Deciphering Black Hole Spin, Inclination angle & Charge From Kerr Shadow

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Abstract: The apparent shape of the black hole shadow provides a full description of the spin, the inclination angle and the charge of a Kerr black hole, without any astrophysical process or underlying theory in the astrophysical process.

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In Boyer-Lindquist coordinates (t, r, θ, ϕ), the line element of the Kerr Space-time has been given by the solutions [1],

$$ds^2 = -\left(\frac{1-2Mr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 - \frac{4MraSin^2\theta}{\rho^2}dtd\phi + \frac{((r^2+a^2)^2 - \Delta a^2 Sin^2\theta)Sin^2\theta}{\rho^2}d\phi^2$$

The metric functions of 2-variables become,

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 Cos^2\theta$$

The valued parameters M and a are BH mass and spin respectively. The horizons of the BH in Kerr space-time could be obtained by approaching the limits of $\Delta(r) = 0$, having the horizon radius at,

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

There is one horizon at $|a| < M$ while the other is at $|a| = M$, however, at $|a| > M$, there is no horizons and this corresponds to a naked singularity.

The 4-momentum of photon could be denoted as, p^μ with the time orientation as $p^t > 0$ provides the timelike trajectories that provides the three conserved quantities,

$$E = -p^t, \quad L = p_\phi$$
$$Q = p_\theta^2 - Cos^2\theta\left(a^2 p_t^2 - p_\phi^2 Csc^2\theta\right)$$

Corresponding the energy at infinity [2,3], spin momentum and Carter integral, respectively, this makes it easy to work in energy rescaled quantities,

$$\xi(r) = \frac{l}{E}, \quad \eta(r) = \frac{Q}{E^2}$$

Where l is the spin angular momentum, E is the energy of the charged test particles, Q is the Carter constant related to the Killing-Yano [4,5] tensor field.

The Null-Geodesics could be given by the solutions of the equations as,

$$\rho^2\frac{dt}{d\lambda} = a(l - aESin^2\theta) + \frac{r^2+a^2}{\Delta}(E(r^2+a^2) - al)$$
$$\rho^2\frac{dr}{d\lambda} = \pm_r\sqrt{\zeta(r)}$$
$$\rho^2\frac{d\theta}{d\lambda} = \pm_\theta\sqrt{\Theta(\theta)}$$
$$\rho^2\frac{d\phi}{d\lambda} = (lCsc^2\theta - aE) + \frac{a}{\Delta}(E(r^2+a^2) - al)$$

Where λ is the affine parameter with the consecutive "potentials" as,

$$\zeta(r) = (r^2 + a^2 - a\lambda)^2 - \Delta(r)[\eta + (\lambda - a)^2]$$
$$\Theta(\theta) = \eta + a^2Cos^2\theta - \lambda^2 Cot^2\theta$$

The null geodesics helps us in finding the two celestial coordinates α and β which describes the shadow of the BH, that the observer should see in the sky. For the observer at infinity, this corresponds to the equations,

$$\alpha = -\xi Csc\theta_0$$
$$\beta = \pm\sqrt{\eta + a^2Cos^2\theta_0 - \xi^2 Cot^2\theta_0}$$

In the equatorial plane, where $\theta_0 = \frac{\pi}{2}$, the parameters are,

$$\alpha = -\xi$$
$$\beta = \pm\sqrt{\eta}$$

Then, the sub parameters are [6],

$$\xi(r) = \frac{(3M - r_0)r_0^2 - a^2(M + r_0)}{a(r_0 - M)}$$
$$\eta(r) = \frac{r_0^3(4a^2M - r_0(3M - r_0)^2)}{a^2(r_0 - M)^2}$$

Where r_0 is the photon ring radius between the prograde and retrograde photon spheres of the BH shadow.

