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ARTICLE TEMPLATE

Dynamics of a System of Higher Order Difference Equations with Quadratic Terms

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ABSTRACT
In this paper we investigate the global asymptotic stability of following system of higher order difference equations with quadratic terms:

$$x_{n+1} = A + B\frac{y_n}{y_{n-m}^2}, y_{n+1} = A + B\frac{x_n}{x_{n-m}^2}$$

where A and B are positive numbers and the initial values are positive numbers. We also study the boundedness, rate of convergence and oscillation behaviour of the solutions of related system.

KEYWORDS
Difference equations; global asymptotic stability; boundedness; rate of convergence; oscillation

1. Introduction

Nowadays, investigation of dynamic behaviour of recursive sequences (also called difference equations) have attracted significant interest. This attention concern with applications to different fields of sciences. Applied sciences need many mathematical models. Mathematical models easily can be create from difference equations or their systems. Although the appearance of the difference equations is simple, the solutions of these equations are difficult to understand and are different from each other. Particularly since the solutions of the higher order difference equations differ in every order, it becomes difficult to understand the behavior of their solutions. Especially researchers have studied the global asymptotic stability, boundedness and oscillatory behavior of system of difference equations. There are many examples related to difference equations or systems. Therefore, studies on difference equations are increasing day by day and will continue to increase. There are many articles and books in the literature on the theory of difference equations and systems, see [1]-[17]. Additionally, there are many papers related to our study as follows:

In [13], Abualrub et al. discussed the global behavior and semi-cycles of positive

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solutions of the system of difference equations

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, y_{n+1} = A + \frac{x_n}{x_{n-k}},$$

where $A > 0$ and the initial values are arbitrary positive numbers.

In [6], Hadžiabdić et al. dealt with the global dynamics of the system of difference equations

$$x_{n+1} = \frac{b_1 x_n^2}{A_1 + y_n^2}, y_{n+1} = \frac{a_2 + c_2 y_n^2}{x_n^2},$$

where the parameters b_1, a_2, A_1, c_2 are positive numbers and the initial condition y_0 is an arbitrary nonnegative number and x_0 is a positive number.

In [7], Khan et al. studied the local asymptotic stability, instability, global asymptotic stability of equilibrium points, and rate of convergence of positive solutions of following two systems of difference equations:

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_{n-k}}{\beta + \gamma y_{n-k+1}^2}, y_{n+1} = \frac{\alpha_1 y_{n-k}}{\beta_1 + \gamma_1 x_{n-k+1}^2}, \\ x_{n+1} &= \frac{a y_{n-k}}{b + c x_{n-k+1}^2}, y_{n+1} = \frac{a_1 x_{n-k}}{b_1 + c_1 y_{n-k+1}^2}, \end{aligned}$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, a, b, c, a_1, b_1, c_1$ and the initial conditions are positive real numbers.

In [15], Papaschinopoulos et al. investigated the boundedness, persistence of positive solutions and global asymptotic stability of following system of difference equations

$$x_{n+1} = \sum_{i=0}^{k} \frac{A_i}{y_{n-i}^{p_i}}, y_{n+1} = \sum_{i=0}^{k} \frac{B_i}{x_{n-i}^{q_i}},$$

where A_i, B_i and the initial values are positive numbers, and p_i, q_i are positive constants for $i = 0, 1, \cdots, k$.

In [16], Stević et al. handled the boundedness of positive solutions of system of difference equations

$$x_{n+1} = A + \frac{y_n^p}{x_{n-1}^p}, y_{n+1} = A + \frac{x_n^p}{y_{n-1}^p},$$

where parameters A, p and q are positive numbers.

In [14], Bao investigated the local stability, oscillation of positive solutions to the system of difference equations

$$x_{n+1} = A + \frac{x_{n-1}^p}{y_n^p}, y_{n+1} = A + \frac{y_{n-1}^p}{x_n^p},$$

where $A > 0$, $p > 1$ and the initial values are positive numbers.

In [17], Stević et al. handled the boundedness character of positive solutions of the following system of difference equations:

$$x_{n+1} = A + \frac{y_n^p}{x_{n-3}^r}, y_{n+1} = A + \frac{x_n^p}{y_{n-3}^r},$$

where $\min\{A, r\} > 0$ and $p \geq 0$.

