NEW BOUNDS FOR THE HAUSDORFF DIMENSION OF A DYNAMICALLY DEFINED CANTOR SET

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Abstract. In this paper we use the additive thermodynamic formalism to obtain new bounds of the Hausdorff and box-counting dimension of certain non conformal hyperbolic repellers defined by piecewise smooth expanding maps on a d-dimensional smooth manifold M.

1. Introduction

In many important situations the study of the time evolution of a system can be reduced to the iterations of a map f over a phase space M. This iterations produce recursive sequences, so called orbits, which are mathematical representions of the time evolution of the system. If we are interested in a qualitative description of the dynamics we may direct our inquiries to the geometry of these orbits. This approach motivate the study of topological properties of dynamical systems, which is the subject of the present volume. The idea goes back to the work of Poincaré and is the subject of the modern theory of dynamical systems. See [7]

In this contribution we study some geometric properties of a class of sets originated from the iteration of piecewise differentiable maps f over smooth manifolds M. Concretely we are interested in the fractal geometry of Cantor sets defined by iterations of a piecewise C^r $(r > 1)$ uniformly expanding maps $f : \bigcup_i U_i \to M$. For instance, the classical ternary Cantor set can be defined as the set of points whose orbits do not escape from the intervals $[0, 1/3] \cup [2/3, 1]$ under the iterations of the expanding map $f(x) = 3x \mod 1$ acting on the circle $M = \mathbb{S}^1$.

The fractal geometry of sets and measures studies irregular sets such as non rectifiable curves and Cantor sets. To do this, important dimensional characteristics as the Hausdorff and box-counting dimensions are introduced. Theses quantities capture some important geometric properties of fractal sets. It is also important to determine whether or not the Hausdorff measure, used to calculate the Hausdorff dimension, is finite and positive as well as the continuity of these quantities under small perturbations, among other things. We refer to [4] and [12] for a complete exposition of the theory and some of its applications.

The application of ideas and methods from fractal geometry to dynamical systems had been considered widely in the literature and originated a number of remarkable results as those appearing in the study of self-similar and self affine subsets of the plane, dynamically defined Cantor sets produced by piecewise expanding smooth maps of the interval –which are non linear analogs of the ternary Cantor set–, conformal and non conformal hyperbolic repellers as well as the Cantor sets obtained as transversal sections of the stable and unstable laminations of hyperbolic horseshoes. We refer to [2] and [15] for a comprehensive exposition of the dimension theory of dynamical systems.

In this paper we consider the case of non conformal hyperbolic repellers defined by piecewise C^r, $r > 1$ differentiable expanding maps. We refer to [1] and [16] for an overview of problems and results on the subject. Concretely we use the additive thermodynamic formalism to obtain some useful bounds for the Hausdorff and box-counting dimension of dynamically defined Cantor sets.

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The problem had been discussed previously in [6], [9], [10], [11], [13], [14], [18]. Our approach will rely upon classical and well known ideas from the additive thermodynamic formalism which we will explain below.

2. Thermodynamics and dimension: the setting

Let us start recalling the notion of Hausdorff dimension of a set. Let (X, d) a complete metric space and $Z \subset X$ a subset. We say that a countable collection of open sets $\mathcal{U} = \{U_i\}$ of Z is a **δ-covering** if $Z \subset \bigcup_i U_i$ and $\mathrm{diam}(U_i) < \delta$ for every i. We define an outer measure:

$$\mathcal{H}_{a,\delta}(Z) = \inf_{\mathcal{U}} \sum_{i=1}^{+\infty} \mathrm{diam}(U_i)^a$$

where infimum is taken over all the δ-coverings of Z. The outer measure $\mathcal{H}_{a,\delta}(Z)$ is a non decreasing function of δ so we define the **a-Hausdorff measure of Z** as

$$\mathcal{H}_a(Z) = \lim_{\delta \to 0^+} \mathcal{H}_{a,\delta}(Z) = \sup_{\delta > 0} \mathcal{H}_{a,\delta}(Z)$$