To parameterize the shadow, we have to identify several points on the curvature boundary of the shadow, however, the brevity of the mathematics lies in minimizing the amounts of points on the shadow boundary and we have seen that, for each point in the celestial coordinates (α, β), the shadow has a point on r_0, then to obtain the curvature radius of that point, we simply have to consider 3 points of the boundary to establish the curvature radius, where the points are,

$$\alpha(r_0 - \omega), \beta(r_0 - \omega)$$
$$\alpha(r_0), \beta(r_0)$$
$$\alpha(r_0 + \omega), \beta(r_0 + \omega)$$

To plot a circle of radius $R(r_0, \omega)$, if the limit could be taken as $\omega \to 0$, then all the three points would approach at the curvature radius of the point, $\alpha(r_0), \beta(r_0)$, and then the curvature radius of the
Kerr BH shadow reads [7,8],

$$R_s = \frac{64\sqrt{M}(r_0^3 - a^2 r_0 Cos^2\theta_s)^{3/2}[r_0(r_0 - 3Mr_0 + 3M^2) - a^2M^2]}{(r_0 - M)^3[3(8r_0^4 - a^4 - 8a^2r_0^2) - 4a^2(6r_0^2 + a^2)Cos2\theta_s - a^4Cos(4\theta_s)]}$$

The above relation can be summarized as,

$$R_s = \frac{(a_t - a_r)^2 + \beta_t^2}{2|a_t - a_r|}$$

Now, with the curvature radius R_s with the length parameter λ of the shadow, we get an invariant topological quantity as,

$$\delta_s = \int \frac{d\lambda}{R(\lambda)} + \sum_i \theta_i = \frac{D_0}{R_s} \ \forall\ D_0 = d\lambda + \sum_i \theta_i$$

BH has another parameter which is charge and that can be found in the "Braneworld" rotating BH having its line element in the Boyer-Lindquist coordinates being same as that of a "normal" Kerr BH, except a few changes like,

$$\Delta = r^2 - 2Mr + a^2 + Q$$

Where Q is the "Tidal charge" that could be either positive or negative having its horizon element slightly different than the "normal" Kerr BH as,

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q}$$

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We could have a naked singularity but since $Q \leq Q_c = M^2 - a^2$ we have a BH.

The following conditions hold;

- For Q=Q_c its an extremal BPS BH.
- For Q>0 R_s reduces
- For Q<0 R_s increases and δ_s reduces