In [3], Beso et al. investigated the boundedness and global asymptotic stability of solutions of following difference equation

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2},$$

where γ, δ are positive real numbers and the initial conditions x_{-1} and x_0 are positive real numbers.

In [11], Taşdemir studied the periodicity, boundedness, semi-cycles, global asymptotically stability and rate of convergence of solutions of the following higher order difference equation

$$x_{n+1} = A + B \frac{x_n}{x_{n-m}^2},$$

where A and B are positive numbers and the initial values are positive numbers.

In [12], Taşdemir investigated the oscillation, global asymptotic stability and rate of convergence of solutions of following system of second order difference equations with quadratic terms

$$x_{n+1} = A + B \frac{y_n}{y_{n-1}^2}, y_{n+1} = A + B \frac{x_n}{x_{n-1}^2},$$

where A and B are positive numbers and the initial values are positive numbers.

In this paper, we study the system of higher order difference equations with quadratic terms

$$x_{n+1} = A + B \frac{y_n}{y_{n-m}^2}, y_{n+1} = A + B \frac{x_n}{x_{n-m}^2}, \tag{1}$$

where A and B are positive numbers and the initial values are positive numbers and $m \in \{2, 3, \cdots\}$. We especially investigate the stability, global behaviours and rate of convergence of solutions of system (1). We also study the oscillation behaviour of solutions of related system.

Now, we give three theorems which are used during this study.

Let us consider the following system of difference equations:

$$E_{n+1} = (A + B(n)) E_n, \tag{2}$$

where E_n is a $k-$dimensional vector, $A \in C^{k \times k}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{k \times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0, \tag{3}$$

as $n \to \infty$, where $\|\cdot\|$ denotes any matrix norm that is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Theorem 1.1 (Linearized Stability Theorem [9], p.11)**.** *Assume that*

$$X_{n+1} = F(X_n), n = 0, 1, \cdots,$$

is a system of difference equations such that $\bar{X}$ is a fixed point of F.

(i) *If all eigenvalues of the Jacobian matrix B about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, that is, if all of them have absolute value less than one, then $\bar{X}$ is locally asymptotically stable.*

(ii) *If at least one of them has a modulus greater than one, then $\bar{X}$ is unstable.*

Theorem 1.2 ([4])**.** *Let $n \in N_{n_0}^{+}$ and $g(n, u, v)$ be a nondecreasing function in u and v for any fixed n. Suppose that for $n \geq n_0$, the inqualities*

$$\begin{aligned} y_{n+1} &\leq g(n, y_n, y_{n-1}) \\ u_{n+1} &\geq g(n, y_n, y_{n-1}) \end{aligned}$$

hold. Then

$$y_{n_0-1} \leq u_{n_0-1}, y_{n_0} \leq u_{n_0}$$

implies that

$$y_n \leq u_n, n \geq n_0.$$

Theorem 1.3 (Perron's Theorem, [10])**.** *Assume that condition (3) holds. If E_n is a solution of (2), then either $E_n = 0$ for all $n \to \infty$, or*

$$\lim_{n\to\infty} \sqrt[n]{\|E_n\|},$$

or

$$\lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|},$$

exists and is equal to modulus of one of the eigenvalues of matrix A.

2. Boundedness of System (4)

Now, we apply the following change of the variables for system (1):

$$t_n = \frac{x_n}{A}, z_n = \frac{y_n}{A}.$$

Hence, we obtain the new system as:

$$t_{n+1} = 1 + p\frac{z_n}{z_{n-m}^2}, z_{n+1} = 1 + p\frac{t_n}{t_{n-m}^2}, \tag{4}$$

where $p = \frac{B}{A^2} > 0$. From here on, we consider the system (4).

The system (4) has a unique positive equilibrium point such as

$$(\bar{t}, \bar{z}) = \left(\frac{1+\sqrt{1+4p}}{2}, \frac{1+\sqrt{1+4p}}{2}\right),$$

where $p > 0$.

