ve-degree, ev-degree and First Zagreb Index Entropies of Graphs

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Abstract. Chellali et al. introduced two degree concepts, ve-degree and ev-degree (Chellali et al, 2017). The ve-degree of a vertex v equals to number of different edges which are incident to a vertex from the closed neighborhod of v. Moreover the ev-degree of an edge $e = ab$ equals to the number of vertices of the union of the closed neighborhoods of a and b. The most private feature of these degree concepts is, total number of ve-degrees and total number of ev-degrees equal to first Zagreb index of the graphs for triangle-free graphs. In this paper we introduce ve-degree entropy, ev-degree entropy and investigate the relations between these entropies and the first Zagreb index entropy. Finally we obtain the maximal trees with respect to ve-degree irregularity index.

Keywords. ve-degree, ev-degree, entropy, information functional

Mathematics Subject Classification. 05C07, 05C90

1. Introduction

There are many studies to determine the complexity of the networks in the last years. Computation of graph entropy measures is used in interdisciplinary researchs for example chemistry, information science and biology $[1-6]$. In order to determine the complexity of the graphs, most of the graph entropies are based on entropy of Shannon $[7]$.

In the literature there are several graph entropy measures by using the order of the graphs, degree sequence of the graphs, distance of the graphs, eccentricity of the graphs, characteristic polynomials and other graph polynomials of the graphs $[7-14]$. Graph entropies which are related to molecular descriptors are introduced in the last years. Network entropies which are based on degree powers of graphs were further studied in the last years $[15-18]$. Calculation of graph entropy measures which are based on matchings and independent sets was defined $[19]$ and furter studied $[20]$. Some relations between Randic index and Randic information were studied by Gutman et al. $[21]$. Some extremal properties of general graph entropies studied by Eliasi $[22]$. Entopies of fullerene graphs which are based on eccentricity of vertices were studied by Ghorbani et al. $[23]$. Entropy of weighted graphs was studied by Kazemi $[24]$. More details about the graph entropies can be found in the book $[25]$.

Vertex-edge domination and end-vertex domination are two mixed type domination invariants. The ve-domination and ev-domination concepts were introduced by Peters $[26]$ and studied in for example $[27,28]$.

Chellali et al. introduced two degree concepts, ve-degree and ev-degree of the graphs [29]. The regularity and irregularity of the graphs about ve-degree and ev-degree were studied by Horoldagva et al. [30]. The ve-degree and ev-degree concepts of the graphs were widely applied to Chemical Graph Theory [31,32]. Approximately forty papers were written about modified versions of the various topological indices with respect to ve-degree and ev-degree. Some chemical materials were investigated with this modified versions of the topological indices [33,34]. It was seen that ve-degree and ev-degree topological indices can be used as possible tools in QSPR researches.

First Zagreb index was introduced by Gutman and Trinajstić in 1972 [35]. The first Zagreb index is a well studied molecular descriptor in the past [36, 37].

In this paper we introduce two graph entropy measures which are based on ve-degrees and ev-degrees of the graphs. Moreover we investigate the relations of these graph entropies with first Zagreb index entropy. Finally we investigate the ve-degree irregularity index and we obtain the maximal trees with respect to ve-degree irregularity index.

2. Preliminaries

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. For a vertex $u \in V(G)$, the open neighborhood of u is defined as $N_G(u) = \{v|\, uv \in E(G)\}$ and the closed neighborhood of u is defined as $N_G[u] = \{u\} \cup N_G(u)$.

The degree of a vertex u is cardinality of $N_G(u)$ and it is denoted by $deg(u)$. The ve-degree of a vertex v equals to number of different edges which are incident to a vertex from the closed neighborhod of v and it is denoted by $deg_{ve}(v)$. Moreover the ev-degree of an edge $e = ab$ equals to the number of vertices of the union of the closed neighborhoods of a and b, it is denoted by $deg_{ev}(v)$.

