tau,z)||_{(\sigma,Z_0,\mathcal{U})}\Big] \leq 1/2$$

The collection of bounds (280), (281), (282) and (283) contingent upon (284) sparks off the second feature (257).

At last, we single out the constants $\check{Z}_0$ and $\check{\upsilon} > 0$ in a way that both constraints (279), (284) occur unitedly. Lemma 9 follows. □

In the lemma to come, the linear Cauchy problem (251), (252) is solved within the Banach space $G^k_{(\sigma,\check{Z}_0,\mathcal{U})}$.

Lemma 10 *Take for granted that the condition (19) hold. We fix the constants $\check{Z}_0 > 0$ and $\check{\upsilon} > 0$ (independently of q in $(1, q_0]$) as in Lemma 9. Then, the linear Cauchy problem (251), (252) carries a solution $\Delta^{1;q}\Theta(u,z)$ that belongs to the Banach space $G^k_{(\sigma,\check{Z}_0,\mathcal{U})}$ for any given $\sigma > 0$. Along with it, a constant $\check{M}_1 > 0$ (unrelated to $q \in (1, q_0]$) can be distinguished with the next property*

$$(285) \qquad ||\Delta^{1;q}\Theta(u,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq \check{M}_1\check{Z}_0^{\kappa}(q-1)$$

Proof Based on Lemma 9, the classical fixed point theorem for shrinking maps on metric spaces can be used for the map $\mathfrak{C} : B_{\check{\upsilon}(q-1)} \to B_{\check{\upsilon}(q-1)}$ due to the fact that $(B_{\check{\upsilon}(q-1)}, d)$ represents a complete metric space for the distance $d(x,y) = ||x-y||_{(\sigma,\check{Z}_0,\mathcal{U})}$. Thus, $\mathfrak{C} : B_{\check{\upsilon}(q-1)} \to B_{\check{\upsilon}(q-1)}$ gets a unique fixed point denoted $\Delta^{1;q}\Xi(u,z)$ inside the ball $B_{\check{\upsilon}(q-1)}$, meaning that

$$\mathfrak{C}(\Delta^{1;q}\Xi) = \Delta^{1;q}\Xi$$

Thereby, a unique solution $\Delta^{1;q}\Xi$ for the equation (254) is confirmed in the ball $B_{\check{\upsilon}(q-1)}$. Furthermore, the proposition 2 conjointly with the decomposition (253) warrant that the map

$$\Delta^{1;q}\Theta(u,z) = \partial_z^{-\kappa}\Delta^{1;q}\Xi(u,z)$$

belongs to $G^k_{(\sigma,\check{Z}_0,\mathcal{U})}$, solves the problem (251), (252) and is submitted to the next upper estimates

$$(286) \quad ||\Delta^{1;q}\Theta(u,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq M_1\check{Z}_0^{\kappa}||\Delta^{1;q}\Xi(u,z)||_{(\sigma,\check{Z}_0,\mathcal{U})} \leq M_1\check{Z}_0^{\kappa}\check{\upsilon}(q-1) \leq \check{M}_1\check{Z}_0^{\kappa}(q-1)$$

for some constant $\check{M}_1 > 0$ which is unattached to q in the range $(1, q_0]$. □

According to Lemma 10, it turns out that the *unique* formal series in z with holomorphic coefficients on $\mathcal{U}$ solution of (251), (252) suffers the bounds (285). Since the difference $\Theta_{;1}(u,z)-$

$\Theta_{;q}(u,z)$, which represents in particular a formal power series in z with holomorphic coefficients on $\mathcal{U}$, is shown to solve (251), (252) in the first step of Proposition 20, it must coincide with the solution $\Delta^{1;q}\Theta(u,z)$ constructed above in the second step and we arrive at the conclusion that the forsought estimates (246) necessarily hold for it. The proof of the proposition is completed. □

5.7 Confluence for the analytic solutions of the problem (21), (22) as $q \to 1$.

In this subsection, we unveil the third and last main result of the work.