Here, we determine the boundedness character of solutions of system (4).

Firstly, let $p > 0$ and $\{(t_n, z_n)\}_{n=-m}^{\infty}$ be a positive solution of system (4). Hence, by simple calculations, we get the followings:

$$t_n > 1, \tag{5}$$

and

$$z_n > 1, \tag{6}$$

for $n \geq 1$.

Theorem 2.1. *Let $p \in (0,1)$. Then, every solution of system (4) has bounded from below and above.*

Proof. Let $p \in (0,1)$ and $\{(t_n, z_n)\}_{n=-m}^{\infty}$ be a positive solution of system (4). From (5), (6) and system (4), we have

$$t_{n+1} < 1 + p + p^2 t_{n-1}, n \geq 1. \tag{7}$$

Now, we consider Theorem 1.2. According to this, $t_n \leq u_n$, $n = 0, 1, ...$, where $\{u_n\}$ satisfy

$$u_{n+1} = 1 + p + p^2 u_{n-1}, n \geq 1, \tag{8}$$

such that

$$u_s = t_s, u_{s+1} = t_{s+1}, s \in \{-1, 0, 1, ...\}, n \geq s.$$

Therefore, the solution u_n of the difference equation (8) is

$$u_n = \frac{1}{1-p} + p^n C_1 + (-p)^n C_2. \tag{9}$$

Moreover, we obtain the relations (7) and (9) imply that

$$t_{n+1} - u_{n+1} \leq p^2 (t_{n-1} - u_{n-1}), n > s, p \in (0,1).$$

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Thus, we get

$$t_n \le u_n, n > s. \tag{10}$$

Hence, we obtain from (5), (6), (9) and (10)

$$1 < t_n \le \frac{1}{1-p} + p^n C_1 + (-p)^n C_2,$$

where

$$\begin{aligned} C_1 &= \frac{1}{2p}\left(pt_0 + t_1 - \frac{1+p}{1-p}\right), \\ C_2 &= \frac{1}{2p}\left(pt_0 - t_1 + 1\right). \end{aligned}$$

Likewise, we have that

$$1 < z_n \le \frac{1}{1-p} + p^n C_3 + (-p)^n C_4,$$

where

$$\begin{aligned} C_3 &= \frac{1}{2p}\left(pz_0 + z_1 - \frac{1+p}{1-p}\right), \\ C_4 &= \frac{1}{2p}\left(pz_0 - z_1 + 1\right). \end{aligned}$$

□

Theorem 2.2. *Let $p \ge 1$ and m is even. Then every solution of system (4) has bounded from above and below as follows:*

$$1 < t_n < 1 + p\,(1+p)^m,$$

and

$$1 < z_n < 1 + p\,(1+p)^m,$$

for $n \ge 2m+2$.

Proof. Let $p \ge 1$ and $\{(t_n, z_n)\}_{n=-m}^{\infty}$ be a positive solution of system (4). From system (4), we have

$$t_n = 1 + p\frac{z_{n-1}}{z_{n-m-1}^2} = 1 + \frac{p}{z_{n-m-1}}\frac{z_{n-1}}{z_{n-m-1}}.$$

Assume that m is even. Then, we obtain from system (4)

$$t_n = 1 + \frac{p}{z_{n-m-1}}\left(\prod_{i=1}^{\frac{m}{2}} \frac{z_{n-2i+1}}{z_{n-2i-1}}\right). \tag{11}$$

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Moreover, we get from system (4)

$$\begin{aligned} z_{n-2i+1} &= 1+p\frac{t_{n-2i}}{t^2_{n-m-2i}}, \\ \frac{z_{n-2i+1}}{t_{n-2i}} &= \frac{1}{t_{n-2i}}+\frac{p}{t^2_{n-m-2i}}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} t_{n-2i} &= 1+p\frac{z_{n-2i-1}}{z^2_{n-m-2i-1}}, \\ \frac{t_{n-2i}}{z_{n-2i-1}} &= \frac{1}{z_{n-2i-1}}+\frac{p}{z^2_{n-m-2i-1}}, \end{aligned} \tag{13}$$

for $i=1,2,\cdots,\frac{m}{2}$. Thus, multiplying (12) and (13), we get the following:

$$\frac{z_{n-2i+1}}{z_{n-2i-1}} = \left(\frac{1}{t_{n-2i}}+\frac{p}{t^2_{n-m-2i}}\right)\left(\frac{1}{z_{n-2i-1}}+\frac{p}{z^2_{n-m-2i-1}}\right). \tag{14}$$