A graph G is ve-regular if all its vertices have the same ve-degree. A graph G is ev-regular if all its edges have the same ev-degree. The paths, cycles and stars of order n are denoted by P_n, C_n and $S_{1,n-1}$, respectively. Moreover complete graphs of order n are denoted by K_n and complete bipartite graphs are deonted by $K_{p,q}$. The double star graphs $DS_{p,q}$are consisted of the stars $S_{1,p}$ and $S_{1,q}$ such that $n = p + q + 2$. The subdivided star S_k^* is obtained from a star $S_{1,k}$ by adding a vertex to every vertex of the star which has degree one.

Definition 2.1. For a connected graph G [35],

$$M_1(G) = \sum_{u \in V(G)} deg^2(u).$$

Definition 2.2. For a connected graph G,

$$\sum_{v \in V(G)} deg_{ve}(v) = \sum_{e \in E(G)} deg_{ev}(e) = M_1(G) - 3\eta_G$$

such that η_G is total number of triangles contained by G [29].

It implies that for a triangle-free graph G,

$$\sum_{v\in V(G)} deg_{ve}(v) = \sum_{e\in E(G)} deg_{ev}(e) = M_1(G).$$

Definition 2.3. Let G be a triangle-free graph. Then for a vertex $u \in V(G)$ [32]

$$deg_{ve}(v) = \sum_{u\in N_G(v)} deg(u).$$

Definition 2.4. Let G be a triangle-free graph. Then for an edge $e = ab \in E(G)$ [29]

$$deg_{ev}(e) = \deg(a) + \deg(b).$$

Definition 2.5. The entropy of a graph G is defined by Dehmer's information functional method [8] where an arbitrary information functional denoted by f as in the following equations

$$I_f(G) = -\sum_{i=1}^{|V|} \frac{f(v_i)}{\sum_{j=1}^{|V|} f(v_j)} \log\left(\frac{f(v_i)}{\sum_{j=1}^{|V|} f(v_j)}\right)$$

$$= \log\left(\sum_{i=1}^{|V|} f(v_i)\right) - \sum_{i=1}^{|V|} \frac{f(v_i)}{\sum_{j=1}^{|V|} f(v_j)} \log f(v_i).$$

Now we can give the definiton of the ve-degree and ev-degree entropies.

Definition 2.6. For a triangle-free graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, we introduce the information functional such that

$$f := deg_{ve}(v_i) \text{ and } p_{ve}(v_i) = \frac{deg_{ve}(v_i)}{\sum_{i=1}^{n} deg_{ve}(v_i)}$$

for $1 \le i \le n$. Then by using Definition 2.5 we obtain the ve-degree entropy

$$I_{ve}(G) = I_f(G) = -\sum_{i=1}^{n} \frac{deg_{ve}(v_i)}{\sum_{i=1}^{n} deg_{ve}(v_i)} \log\left(\frac{deg_{ve}(v_i)}{\sum_{i=1}^{n} deg_{ve}(v_i)}\right)$$

$$= \log(M_1(G)) - \frac{1}{M_1(G)} \sum_{i=1}^{n} deg_{ve}(v_i) \log\left(deg_{ve}(v_i)\right).$$

Definition 2.7. For a triangle-free graph G with $E(G) = \{e_1, e_2, \ldots, e_m\}$, we introduce the information functional such that

$$f := deg_{ev}(e_i) \text{ and } p_{ev}(e_i) = \frac{deg_{ev}(e_i)}{\sum_{i=1}^{m} deg_{ev}(e_i)}$$

for $1 \le i \le m$. Thus

$$I_{ev}(G) = I_f(G) = -\sum_{i=1}^{m} \frac{deg_{ev}(e_i)}{\sum_{i=1}^{m} deg_{ev}(e_i)} \log\left(\frac{deg_{ev}(e_i)}{\sum_{i=1}^{m} deg_{ev}(e_i)}\right)$$

$$= \log(M_1(G)) - \frac{1}{M_1(G)} \sum_{i=1}^{m} deg_{ev}(e_i) \log\left(deg_{ev}(e_i)\right).$$