Theorem 6 *Let $\underline{\mathcal{D}} = \{\underline{\mathcal{T}},\underline{\mathcal{U}}\}$ be the admissible set of sectors distinguished in Proposition 15. We denote $\mathcal{U}$ the unbounded sector singled out in Proposition 15 and $\mathcal{T}$ its corresponding bounded sector in accordance with the requirement 2) of Definition 2.*

We denote $v_{;q}(t,z)$ the bounded holomorphic solution to (21), (22) on the product $\mathcal{T} \times D_{Z_0/2}$, given by a Laplace transform of order k, see (86), constructed in Theorem 2. Besides, we consider the bounded holomorphic solution $v_{;1}(t,z)$ of the nonlinear limit problem (180), (181) on the domain $\mathcal{T} \times D_{Z_{0;1}/2}$ expressed through a Laplace transform of order k, see (186), built up in Proposition 16.

Then, a constant $C > 0$ (unrelated to $q \in (1,q_0]$) can be found such that

$$\sup_{t\in\mathcal{T},z\in D_{\check{Z}_0/2}} |v_{;1}(t,z) - v_{;q}(t,z)| \leq C(q-1) \tag{287}$$

for all $q \in (1,q_0]$. In other words, the solution $v_{;q}(t,z)$ of (21), (22) merges uniformly on $\mathcal{T} \times D_{\check{Z}_0/2}$ to the solution $v_{;1}(t,z)$ of (180), (181) as $q \to 1$.

Proof We express both solutions $v_{;q}(t,z)$ and $v_{;1}(t,z)$ as Laplace transforms

$$v_{;q}(t,z) = k\int_{L_\gamma} \Theta_{;q}(u,z)\exp\big(-(u/t)^k\big)du/u \quad , \quad v_{;1}(t,z) = k\int_{L_\gamma} \Theta_{;1}(u,z)\exp\big(-(u/t)^k\big)du/u$$

along a halfline $L_\gamma = [0,+\infty)e^{\sqrt{-1}\gamma} \subset \mathcal{U}\cup\{0\}$ assigned to the condition

$$\cos\big(k(\gamma - \arg(t))\big) > \Delta \tag{288}$$

for some fixed constant $\Delta > 0$, provided that $t \in \mathcal{T}$, where the *Borel map* $\Theta_{;q}(u,z)$ is outlined in (87) and $\Theta_{;1}(u,z)$ is described in (187) whose Taylor expansion can be displayed as

$$\Theta_{;1}(u,z) = \sum_{\beta\geq 0}\Theta_{;1,\beta}(u)\frac{z^\beta}{\beta!}$$

for $(u,z) \in \mathcal{U} \times D_{Z_{0;1}/2}$.

The deviations bounds reached in Proposition 20 can be rephrased in the next explicit way

$$|\Theta_{;1}(u,z) - \Theta_{;q}(u,z)| \leq \sum_{\beta\geq 0}\big|\Theta_{;1,\beta}(u) - \Theta_{;q,\beta}(u)\big|\frac{|z|^\beta}{\beta!} \tag{289}$$
$$\leq \check{M}_1\check{Z}_0^\kappa(q-1)\frac{|u|}{1+|u|^{2k}}\sum_{\beta\geq 0}\exp(\sigma r_b(\beta)|u|^k)\big(\frac{|z|}{\check{Z}_0}\big)^\beta \leq 2\check{M}_1\check{Z}_0^\kappa(q-1)|u|\exp(\sigma\zeta(b)|u|^k)$$

for all $u \in \mathcal{U} \cup \{0\}$, provided that $z \in D_{\check{Z}_0/2}$.

Thereby, we can govern the difference $v_{;1}(t,z) - v_{;q}(t,z)$ in the following manner

$$|v_{;1}(t,z) - v_{;q}(t,z)| = \left| k \int_{L_\gamma} \big(\Theta_{;1}(u,z) - \Theta_{;q}(u,z)\big) \exp\big(-(u/t)^k\big) du/u \right| \\ \leq 2\check{M}_1 \check{Z}_0^{\kappa}(q-1) \int_0^{+\infty} \exp(\sigma\zeta(b) r^k) \exp\big(-(r/r_{\mathcal{T}})^k \Delta\big) dr \tag{290}$$

for all $t \in \mathcal{T}$, all $z \in D_{\check{Z}_0/2}$, bearing in mind that radius $r_{\mathcal{T}}$ of $\mathcal{T}$ fulfills

$$r_{\mathcal{T}} < (\frac{\Delta}{\sigma\zeta(b)})^{1/k}$$

according to (85). This achieves the expected bounds (287). □

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