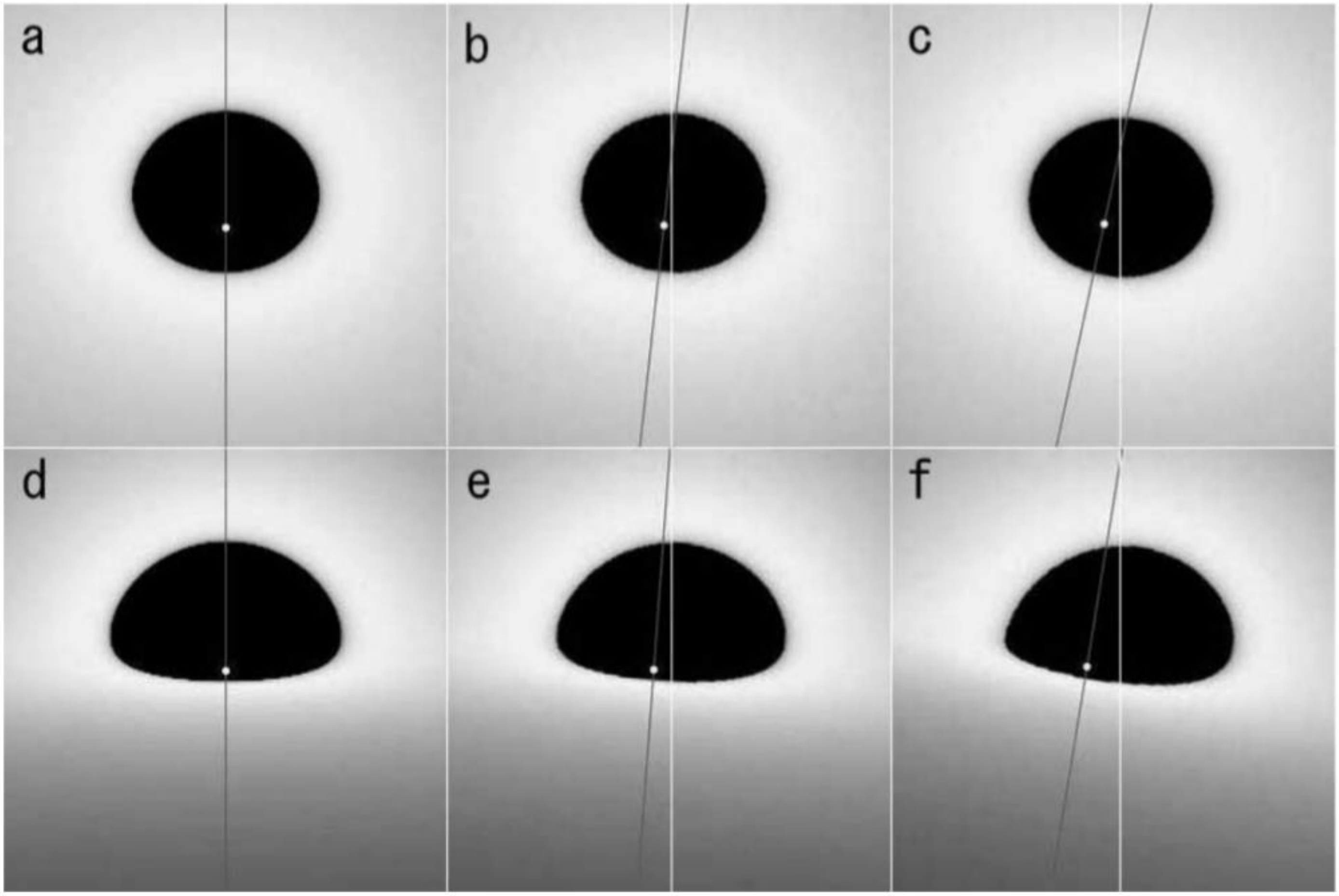
To measure the topological structure of the shadow, δ_s would be useful to determine the BH shadow from a naked singularity.

Inclination angle could be formulated by the parameter,

$$\theta_s = R_s \frac{M}{D_0}$$

For the BH Sgr. A* at the centre of the Milky Way, M = 4.3 * 10^6 $M_\odot$ and D_0 = 8.3 KPc. The values θ_s and δ_s computed as per the below table [9],

a	0				0.9			
Q	-0.5	-0.1	0	0.1	-0.5	-0.1	0	0.1
$\theta_s(\mu as)$	28.605	27.006	26.572	26.120	28.612	27.018	26.586	26.136
$\delta_s(\%)$	0	0	0	0	7.45	11.8	13.9	17.2



BH with same width and flatness but different spin parameters. The shift of the rotation axis can be observed as (b) 0.22, (c) 0.38, (e) 0.50, (f) 0.95. the inclination angles are (b) 6°, (c) 12°, (e) 4°, (f) 9°. the shifts from the rotation axis determines the spin parameters of the BHs. Figure and data from [10]

To determine the spin of the BH from the shadow casted on the accretion disc, it is necessary for a qualitative analysis of the shape and positions of the shadow and how its spin angular momentum is affecting its shadow. Even same mass of BH cast different shadows depending on the spin parameters. Moreover, if the BH is rotating (or Kerr property) as per the paper, then due to the frame dragging effect, the shadow on one side got elongated depending on the direction of the spin making the observing and computing angular momentum more difficult. The spin parameter causes a displacement of the shadow from its rotation axis. Observationally, as its difficult to determine the rotation parameter (or axis of rotation parameter), therefore, its spin effects on the casted shadow is also hard to compute. Being the shape and position of the casted shadow is not symmetric with respect to the rotation axis, the interval between the mass of the BH and the rotation axis is proportional for the spin of the BH, from a fixed inclination angle of the observer. r_{gav} being the gravitational radius, the minimum intervals could extent upto $1.5r_{gav}$. this is realized for the maximal Kerr BH, in which the inner edge is an event horizon [10].

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Author declares that he does not have any competing interests as related to this paper.

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