Definition 2.8. For a graph G with $V(G) = \{v_1, v_2, \dots, v_n\}$, we introduce the information functional such that

$$f := deg^2(v_i) \text{ and } p_{M_1}(v_i) = \frac{deg^2(v_i)}{M_1(G)}$$

for $1 \leq i \leq n$. Therefore

$$I_{M_1}(G) = I_f(G) = -\sum_{i=1}^{n} \frac{deg^2(v_i)}{\sum_{i=1}^{n} deg^2(v_i)} \log\left(\frac{deg^2(v_i)}{\sum_{i=1}^{n} deg^2(v_i)}\right)$$

$$= \log(M_1(G)) - \frac{1}{M_1(G)} \sum_{i=1}^{n} deg^2(v_i) \log(deg^2(v_i)).$$

We obtain very similar three graph entropy measures. Now we can obtain some relations between them in the next section.

3. Main Results

In order to make some comparisons, we use majorization method as in defined by Das and Shi [18]. We consider non-increasing arrangement of each vector in R^n such that for a vector $x = (x_1, x_2, \dots, x_n) \in R^n$. Thus we have $x_1 \geq x_2 \geq \cdots \geq x_n$.

Definition 3.1. For $x, y \in R^n$, $x \prec y$ if ([18])

$$\begin{cases} \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \,, i = 1,2, \dots, n-1, \\ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \,. \end{cases}$$

When , $x \prec y$, x is said to be majorized by y (y majorizes x).

Let $p_\alpha(G) = \big(p_\alpha(v_1), p_\alpha(v_2), \dots, p_\alpha(v_n)\big)$ is a probably vector of the graph G with respect to a parameter α. Since the information entropy $I_\alpha(G) = -\sum_{i=1}^{n} p_\alpha(v_i) \log p_\alpha(v_i)$,it is obtained that the fuction $h(x) = -x \log x$ is a concave function for $x > 0$. Therefore we can give an essential theorem as used in [18].

Theorem 3.2. Let H and G be two non-isomorphic graphs of order n and $p_\alpha(H), p_\alpha(G)$ be the probability vectors of H and G, respectively. If $p_\alpha(H) \prec p_\alpha(G)$, then we obtain that $I(G) \leq I(H)$.

Example 3.3. For a path graph $P_7\colon v_1 v_2 \dots v_7$, we obtain that $\deg(v_1) = \deg(v_7) = 1$ and the other vertices have degree two. Thus $M_1(P_7) = 22$.

Moreover, $deg_{ve}(v_1) = deg_{ve}(v_7) = 2$, $deg_{ve}(v_2) = deg_{ve}(v_6) = 3$ and $deg_{ve}(v_3) = deg_{ve}(v_4) = deg_{ve}(v_5) = 4$.

The ev-degrees are as follows. $deg_{ev}(v_1 v_2) = deg_{ev}(v_6 v_7) = 3$ and $deg_{ev}(v_2 v_3) = deg_{ev}(v_3 v_4) = deg_{ev}(v_4 v_5) = deg_{ev}(v_5 v_6) = 4$. Then,

$$I_{ve}(P_7) = -(2 \times \frac{2}{22}\log\frac{2}{22} + 2 \times \frac{3}{22}\log\frac{3}{22} + 3 \times \frac{4}{22}\log\frac{4}{22}) = 2{,}739$$

$$I_{M_1}(P_7) = -\left(2 \times \frac{1}{22}\log\frac{1}{22} + 5 \times \frac{4}{22}\log\frac{4}{22}\right) = 2{,}626$$

$$I_{ev}(P_7) = -\left(2 \times \frac{3}{22}\log\frac{3}{22} + 4 \times \frac{4}{22}\log\frac{4}{22}\right) = 2{,}562$$

In the following theorem we obtain a general relation between ve-degree entropy and first Zagreb index entropy of the trees.

Theorem 3.4. Let T be a tree with order n. Then $I_{M_1}(T) \leq I_{ve}(T)$.