since $\mathcal{H}_{a,\delta}(Z)$ is non decreaqsing in δ. This is the well known Carathéodory's method. The same procedure with closed and convex sets produces the same measure, in particular, $\mathcal{H}_a$ is a Borel regular measure. It holds that, if $\mathcal{H}_a(Z) > 0$ the $\mathcal{H}_s(Z) = +\infty$ for every $s < a$. See [12, Theorem 4.7, Chapter 4]. Similarly so if $\mathcal{H}_a(Z) = 0$ then $\mathcal{H}_s(Z) = 0$ for every $s > a$. We therefore define the **Hausdorff dimension of Z as**

$$\dim_{\mathcal{H}}(Z) = \inf\{a > 0 : \mathcal{H}_a(Z) = 0\} = \sup\{a > 0 : \mathcal{H}_a(Z) = +\infty\}$$

This is a dimensional characteristic and it has the following well known properties:

(1) $\dim_{\mathcal{H}}(Z) \leq \dim_{\mathcal{H}}(Y)$ if $Z \subseteq Y$;
(2) $\dim_{\mathcal{H}}(\{point\}) = 0$ and
(3) $\dim_{\mathcal{H}}\left(\bigcup_{i=0}^{+\infty} Z_i\right) = \sup_{i>0} \dim_{\mathcal{H}}(Z_i)$.

The Hausdorff dimension is invariant under Lipschitz continuous map. See for instance [4] and [12]. Another important dimensional characteristic of a set is the **box countig dimension or limit capacity**: let $Z \subset X$ be a compact set and define:

$$\mathcal{N}(X, \rho) = \inf\{\ \#\mathcal{U} :\ \mathcal{U} = \{B(x_i, \rho)\},\ \text{finite covering of } Z \text{ by } \rho\text{-balls } B(x, \rho)\ \}$$

Then,

$$\overline{\dim_B}(Z) = \limsup_{\rho \to 0^+} -\frac{\log(\mathcal{N}(X, \rho))}{\log(\rho)}$$

is the **upper box-counting dimension** of Z. Similarly so, we define the **lower box-counting dimension**

$$\underline{\dim_B}(Z) = \liminf_{\rho \to 0^+} -\frac{\log(\mathcal{N}(X, \rho))}{\log(\rho)}$$

and, finally, we have the **box-counting dimension**

$$\dim_B(Z) = \lim_{\rho \to 0^+} -\frac{\log(\mathcal{N}(X, \rho))}{\log(\rho)}$$

if the limit exists. Box-counting dimension can also be defined using the Carathéodory method: let $Z \subset X$ and $a > 0$ we define the measure

$$B_{a,\delta}(Z) = \inf_{\mathcal{U}} \sum_i \mathrm{diam}(B(x_i, \delta))^a,$$

where infimum is taken over the family of **finite** open coverings by δ-balls $\mathcal{U} = \{B(x_i, \delta)\}$. Then, up to a constant,

$$B_{a,\delta}(Z) = \mathcal{N}(Z, \delta)\delta^a \quad \text{and we then define} \quad B_a(Z) = \lim_{\delta \to 0^+} B_{a,\delta}(Z).$$

If $\overline{\dim_B}(Z) < a$ then $B_a(Z) = 0$ for, in this case,

$$L = \limsup_{\delta \to 0^+} \frac{\log(\mathcal{N}(Z, \delta)\delta^a)}{\log(\delta^{-1})} < 0$$

and therefore, $\mathcal{N}(Z, \delta)\delta^a \leq \delta^{-L}$ for every sufficiently small $\delta > 0$. Similarly, $a < \underline{\dim_B}(Z)$ implies $B_a(Z) = +\infty$. If $\dim_B(\Lambda) = \underline{\dim_B}(Z) = \overline{\dim_B}(Z)$ then

$$\dim_B(\Lambda) = \inf\{a > 0 : B_a(Z) = 0\} = \sup\{a > 0 : B_a(Z) = +\infty\}.$$

Moreover,

$$\dim_{\mathcal{H}}(Z) \leq \underline{\dim_B}(Z) \leq \overline{\dim_B}(Z)$$

since $\mathcal{H}_{a,\delta} \leq B_{a,\delta}(Z)$, for every $\delta > 0$.