Additionally, we obtain from (5), (6) and (14)

$$\frac{z_{n-2i+1}}{z_{n-2i-1}} < (1+p)^2,$$

for $i=1,2,\cdots,\frac{m}{2}$. Therefore, we have

$$\prod_{i=1}^{\frac{m}{2}}\frac{z_{n-2i+1}}{z_{n-2i-1}} < (1+p)^m. \tag{15}$$

So, we have from (5), (6), (11) and (15)

$$\begin{aligned} t_n &= 1+\frac{p}{z_{n-m-1}}\left(\prod_{i=1}^{\frac{m}{2}}\frac{z_{n-2i+1}}{z_{n-2i-1}}\right) \\ &< 1+p(1+p)^m, \end{aligned}$$

for $n \geq 2m+2$. With similar calculations, we obtain

$$z_n < 1+p(1+p)^m,$$

for $n \geq 2m+2$. $\square$

3. Global Asymptotic Stability of System (4)

Now, we firstly consider a transformation such as:

$$(t_n, t_{n-1}, \cdots, t_{n-m}, z_n, z_{n-1}, \cdots, z_{n-m}) \to (f, f_1, \cdots, f_m, g, g_1, \cdots, g_m)$$

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where $f = 1 + p\frac{z_n}{z_{n-m}^2}, f_1 = t_n, \cdots f_m = t_{n-m}, g = 1 + p\frac{t_n}{t_{n-m}^2}, g_1 = z_n, \cdots, g_m = z_{n-m}$. Hence, we obtain the following jacobian matrix about equilibrium point $(\bar{t}, \bar{z})$:

$$B(\bar{t}, \bar{z}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{p}{\bar{z}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{p}{\bar{t}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{t}^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{(2m+2)\times(2m+2)}.$$

Theorem 3.1. *The positive equilibrium point $(\bar{t}, \bar{z})$ of system (4) is locally asymptotically stable.*

Proof. The linearized system of system (4) about the positive equilibrium point is given by $X_{N+1} = B(\bar{t}, \bar{z})\, X_N$, where

$$X_N = \begin{pmatrix} t_n \\ \vdots \\ t_{n-m} \\ z_n \\ \vdots \\ z_{n-m} \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & c & 0 & \cdots & 0 & -2c \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & 0 & \cdots & 0 & -2c & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{(2m+2)\times(2m+2)},$$

and $c = \frac{p}{\bar{z}^2} = \frac{p}{\bar{t}^2}$. Let $\lambda_1, \lambda_2, \cdots, \lambda_{2m+2}$ denote the $2m+2$ eigenvalues of matrix E. Let $D = diag\,(d_1, d_2, \cdots, d_{2m+2})$ be a diagonal matrix such that

$$d_1 = d_{m+2} = 1, d_k = d_{m+1+k} = 1 - k\varepsilon, 1 \leq k \leq m,$$

and

$$0 < \varepsilon < \frac{3c-1}{(m+1)(c-1)}.$$

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Clearly, D is invertible matrix. In computing matrix DED^{-1}, we get that

$$DED^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{cd_1}{d_{m+2}} & 0 & \cdots & 0 & \frac{-2cd_1}{d_{2m+2}} \\ \frac{d_2}{d_1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{d_{m+1}}{d_m} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{cd_{m+2}}{d_1} & 0 & \cdots & 0 & \frac{-2cd_{m+2}}{d_{m+1}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \frac{d_{m+3}}{d_{m+2}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \frac{d_{2m+2}}{d_{2m+1}} & 0 \end{pmatrix}.$$