Proof. Let T be a tree with order n. We label the vertices of T such that $deg_{ve}(v_1) \geq deg_{ve}(v_2) \geq \cdots \geq deg_{ve}(v_n)$ with probably vectors $p_{ve}(v_i) = \frac{deg_{ve}(v_i)}{M_1(T)}$. Moreover we label the vertices of T such that $deg^2(v_1) \geq deg^2(v_2) \geq \cdots \geq deg^2(v_n)$ with probably vectors $p_{M_1}(v_i) = \frac{deg^2(v_i)}{M_1(T)}$. In these two partitions the orderings of vertices are not same. These probably vectors are obtained from the tree T but the degree concepts are different such that

$$\sum_{v \in V(T)} deg_{ve}(v) = \sum_{v \in V(T)} deg^2(v) = M_1(G),$$

$$\sum_{v \in V(T)} p_{ve}(v) = \sum_{v \in V(T)} p_{M_1}(v) = 1.$$

Therefore we use Theorem 3.2. For a connected tree T

$$\sum_{i=1}^{k} p_{ve}(v_i) < \sum_{i=1}^{k} p_{M_1}(v_i), i = 1,2,\dots,k-1$$

Since T is a tree, at least two vertices of T have degree 1 but the ve-degrees of these vertices are at least 2. Thus summation of ve-degrees for the first $n-1$ vertices is smaller than the summation of squares of the first $n-1$ vertices. Consequently by Theorem 3.2, we obtain that

$$\sum_{i=1}^{n} p_{ve}(v_i) = \sum_{i=1}^{n} p_{M_1}(v_i) = 1,$$

$p_{ve}(T) \prec p_{M_1}(T)$ and $I_{M_1}(T) \leq I_{ve}(T)$.

By Theorem 3.4, we also obtain that $I_{M_1}(P_7) \leq I_{ve}(P_7)$ as in computed in Example 3.3.

We know that the $S_{1,n-1}$ is the only ve-regular tree such that all its vertices have same ve-degree $n-1$. Moreover $S_{1,n-1}$ is the only ev-regular tree such that all its edges have same ev-degree n. The cycle $C_n (n \geq 4)$ is the unique unicyclic graph which is ve-regular and ev-regular.

For simplicity, a ve-regular graph each of whose vertices has ve-degree r is called r_{ve}-regular and an ev-regular graph each of whose edges has ev-degree r is called r_{ev}-regular [29]. For example the cycle graph is 4_{ve}-regular and 4_{ev}-regular for $n \geq 4$. Furthermore the complete graph K_n is m_{ve}-regular such that the size $m = n(n-1)/2$ and it is n_{ev}-regular. A complete bipartite graph $K_{p,q}$ is pq_{ve}-regular and $(p+q)_{ev}$-regular.

Das and Shi [18] obtained that first degree based entropy of the graphs have the maximum value for regular graphs. By a similar way we obtain the following theorems about ve-regular and ev-regular graphs.

Theorem 3.5. Let G be a ve-regular graph and H be an arbitrary graph of order n. Thus $I_{ve}(H) \leq I_{ve}(G) = \log n$, the equality holds if and only if $G \cong H$.

Theorem 3.6. Let G be a ev-regular graph and H be an arbitrary graph of size m. Thus $I_{ev}(H) \leq I_{ev}(G) = \log m$, the equality holds if and only if $G \cong H$.

Result 3.7. Let T be a tree with order n. Thus $I_{ve}(T) \leq I_{ve}\left(S_{1,n-1}\right) = \log n$, the equality holds if and only if $T \cong S_{1,n-1}$.

Result 3.8. Let T be a tree with order n. Thus $I_{ev}(T) \leq I_{ev}\left(S_{1,n-1}\right) = \log(n-1)$, the equality holds if and only if $G \cong S_{1,n-1}$.

Result 3.9. Let G be a unicyclic graphs with order n. Thus $I_{ve}(G) \leq I_{ve}(C_n) = \log n$, the equality holds if and only if $G \cong C_n$ for $n \geq 4$.

Result 3.10. Let G be a unicyclic graphs with order n. Thus $I_{ev}(G) \leq I_{ev}(C_n) = \log n$, the equality holds if and only if $G \cong C_n$ for $n \geq 4$.

Theorem 3.11. Let G be a regular graph. If G is ev-regular, then it is ve-regular ([30]).