We are concerned with the Hausdorff dimension of the limit set of certain geometrical constructions. Let us describe our basic model. Let $N \subset M$ be a compact domain with non empty interior. What we have in mind is basically a closed d-cube $[0,1]^d$ embedded in M with piecewise smooth border ∂N, where $d = \dim(M)$. We also need finitely many one-to-one maps $g_i : N \to N$, $i = 1, \cdots, s$ satisfying the following conditions:

- the maps g_i, $i = 1, \cdots s$ are C^r $(r > 1)$ **contractions**:
$$\|Dg\| := \max_{1 \leq i \leq s} \sup_{x \in U} \|Dg_i(x)\| < 1;$$
- **open condition**: there exists an open neighborhood $N \subset U$ such that $g_i(U) \subset int(N)$ are pairwise disjoint: $g_i(U) \cap g_j(U) = \emptyset$ for $i \neq j$;
- **border condition**: we will suppose in addition that $\partial N_i \cap \partial N = \emptyset$. We introduce this condition as we need it in some part of our arguments. However it is not too restrictive as long as we can approximate a generic geometric construction by a construction satisfying this property. See below.

By a well known theorem due to Hutchinson, there exists a limit set

$$\Lambda = \bigcap_{n=1}^{+\infty} \bigcup_{(i_1, \ldots, i_n) \in s^n} g_{i_1} \circ \cdots \circ g_{i_n}(N),$$

which is an attractor of the IFS $\mathbf{g} = \{g_i\}$ acting upon the (complete) metric space of compact subsets of N with the Hausdorff distance as a set map

$$\mathbf{g}(A) = \bigcup_i g_i(A) \quad \text{with a composition law} \quad \mathbf{g} \circ \mathbf{g}(A) = \bigcup_{i,j} g_i \circ g_j(A).$$

The family of sets $\wp_n = \{g_{i_1} \circ \cdots \circ g_{i_n}(N) : (i_1, \ldots, i_n) \in s^n\}$, defined at the n-stage of the construction, will be called **atoms of generation** n:

$$\mathbf{g}^n(N) = \bigcup_{P \in \wp_n} P.$$

The family of sets of first generation will be called the **template** of the construction. Topologically, the attractor of a geometric construction Λ satisfying the open and border condition is a Cantor set, that is, a compact, totally disconnected, perfect set. In particular Λ has zero topological dimension. Furthermore, our method requires that there exists a well defined bounded relation between diameter and volume in the following sense.

Definition 2.1. We say that a subset $X \subset M$ is **volume reducible** if it can be covered with **at most countably many smooth subsets** B_i **of positive** k**-volume**, where $\mathrm{Vol}_k(B_i) \asymp radius(B)^k$ uniformly, for some $k < d$. By smooth we mean that $B_i \subset M$ is an smoothly embedded k-disc. A subset is **volume irreducible** if it is not volume reducible.

A cartesian product of Cantor sets in the plane is volume irreducible since it can not be covered by countably many intervals. Similarly so a Cantor set $\Lambda \subset \mathbb{R}^d$ whose projections onto the every k dimensional coordinate plane is a Cantor set. Moreover, if some of these projections have positive Lebesgue measure, the set is volume irreducible. A sufficient condition for a limit set Λ to be volume reducible is that there exists an embedded smooth submanifold $Z \subset M$ of codimension ≥ 1 such that $\Lambda \subset Z$. This situation appears in some examples of self-affine subsets, depending of the initial configuration of the template. Volume reducible configurations are exceptional in that they can be removed by an small perturbation of the initial template of the IFS.

If the construction converges to a non rectifiable connected set, as in the case of Weierstrass-like graphs, the limit set is volume irreducible in that it can not be covered by countably many **smooth** k-discs. This type of examples are not directly tractable with our methods; however, in some cases, we can approximate its attractor by volume irreducible Cantor sets, as we will see in brief. Therefor, we will suppose, for simplicity, that the limit of our constructions are volume irreducible Cantor sets in the d dimensional ambient manifold M.

Definition 2.2. We say that a contractive Iterated Function System (IFS) $\{g_i\}$ satisfying the open and border condition, whose limit set is volume irreducible, is a **regular geometric construction**.