From the following two inequalities

$$\begin{aligned} 1 &= d_1 > d_2 > \cdots > d_m > d_{m+1} > 0, \\ 1 &= d_{m+2} > d_{m+3} > \cdots > d_{2m+1} > d_{2m+2} > 0, \end{aligned}$$

it implies that

$$\begin{aligned} d_2 d_1^{-1} &< 1, d_3 d_2^{-1} < 1, \cdots, d_{m+1} d_m^{-1} < 1, \\ d_{m+3} d_{m+2}^{-1} &< 1, d_{m+4} d_{m+3}^{-1} < 1, \cdots, d_{2m+2} d_{2m+1}^{-1} < 1. \end{aligned}$$

Moreover, we have that

$$\begin{aligned} cd_1 d_{m+2}^{-1} + 2cd_1 d_{2m+2}^{-1} &= c\left(1 + \frac{2}{1-(m+1)\varepsilon}\right) < 1, \\ cd_{m+2} d_1^{-1} + 2cd_{m+2} d_{m+1}^{-1} &= c\left(1 + \frac{2}{1-(m+1)\varepsilon}\right) < 1. \end{aligned}$$

It is a well-known fact that E has the same eigenvalues as DED^{-1}. Thus, we get

$$\begin{aligned} \max_{1\le k\le 2m+2} |\lambda_k| &= \left\| DED^{-1} \right\| \\ &= \max \left\{ \begin{array}{c} d_2 d_1^{-1}, \cdots, d_{m+1} d_m^{-1}, \\ d_{m+3} d_{m+2}^{-1}, \cdots, d_{2m+2} d_{2m+1}^{-1}, \\ cd_1 d_{m+2}^{-1} + 2cd_1 d_{2m+2}^{-1}, \\ cd_{m+2} d_1^{-1} + 2cd_{m+2} d_{m+1}^{-1} \end{array} \right\} \\ &< 1. \end{aligned}$$

So, the positive equilibrium point $(\bar{t}, \bar{z})$ of system (4) is locally asymptotically stable. □

Theorem 3.2. *Suppose that* $0 < p < \frac{1}{2}$. *Then the positive equilibrium point of system (4) is globally asymptotically stable.*

Proof. From (5) and (6), we know that

$$\begin{aligned} 1 &< l_1 = \liminf_{n\to\infty} t_n, \\ 1 &< l_2 = \liminf_{n\to\infty} z_n, \\ 1 &< L_1 = \limsup_{n\to\infty} t_n, \\ 1 &< L_2 = \limsup_{n\to\infty} z_n. \end{aligned}$$

Thus, we have the followings for system (4)

$$\begin{aligned} L_1 &\leq 1 + p\frac{L_2}{l_2^2}, l_1 \geq 1 + p\frac{l_2}{L_2^2}, \\ L_2 &\leq 1 + p\frac{L_1}{l_1^2}, l_2 \geq 1 + p\frac{l_1}{L_1^2}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} L_1 + p\frac{l_1}{L_1} &\leq L_1 l_2 \leq l_2 + p\frac{L_2}{l_2}, \\ L_2 + p\frac{l_2}{L_2} &\leq L_2 l_1 \leq l_1 + p\frac{L_1}{l_1}. \end{aligned}$$

Hence, we get that

$$\begin{aligned} L_1 + p\frac{l_1}{L_1} + L_2 + p\frac{l_2}{L_2} \leq l_2 + p\frac{L_2}{l_2} + l_1 + p\frac{L_1}{l_1}, \\ L_1 + p\frac{l_1}{L_1} + L_2 + p\frac{l_2}{L_2} - l_2 - p\frac{L_2}{l_2} - l_1 - p\frac{L_1}{l_1} \leq 0, \\ (L_1 - l_1)\left(1 - p\left(\frac{1}{l_1} + \frac{1}{L_1}\right)\right) + (L_2 - l_2)\left(1 - p\left(\frac{1}{l_2} + \frac{1}{L_2}\right)\right) \leq 0. \end{aligned}$$

From $l_1, L_1, l_2, L_2 > 1$, we have

$$\frac{1}{l_1} + \frac{1}{L_1} < 2,$$

and

$$\frac{1}{l_2} + \frac{1}{L_2} < 2.$$

Hence, we get

$$\begin{aligned} 1 - p\left(\frac{1}{l_1} + \frac{1}{L_1}\right) &> 1 - 2p, \\ 1 - p\left(\frac{1}{l_2} + \frac{1}{L_2}\right) &> 1 - 2p. \end{aligned}$$

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Meanwhile, we know that $L_1 \geq l_1$ and $L_2 \geq l_2$. Therefore, if $1-2p > 0$, then we obtain

$$L_1 - l_1 \leq 0, L_2 - l_2 \leq 0.$$

So, $L_1 = l_1$ and $L_2 = l_2$. □

4. Rate of Convergence of System (4)

This section, we investigate the rate of convergence of system (4).