The complete bipartite graph $K_{p,q}$ $(p \neq q)$ is both ev-regular and ve-regular, but not regular.

Theorem 3.12. For the cycle C_n other than C_3, $I_{ve}(C_n) = I_{ev}(C_n) = I_{M_1}(C_n) = \log n$.

Proof. By Theorem 3.11, we know that the C_n is 2-regular, 4_{ve}-regular and 4_{ev}-regular. Thus

$$p_{ve}(v_i) = p_{ev}(v_i) = p_{M_1}(v_i) = \frac{4}{4n} = \frac{1}{n}$$

and

$$I_{ve}(C_n) = I_{ev}(C_n) = I_{M_1}(C_n) = -n\frac{1}{n}\log\frac{1}{n} = \log n.$$

Theorem 3.13. Let T be a tree of order n. Then it is obtained that $I_{M_1}(T) \leq I_{M_1}(P_n)$, the equality holds if and only if $T \cong P_n$; $I_{M_1}(T) \geq I_{M_1}(S_n)$, the equality holds if and only if $T \cong S_n$ ([15, 18]).

Theorem 3.14. Let $K_{p,q}$ be a complete bipartite graph. Then $I_{ve}\left(K_{p,q}\right) \leq I_{ev}\left(K_{p,q}\right)$, the equality holds if and only if $T \cong K_{2,2}$.

Proof. A complete bipartite graph $K_{p,q}$ is ve-regular and ev-regular. Thus $I_{ve}\left(K_{p,q}\right) = \log(p+q)$ and $I_{ev}\left(K_{p,q}\right) = \log(pq)$. Since $\log x$ is an increasing function for $x > 0$, $I_{ve}\left(K_{p,q}\right) \leq I_{ev}\left(K_{p,q}\right)$. For the equality, it has to be attained that

$$p + q = pq$$

$$\frac{p+q}{pq} = 1$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

Now we use the $p + q = pq$ and it means $p = q(p - 1)$. Therefore, q divides p. This relation can be obtained as p divides q. Consequently, $p = q$. Then the equality $\frac{1}{p} + \frac{1}{q} = 1$ is obtained for $p = q = 2$.

Theorem 3.15. Let T ($\ncong S_{1,n-1}$) be a tree of order n. Thus

$$i)\ \frac{deg_{ve}(v_n)}{M_1(T)} \geq \frac{2}{n^2 - 3n + 6}$$

$$ii)\ \frac{deg_{ev}(e_{n-1})}{M_1(T)} \geq \frac{3}{n^2 - 3n + 6}$$

with equalities hold if and only if $T \cong DS_{n-3,1}$.

Proof. Let T be a tree of order n with probably vectors

$p_{ve}(T) = (p_{ve}(v_1), p_{ve}(v_2), \ldots, p_{ve}(v_n))$ as $p_{ve}(v_1) \geq p_{ve}(v_2) \ldots \geq p_{ve}(v_n)$ and

$p_{ev}(T) = (p_{ev}(e_1), p_{ev}(e_2), \ldots, p_{ev}(e_{n-1}))$ as $p_{ev}(e_1) \geq p_{ev}(e_2) \ldots \geq p_{ev}(e_{n-1})$.

Let T be a double star graph which is obtained from a star graph $S_{1,n-3}$ with the central vertex u and a path P_2 xy such that u is joined to x. Thus $deg_{ve}(y) = 2$, $deg_{ve}(x) = deg_{ve}(u) = n - 1$ and the remaining $(n-3)$ vertices have ve-degree $(n-2)$. Moreover $deg_{ev}(xy) = 3$, $deg_{ev}(ux) = n$ and the remaining $(n-3)$ edges have ev-degree $(n-1)$. Then $M_1(G) = n^2 - 3n + 6$ and the $p_{ve}(v_n) = \frac{deg_{ve}(v_n)}{M_1(T)} = \frac{2}{n^2-3n+6}$ and $p_{ev}(e_{n-1}) = \frac{deg_{ev}(e_{n-1})}{M_1(T)} = \frac{3}{n^2-3n+6}$