In principle, our definition exclude some interesting examples: let $n > m$ be two positive integers and divide the unit square $\mathsf{R} = [0,1]^2$ into rectangles of width $1/m$ and heigth $1/n$. Then choose finitely many rectangles R_i, $i = 1, \cdots, s$ from this collection of nm rectangles and introduce affine contractions

$$g_i(x,y) = \begin{bmatrix} m^{-1} & 0 \\ 0 & n^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + (x_i, y_i) \quad \text{such that} \quad \mathsf{R}_i = g_i(\mathsf{R})$$

The limit Λ of the IFS $\{g_i\}$ is known as a **Bedford-McMullen carpet** and is a well understood non conformal self-affine subset of the plane. We refer to the survey [5] for a review of results on fractal geometry of these interesting self-affine subsets. It might happen however that, depending on the choice of the rectangles R_i, this geometric construction do not satisfy our hypothesis of regularity. Moreover, depending on the initial configuration of the template it might happen that the geometric construction converges to a Cantor set in an interval or to a non rectifiable curve, as a Weierstrass-like graph. However, we can slightly perturb the IFS to get a regular construction converging to a Cantor set. For this we choose a small $\epsilon > 0$ an define the contraction:

$$g_i^{\epsilon}(x,y) = \begin{bmatrix} (m+\epsilon)^{-1} & 0 \\ 0 & (n+\epsilon)^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + (x_i, y_i) \quad \text{such that} \quad \mathsf{R}_i^{\epsilon} = g_i^{\epsilon}(\mathsf{R})$$

such that R_i^{ϵ}, $i = 1, \cdots, s$ are pairwise disjoint and satisfy the open and border condition so that we can approximate from inside the attractor Λ by Cantor sets. To see this let $\epsilon_n \downarrow 0$ be a decreasing sequence of positive numbers and define Λ_n as the limit of the corresponding IFS $\{g_i^{\epsilon_n}\}$. Then

$$\Lambda = \overline{\bigcup_{n>0} \Lambda_n}, \qquad \text{where} \qquad \Lambda_n \subset \Lambda_{n+1}, \quad \text{for every} \quad n > 0.$$

In other words, generically, Bedford-McMullen carpets are limits of increasing sequences of Cantor sets defined by suitable regular geometric constructions, in the sense of our definition. This cover the case of the non rectifiable Weierstrass-like graphs mentioned before.

Moreover, we can slightly displace the rectangles, horizontally or vertically, to get a regular construction. Of course the Hausdorff dimension of these new configurations vary and perhaps there might not exist known exact formulas for these perturbations. However, as we will see below, our methods produce robust bounds for the Hausdorff and box-counting dimension which are still valid for slightly perturbed geometric constructions. This could be enough for some purposes.

Dynamically, our limits sets are hyperbolic repellers of a suitable piecewise expanding map. Let $f : \bigcup_i U_i \to U$ be defined a piecewise C^r, $r > 1$ differentiable map defined as $f|U_i = g_i^{-1}$, where $U_i = g_i(U)$, $i = 1, \cdots, s$ and $N \subset U$ is the open neighborhood provided by the open condition. Then,

$$\Lambda = \bigcap_{n=0}^{+\infty} f^{-n} N$$

is the locally maximal f-invariant subset of f in $\bigcup_i N_i$. Λ is the set of point which never escapes from the initial template $\{N_i\}$. This is the simplest model of a non linear hyperbolic repeller topologically conjugated to a full-shift on s-symbols: there are constants $C > 0$ and $\lambda > 1$ such that:

$$\|Df^n(x)v\| \geq C\lambda^n \|v\| \quad \forall v \in T_xM, \ x \in \Lambda$$

and there exists a homeomorphism $h : B^+(s) \to \Lambda$ intertwining f and the shift σ: $f \circ h = h \circ \sigma$, where $\sigma(\omega)(n) = \omega(n+1)$, $\omega \in B^+(s) = \{1, \cdots, s\}^{\mathbb{Z}^+}$ with the product topology. In contrast with other approaches we do not use directly the symbolic dynamics.

There exists a deep connection between the dimension theory of dynamical systems and thermodynamic formalism. Let us recall some crucial notions of the theory.