Theorem 4.1. *Assume that $0 < p < \frac{1}{2}$ and $\{(t_n, z_n)\}_{n=-m}^{\infty}$ be a solution of the system (4) such that $\lim_{n\to\infty} t_n = \bar{t}$ and $\lim_{n\to\infty} z_n = \bar{z}$. Then, the error vector*

$$E_n = \begin{pmatrix} e_n^1 \\ \vdots \\ e_{n-m}^1 \\ e_n^2 \\ \vdots \\ e_{n-m}^2 \end{pmatrix} = \begin{pmatrix} t_n - \bar{t} \\ \vdots \\ t_{n-m} - \bar{t} \\ z_n - \bar{z} \\ \vdots \\ z_{n-m} - \bar{z} \end{pmatrix},$$

of every solution of system (4) satisfies both of the following asymptotic relations:

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{\|E_n\|} &= |\lambda F_J(\bar{t}, \bar{z})|, \\ \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} &= |\lambda F_J(\bar{t}, \bar{z})|, \end{aligned}$$

where $\lambda F_J(\bar{t}, \bar{z})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{t}, \bar{z})$.

Proof. We firstly set to find the error terms,

$$\begin{aligned} t_{n+1} - \bar{t} &= \sum_{i=0}^{k} A_i (t_{n-i} - \bar{t}) + \sum_{i=0}^{k} B_i (z_{n-i} - \bar{z}), \\ z_{n+1} - \bar{z} &= \sum_{i=0}^{k} C_i (t_{n-i} - \bar{t}) + \sum_{i=0}^{k} D_i (z_{n-i} - \bar{z}), \end{aligned}$$

and $e_n^1 = t_n - \bar{t}, e_n^2 = z_n - \bar{z}$. Hence we obtain

$$\begin{aligned} e_{n+1}^1 &= \sum_{i=0}^{k} A_i e_{n-i}^1 + \sum_{i=0}^{k} B_i e_{n-i}^2, \\ e_{n+1}^2 &= \sum_{i=0}^{k} C_i e_{n-i}^1 + \sum_{i=0}^{k} D_i e_{n-i}^2, \end{aligned}$$

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where $A_i = 0$ and $D_i = 0$ for $i = 0, 1, \cdots, m$,

$$\begin{aligned} B_0 &= \frac{p}{z_{n-m}^2}, B_i = 0, i \in \{1, 2, \cdots, m-1\}, B_m = \frac{-p\left(\bar{z} + z_{n-m}\right)}{\bar{z} z_{n-m}^2}, \\ C_0 &= \frac{p}{t_{n-m}^2}, C_i = 0, i \in \{1, 2, \cdots, m-1\}, C_m = \frac{-p\left(\bar{t} + t_{n-m}\right)}{\bar{t} t_{n-m}^2}. \end{aligned}$$

Taking the limits, we have $\lim_{n\to\infty} A_i = \lim_{n\to\infty} D_i = 0$ for $i \in \{0, 1, \cdots, m\}$ and $\lim_{n\to\infty} B_i = \lim_{n\to\infty} C_i = 0$ for $i \in \{1, \cdots, m-1\}$. Moreover, we obtain that

$$\begin{aligned} \lim_{n\to\infty} B_0 &= \frac{p}{\bar{z}^2}, \lim_{n\to\infty} B_m = \frac{-2p}{\bar{z}^2}, \\ \lim_{n\to\infty} C_0 &= \frac{p}{\bar{t}^2}, \lim_{n\to\infty} C_m = \frac{-2p}{\bar{t}^2}. \end{aligned}$$