It means that the equalities are obtained. Otherwise, let $T \ncong DS_{n-3,1}$. It is obtained that $M_1(T) = n^2 - 3n + 6$ is the maximum value of the first Zagreb index with maximum degree $(n-2)$ ([37]). We know that ve-degree of a vertex is minimum 2 and ev-degree of an edge is 3 for $n \geq 3$.Therefore,

$$p_{ve}(v_n) = \frac{deg_{ve}(v_n)}{M_1(T)} > \frac{2}{n^2 - 3n + 6}$$

$$p_{ev}(e_{n-1}) = \frac{deg_{ev}(e_{n-1})}{M_1(T)} > \frac{3}{n^2 - 3n + 6}.$$

ve-irregular graphs and ev-irregular graphs were studied Chellali et al. [29] and Horoldagva et al. [30]. Therefore we can intrdoduce the ve-degree Albertson index [39] for further investigations.

Definition 3.16. Let G be a graph of order n. Then ve-degree irregularity of G is computed by

$$irr_{ve}(G) = \sum_{uv \in E(G)} |deg_{ve}(u) - deg_{ve}(v)|.$$

It is clear that $irr_{ve}(G) = 0$ for ve-irregular graphs.

Theorem 3.17. Let T be a simple, connected tree with order n, then

$$i) irr_{ve}(T) \leq \frac{n^2 - 4n + 3}{2}$$

if $n = 2k + 1$ and the equality holds if and only if $T \cong S_k^*$.

$$ii) irr_{ve}(T) \leq \frac{n^2 - 4n + 4}{2}$$

if $n = 2k + 2$ and the equality holds if and only if $T \cong T_6$ depicted in Figure 1.

Proof. In order to prove the equalities we apply some operations to $S_{1,n-1}$ graphs. Clearly the star graphs are ve-regular graphs and they have maximum value of total number of the ve-degrees in trees.

i) Assume that $n = 2k + 1$. If we remove a vertex of degree 1 from a star $S_{1,n-1}$ and attach it to an other vertex which has degree one, we obtain double star graph $T_1 = DS_{n-3,1}$. It is used in the proof of Theorem 3.15. Therefore $irr_{ve}(T_1) = 2n - 6$.

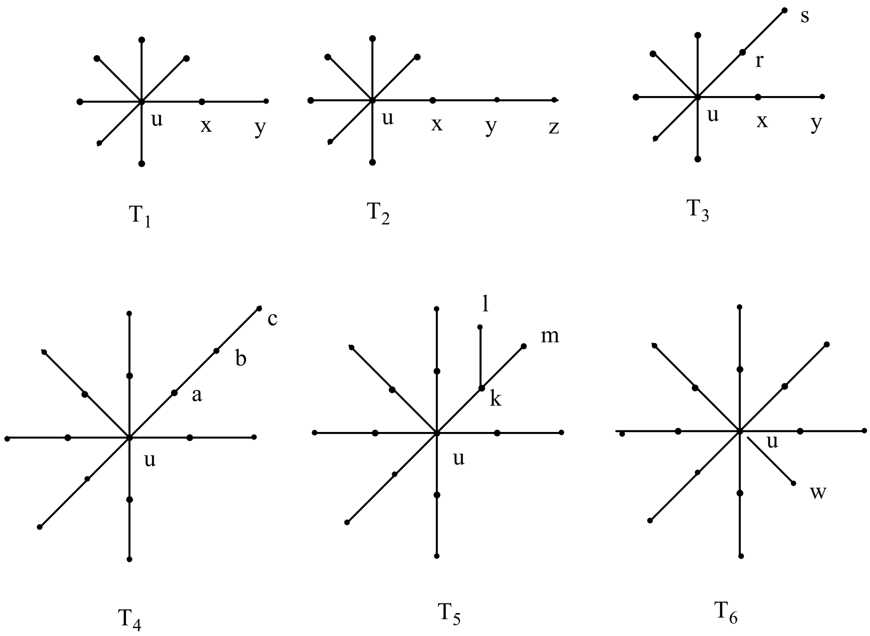

Figure 1. The trees $T_1, T_2, T_3, T_4, T_5, T_6$.