Let (X, d) the a compact metric space and $f : X \to X$ a continuous self map. We say that a subset $E \subset X$ is (ϵ, n)-separated if for every pair of points $x, y \in E$ there exists $0 \leq k < n$ such that $d(f^k(x), f^k(y)) > \epsilon$. The cardinal of a maximal (ϵ, n)-separated subset is essentially the number of dynamically different orbits up to time n with precision $\epsilon > 0$. The topological entropy, a quantitative indicator of the complexity of the dynamics, can be computed as the rate of growing of dynamically different orbits up to finite arbitrarily small precision. The **topological pressure** of a continuous potential ϕ is a generalization of the topological entropy and can be defined as a the rate of growing of weighted dynamically different orbits up to finite time and small precision,

$$P(\phi) = \lim_{\epsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log \left(\sup_E \sum_{x \in E} e^{S_n \phi(x)} \right),$$

where supremum is taken over the family of (ϵ, n)-separated subsets and $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(f^k(x))$. See [3] and [20]. This important quantity is invariant under topological conjugation and can be used as a dynamical indicator of the statistical properties of the system. The following **variational principle** is a cornestone of the thermodynamic formalism:

$$P(\phi) = \sup_{\mu \in \mathcal{M}_f} \left\{ h(\mu) + \int \phi d\mu \right\}$$

where $\mathcal{M}_f$ is the set of f-invariant Borel probabilites, $h(\mu)$ the **Kolmogorov-Sinai** metric entropy and

$$P_\mu(\phi) = h(\mu) + \int \phi d\mu$$

is the **free energy**, also called the **measure-theoretical pressure**. We say that $\mu \in \mathcal{M}_f$ is an **equilibrium state** for the potential ϕ if it maximizes the free energy, that is:

$$P(\phi) = h(\mu) + \int \phi d\mu.$$

An important problem of the thermodynamic formalism is to give sufficient conditions for a potential ϕ to have a unique or at most finitely many equilibrium states. These ideas came from the statistical physics on infinite one-dimensional gas whose phase state is modeled on the (bilateral) shift. We refer to [3] and [8] for a comprehensive introduction to the subject.

It is the case that the thermodynamic formalism of C^r $(r > 1)$ hyperbolic repellers is well understood. It can be proved that, for any continuous function $\phi : \Lambda \to \mathbb{R}$, the topological pressure is:

$$P(f|\Lambda, \phi) = \limsup_{n\to+\infty} \frac{1}{n} \log \left(\sum_{P\in\wp_n} \sup_{x\in P} \exp S_n\phi(x) \right).$$

Moreover, is ϕ is Hölder continuous, then there exists a unique equilibrium state μ_ϕ which is is a **Gibbs measure**: there exists a constant $G > 1$ such that:

$$G^{-1} \leq \frac{\mu_\phi(\wp_n(x))}{exp(-P(\phi) + S_n\phi(x)} \leq G \tag{1}$$

for every $x \in \Lambda$ and for every $n > 0$. See [3] and [8].

As pointed out in [15], several pressure-like dynamical indicators constructed in several settings of the termodynamic formalism are dimensional indicators associated to a suitable Carathéodory's structure. This remark had originated a number of useful generalizations of the above thermodynamic formalism to include more general classes of potentials, as subadditive and almost subadditive sequences giving rise to a rich theory for non conformal sets. See for instance [1], [2], [15] and [16]. Let us show how thermodynamic formalism can be used to get dimensional indicators from the dynamics. Let

$$\wp = \bigcup_{n\geq 1} \wp_n.$$

be the generating net for Λ: for every $x \in \Lambda$ there exists a decreasing sequence $\wp_{n+1}(x) \subset \wp_n(x)$, $n \geq 1$, such that:

$$\{x\} = \bigcap_{n=1}^{+\infty} \wp_n(x)$$

where $\wp_n(x) \in \wp_n$ is the atom of generation n containing x. We use the generating familiy $\wp$ and the volume as a set function to define a suitable **Carathéodory's structure** with a dimensional indicator as follows. We say that a countable family of subsets $\mathcal{U} = \{P_i\}$, with $P_i \in \wp$, is a $(\wp, \delta)$-covering of X if a) $X \subset \bigcup_i P_i$ and $\text{diam}(P_i) < \delta$. Then we use this family of coverings to define an outer measure:

$$\mathcal{D}_{a,\delta}(X) = \inf_{\mathcal{U}} \sum_{i=1}^{+\infty} \text{Vol}\,(P_i)^a$$

where infimum is taken over the family of $(\wp, \delta)$-coverings. As usual, $\mathcal{D}_{a,\delta}(X)$ is non decreasing in δ so we introduce the following **dynamical a-measure**:

$$\mathcal{D}_a(X) = \sup_{\delta>0} \mathcal{D}_{a,\delta}(X).$$

This is a Borel regular measure and its dimensional indicator is the **dynamical dimension**

$$\dim_{\mathcal{D}}(X) = \inf\{a > 0 : \mathcal{D}_a(X) = 0\} = \sup\{a > 0 : \mathcal{D}_a(X) = +\infty\}$$

It holds out that $\alpha = \dim_{\mathcal{D}}(\Lambda)$ is the solution to the Bowen equation

$$P(f|\Lambda, -\alpha \log(Jf)) = 0, \tag{2}$$

where $Jf = |\det(Df)|$ is the Jacobian f with respect to the Riemannian volume. Indeed, it is well known that, by the bounded distortion property of the volume under f (see below), there exists a constant $A > 1$, uniform, such that

$$A{-}1 \leq \mathrm{Vol}\,(P) Jf^n(x) \leq A \quad \forall\, x \in P,\ P \in \wp_n,\ n \geq 1.$$

This is also known as the volume lemma. Therefore, $\mathrm{Vol}\,(P)^a \asymp \sup_{x\in P} \exp S_n\phi(x)$, meaning that the ratios of these quantities are uniformly bounded, independent of $P \in \wp_n$ and $n \geq 1$ by suitable constants. Here $\phi(x) = -a \log Jf(x)$ is the so called **geometric potential**. This is a Hölder continuous function, by our assumptions on the differentiability of the map. By the thermodynamic formalism there exists a unique ergodic Borel probability μ_α supported on Λ which is an equilibrium state for the geometrical potential with parameter α. Using the volume lemma and the Gibbs property we find a constant $B > 1$ such that

$$B^{-1} \leq \frac{\mu(P)}{\mathrm{Vol}\,(P)^\alpha} \leq B \quad \forall\, P \in \wp.$$

Then, it is proven that $0 < \mathcal{D}_\alpha(\Lambda) < +\infty$. In particular,

$$\dim_{\mathcal{D}}(\Lambda) = \frac{h(\mu_\alpha)}{\chi^+(\mu_\alpha)},$$

where $\chi^+(\mu) = \int \log |Jf| d\mu$ is the Lyapunov exponent of the Borel probability μ. There are two main settings under which the above approach can be succesfully used to study the fractal geometry of a hyperbolic repeller:

- piecewise expanding interval maps and
- conformal piecewise expanding maps $f : M \to M$ of a d-dimensional compact Riemannian manifold,

both in the C^r, $(r > 1)$ category. In these cases, the following facts are well known:

(1) $\dim_{\mathcal{H}}(\Lambda) = \dim_B(\Lambda)$;
(2) $\dim_{\mathcal{H}}(\Lambda) = d\alpha$, $d \geq 1$ is the unique solution to the Bowen equation

$$P(f|\Lambda, -\alpha \log |f'|) = 0,$$

for the interval map $(d = 1)$ and

$$P(f|\Lambda, -d\alpha \log |a|) = 0,$$

for a multidimensional conformal map, where $Df(x) = a(x) id_{T_xM}$ is the derivative at x;
(3) the Hausdorff measure is finite and positive $0 < \mathcal{H}_{d\alpha}(\Lambda) < +\infty$;
(4) there exists a unique Borel probability of maximal dimension μ_Λ, such that:

$$\dim_{\mathcal{H}}(\Lambda) = \frac{h(\mu_\Lambda)}{\chi^+(\mu_\Lambda)} = \sup_{\mu\in\mathcal{M}_f} \frac{h(\mu)}{\chi^+(\mu)};$$

(5) the Hausdorff dimension $\dim_{\mathcal{H}}(\Lambda_f)$ varies smoothly with f.