Hence,

$$\begin{aligned} B_0 &= \frac{p}{\bar{z}^2} + a_n, B_m = \frac{-2p}{\bar{z}^2} + b_n, \\ C_0 &= \frac{p}{\bar{t}^2} + c_n, C_m = \frac{-2p}{\bar{t}^2} + d_n, \end{aligned}$$

where $a_n \to 0, b_n \to 0, c_n \to 0, d_n \to 0$ as $n \to \infty$. Thus, we get the system of the form (2)

$$E_{n+1} = \left(A + B\left(n\right)\right) E_n,$$

where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{p}{\bar{z}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{p}{\bar{t}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{t}^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$B(n) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & a_n & 0 & \cdots & 0 & b_n \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c_n & 0 & \cdots & 0 & d_n & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

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and $\|B(n)\| \to 0$ as $n \to \infty$. Therefore, we can write the limiting system of error terms about the equilibrium point $(\bar{t}, \bar{z})$ as follows:

$$\begin{pmatrix} e^1_{n+1} \\ e^1_n \\ \vdots \\ e^1_{n-m+1} \\ e^2_{n+1} \\ e^2_n \\ \vdots \\ e^2_{n-m+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{p}{\bar{z}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{p}{\bar{t}^2} & 0 & \cdots & 0 & \frac{-2p}{\bar{t}^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ \vdots \\ e^1_{n-m} \\ e^2_n \\ e^2_{n-1} \\ \vdots \\ e^2_{n-m} \end{pmatrix},$$

which is same as linearized system of system (4) about equilibrium point $(\bar{t}, \bar{z})$. $\square$

5. Oscillatory of Solutions of System (4)

We now study the oscillatory behavior of solutions of system(4).

Theorem 5.1. *Suppose that $\{(t_n, z_n)\}_{n=-m}^{\infty}$ be a positive solution of system (4) and $p > 0$. Then, either $\{(t_n, z_n)\}_{n=-m}^{\infty}$ solution of system (4) has a single semicycle or $\{(t_n, z_n)\}_{n=-m}^{\infty}$ solution of system (4) has semicycles with at most m terms.*

Proof. Let $\{(t_n, z_n)\}_{n=-m}^{\infty}$ solution of system (4) has at least two semicycles. Hence, there exists $N \geq 0$ such that either

$$t_N, z_{N+1} < \bar{t} = \bar{z} < t_{N+1}, z_N$$

or

$$t_{N+1}, z_N < \bar{t} = \bar{z} < t_N, z_{N+1}.$$

Firstly, we assume that $t_N, z_{N+1} < \bar{t} = \bar{z} < t_{N+1}, z_N$. Moreover, we suppose that positive semicycle have m terms and it begins with the term (t_{N+1}, z_{N+1}). Thus, we obtain the followings

$$\begin{aligned} t_N &< \bar{t} = \bar{z} < t_{N+m}, \\ z_{N+m} &< \bar{t} = \bar{z} < z_N. \end{aligned}$$

From this, we get

$$\begin{aligned} t_{N+m+1} &= 1 + p\frac{z_{N+m}}{z_N^2} < \bar{t} = \bar{z}, \\ z_{N+m+1} &= 1 + p\frac{t_{N+m}}{t_N^2} > \bar{t} = \bar{z}. \end{aligned}$$

$\square$

6. Numerical examles

In this here, we provide two examples to verify our theoretical results.

Example 6.1. We handle system (4) with $m = 3$ and $p = 0.49$. Therefore, we obtain the following fourth order system of difference equations

$$t_{n+1} = 1 + 0.49\frac{z_n}{z_{n-3}^2}, z_{n+1} = 1 + 0.49\frac{t_n}{t_{n-3}^2}. \tag{16}$$

Now, we consider this system with the initial values $t_{-3} = 6, t_{-2} = 1, t_{-1} = 0.8, t_0 = 4, z_{-3} = 0.4, z_{-2} = 5, z_{-1} = 3$ and $z_0 = 10$. Thus, positive equilibrium point $(\bar{t}, \bar{z}) = (1.36, 1.36)$ of system (16) is globally asymptotically stable. Figure 1 verifies to our theoretical results.