If we remove a vertex of degree one and attach it (z) to the vertex y on T_1, we obtain the tree T_2. It is obtained that $deg_{ve}(z) = 2, deg_{ve}(y) = 3, deg_{ve}(x) = n - 1, deg_{ve}(u) = n - 2$ and the remaining $(n - 4)$ vertices have ve-degree $(n - 3)$ for T_2. Then $irr_{ve}(T_2) = 2n - 6$. We can see that $irr_{ve}(T_1) = irr_{ve}(T_2)$.

Now we remove a vertex of degree one and attach it to a vertex (r) which is incident to the central vertex u on T_1. Thus we obtain the tree T_3. For the tree T_3, $deg_{ve}(y) = deg_{ve}(s) = 2$, $deg_{ve}(x) = deg_{ve}(r) = n-2, deg_{ve}(u) = n-1$ and the remaining $(n-5)$ vertices have ve-degree $(n-3)$. Then $irr_{ve}(T_3) = 4n - 16$. Consequently, $irr_{ve}(T_3) > irr_{ve}(T_1) = irr_{ve}(T_2)$.

By this way we obtain the subdivided star graph S_k^* which is maximal tree with respect to ve-degree irregularity index with order $n = 2k+1$. For S_k^*, ve-degree of the central vertex u is $2k$, ve-degree of the vertices at distance 1 from the u is $(k+1)$ and ve-degree of the vertices at distance 2 from the u is 2. Therefore

$$irr_{ve}(S_k^*) = 2k(k-1) = \frac{n^2 - 4n + 3}{2}.$$

ii) We investigate the second case for $n = 2k+2$. It means that a vertex of degree one has to be attached to a subdivided star graph S_k^*. Thus we observe the tree subcase for T_4, T_5, T_6.

For the tree T_4, $deg_{ve}(c) = 2, deg_{ve}(b) = 3$, $deg_{ve}(a) = k+2, deg_{ve}(u) = 2k$, ve-degree of the $(k-1)$ vertices at distance 1 from the u is $(k+1)$ and ve-degree of the $(k-1)$ vertices at distance 2 from the u is 2. Thus

$$irr_{ve}(T_4) = 2(k-1)^2 + 2(k-2)$$

$$= 2k^2 - 2k$$

$$= \frac{n^2 - 6n + 8}{2}$$

For the tree T_5, $deg_{ve}(l) = deg_{ve}(m) = 3, deg_{ve}(k) = k+2, deg_{ve}(u) = 2k+1$, ve-degree of the $(k-1)$ vertices at distance 1 from the u is $(k+1)$ and ve-degree of the $(k-1)$ vertices at distance 2 from the u is 2. Thus

$$irr_{ve}(T_5) = (k-1)(2k-1) + 3(k-1)$$

$$= (2k+2)(k-1)$$

$$= \frac{n^2 - 4n}{2}$$

For the tree T_6, $deg_{ve}(w) = k+1, deg_{ve}(u) = 2k+1$, ve-degree of the k vertices at distance 1 from the u is $(k+2)$ and ve-degree of the k vertices at distance 2 from the u is 2. Thus

$$irr_{ve}(T_6) = k(2k-1) + k$$

$$= 2k^2$$

$$= \frac{n^2 - 4n + 4}{2}$$

This completes the proof.

Theorem 3.18. $irr_{ve}(P_n) = \begin{cases} 0, & n = 2,3 \\ 2, & n = 4 \\ 4, & n \geq 5 \end{cases}$

Proof. The proof is clear and we omit the details.

Conclusion

In this paper we initiated to study ve-degree and ev-degree entropies of the graphs. Moreover we introduced the ve-degree Albertson index and we obtain the maximal trees for this modified version. There are many open problems in this topics. We obtain that ve-regular and ev-regular trees are maximal trees with respect to ve-degree and ev-degree entropies of the trees. The minimal trees should be found. We also know that there is no ve-regular bicyclic graphs [29]. The extremal trees should be found for bicyclic graphs. Finally, ve-degree irregularity should be extended to other graphs.

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