It occurs that, in the conformal case, the atoms $P \in \wp$ have bounded geometric distortion. Indeed, let

$$r_{inner}(P) = \sup\{r > 0 : \exists\, x \in P,\ B(x, r) \subset P\}$$
$$r_{outer}(P) = \inf\{r > 0 : \exists\, x \in P,\ P \subset B(x, r)\}$$

be the inner and outer diameters of the set P, respectively. It can be proved that, if f is conformal, then $r_{inner}(P) \asymp r_{outer}(P)$ are uniformly comparable for every $P \in \wp$ and therefore $\mathrm{Vol}\,(P) \asymp \mathrm{diam}(P)^d$, uniformly in $P \in \wp$. Therefore in these cases the dynamical measure $\mathcal{D}_a$ is equivalent to the Hausdorff measure $\mathcal{H}_a$, for every $a > 0$ and thus the dynamical dimension and the Hausdorff dimension are equal, up to the constant d, the dimension of the ambient space. Moreover, the Hausdorff measure is non trivial, there exists a unique measure of maximal dimension, which is an equilibrium state of the geometric potential and the dimensional characteristic varies smoothly with the map f, by differentiability results of the thermodynamic formalism.

When f is non conformal this approach fail, originating a number of interesting and difficult problems. The point is that, for non conformal maps, the geometric distortion of the generating sets $P \in \wp$ is unbounded and there is no longer a clear connection between the dynamical and Hausdorff measures. A large body of literature had been produced trying to dilucidate how the dynamics interacts with the geometry of a non conformal repeller, letting a wide field of open problems on the subject. See [1], [16] for a review of this type of questions.

It is the main goal of this paper to show that, nevertheless, the standard additive thermodynamic formalism and the Bowen equation can still be used, if not to get new exact formulas for the Hausdorff dimension of a non confomal repeller, at least to get upper and lower bounds involving the dynamical dimension.

3. Statement of results

Before to state our main result we need to introduce the following quantities:

- fix small positive numbers $\rho, \delta > 0$;
- define $0 < \epsilon < 1$ as:

$$\epsilon = (1-\delta)\left(\frac{\|\underline{Jg}\|}{\|Dg\|^{d-1+\delta}}\right)^{\frac{1}{1-\delta}},$$

  where:

$$\|\underline{Jg}\| = \min_{i=1,\cdots,s} \inf_{x\in N_i} Jg_i(x) \quad \text{and} \quad \|Dg\| = \max_{i=1,\cdots,s} \sup_{x\in N_i} \|Dg_i(x)\|$$

  where $Jg_i(x) = |\det(Dg_i(x))|$ is the Jacobian and $Dg_i(x)$ the derivative at x and $\delta > 0$ a small positive number;
- let λ_0 and λ_1 be given by the volume expansion of f:

$$\lambda_0 = \log(\inf_{x\in\Lambda} Jf(x) - \rho) \quad \text{and} \quad \lambda_1 = \log(\sup_{x\in\Lambda} Jf(x) + \rho),$$

  where we suppose that $0 < \rho < \inf_{x\in\Lambda} Jf(x)$ and $Jf(x) = |\det(Df(x)|$ is the Jacobian of f with respect to the Riemannian volume.

We will prove the following

Theorem 1. *Let $g_i : N \to N$ be a SIF generating a regular geometric construction, $\Lambda \subset N$ its attractor and $f : \bigcup_i U_i \to U$ be the non conformal C^r $(r > 1)$ piecewise differentiable expanding map generated by the inverse branches $\{g_i\}$. We suppose in addition that:*

$$\dim_B(\Lambda) = \overline{\dim_B}(\Lambda) = \underline{\dim_B}(\Lambda). \tag{3}$$