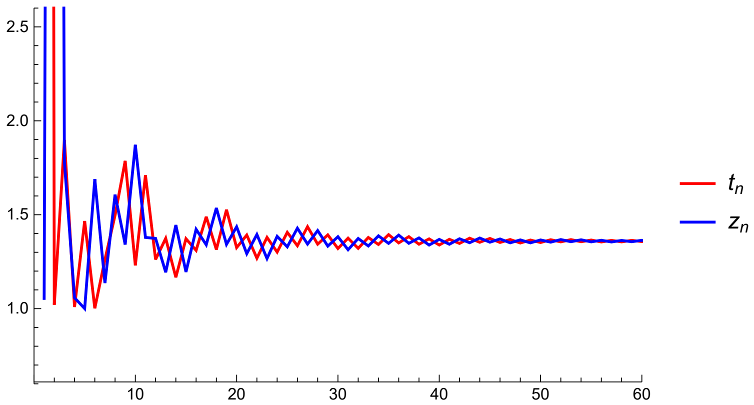

Figure 1. Plot of system (4) with $m = 3$ and $p = 0.49$.

Example 6.2. We take system (4) with $m = 2$ and $p = 1.5$. Hence, we have the following third order system of difference equations

$$t_{n+1} = 1 + 1.5\frac{z_n}{z_{n-2}^2}, z_{n+1} = 1 + 1.5\frac{t_n}{t_{n-2}^2}. \tag{17}$$

We also consider this system with the initial values $t_{-2} = 3, t_{-1} = 4, t_0 = 0.6, z_{-2} = 2, z_{-1} = 0.4$ and $z_0 = 3$. Therefore, the solutions of system (17) oscillate about positive equilibrium point $(\bar{t}, \bar{z}) = (1.82, 1.82)$. Additionally, the every solutions of system (17) has bounded from below and above. Figure 2 verifies our theoretical results.

7. Conclusions and an Open Problem

In this study, we analysed the dynamics of system (4) of higher order difference equations with quadratic terms. We firstly examined the equilibrium point of system (4). Then, we investigated the existence of bounded solutions of system (4). We also studied the stability analysis of solutions of system (4). As a result of this, we obtained that positive equilibrium point of system (4) is globally asymptotically stable when $0 < p < \frac{1}{2}$. Additionally, we investigated the rate of convergence of system (4). Moreover, we revealed an oscillatory result such that the solution of system (4) has a single

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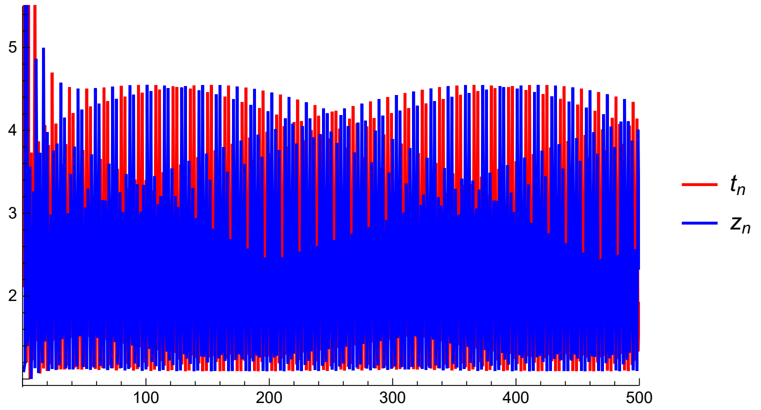

Figure 2. Plot of system (4) with $m = 2$ and $p = 1.5$.

semicycle or the solution of system (4) has semicycles with at most m terms. In addition, we presented two numerical examples to verify our theoretical results.

Open Problem: Investigate the dynamics of following non-symmetric system of difference equations

$$x_{n+1} = A + B\frac{y_n}{y_{n-m}^2}, y_{n+1} = C + D\frac{x_n}{x_{n-m}^2},\ n = 0, 1, ...,$$

where the initial values are real numbers and both $A \neq B$ and $C \neq D$ are real numbers.

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