(1) *Let $s = \alpha$ be the solution to the Bowen equation $P(f|\Lambda, -s\log Jf) = 0$. Then*

$$\dim_{\mathcal{H}}(\Lambda) \le \dim_B(\Lambda) \le d\bar{\beta};$$

where $\bar{\beta}$ is the solution to the equation

$$e^{\lambda_0(1-\alpha)}\epsilon^{d(1-\bar{\beta})} = 1;$$

(2) *There exists a non negative constant $\overline{\Sigma} > 0$ such that, if $s = \underline{\alpha}$ is the solution to the* ***modified Bowen equation***

$$P\left(f|\Lambda, -s\log(e^{\overline{\Sigma}}Jf)\right) = 0$$

then,

$$d\underline{\beta} \leq \dim_{\mathcal{H}}(\Lambda),$$

where $\underline{\beta}$ is the solution to the equation

$$e^{\lambda_1(1-\underline{\alpha})}\epsilon^{d(1-\underline{\beta})} = 1;$$

Explicitly, the upper $U = d\bar{\beta}$ and lower $L = d\underline{\beta}$ bounds are:

$$U = d + \frac{\lambda_0(1-\alpha)}{\log\epsilon} \quad \text{and} \quad L = d + \frac{\lambda_1(1-\underline{\alpha})}{\log\epsilon};$$

These quantities are continuous functions of ρ, δ, ϵ, α and $\underline{\alpha}$ and reflects a delicate interplay between the volume and distance contraction of the inverse branches $g_i : N \to N$ and the geometry of Λ. The choice of constants ρ and δ is quite arbitrary, as long as they are sufficiently small positive numbers. For ρ we have the upper bound $0 < \rho < \|\underline{Jf}\|$ and for δ we will find a small $\delta_1 > 0$ such that $0 < \delta < \delta_1$ is required for the argument to hold on. The number $\epsilon > 0$ is a sort of characteristic scale at which the dynamical measure and the box-counting measure generated by suitable $2\epsilon^n$-coverings are uniformly comparable over atoms $P \in \wp_n$ at the n-th stage of the construction, independent of $n > 0$. The constant $\overline{\Sigma} = \sup_{x\in\Lambda}\Sigma(x)$ is the supremum of a Borel measurable function $\Sigma = \Sigma(x)$ which describes the rate of exponential decay, as $n \to +\infty$, of the fraction of the volume of an atom $P \in \wp_n$ of generation n which is covered by a suitable efficient covering of $\Lambda \cap P$. Moreover, we will prove that, for $0 < \delta < \delta_1$,

$$0 < \log\left(\frac{\inf_{x\in\Lambda} Jf(x)}{\epsilon^{-(d(\Lambda)+\delta-d)}}\right) \leq \Sigma(x) \leq \log\left(\frac{\sup_{x\in\Lambda} Jf(x)}{(\|Df\|\epsilon)^{-(d_B(\Lambda)-\delta)}\epsilon^d}\right) < +\infty,$$

where $d_B(\Lambda) = \dim_B(\Lambda)$ is the box-counting dimension of Λ.

Let us underline that, due to the smoothness of the additive pressure $P(\phi)$ with respect to the potential ϕ and the robustness and structural stability of hyperbolic repellers, the above estimatives are robust with respect to small perturbations of the IFS $\{g_i\}$. This open de door to possible applications to more general hyperbolic repellers which can be obtained as limits of regular constructions, as in the case of Bedford-McMullen carpets. Namely, we have

Corollary 3.1. *Let Λ be the limit of regular geometric construction given by a C^r $(r > 1)$ IFS $\mathcal{G} = \{g_i\}$. Then, given $\zeta > 0$, there exists $\eta > 0$ such that,*

$$|L - L(\eta)| < \zeta \quad \textit{and} \quad |U - U(\eta)| < \zeta,$$

for every η-C^1 near IFS $\mathcal{G}^\eta = \{g_i^\eta\}$, where $L = L(\lambda_1, \epsilon, \underline{\alpha})$ (resp. $U = U(\lambda_0, \epsilon, \alpha)$) are the lower bound (resp. upper bound) of the Hausdorff and box-counting dimension of the attractor $\Lambda = \Lambda(\mathcal{G})$ and $L(\eta)$ and $U(\eta)$ the corresponding upper and lower bounds for $\Lambda(\eta) = \Lambda(\mathcal{G}^\eta)$.

Conclusion: The Hausdorff dimension might not be continuous but the upper U and lower L bounds are.

Outline of the proof:

The idea is to use the volume as the subject of a trade off between the geometry and the dynamics. We start choosing a particular scale $\epsilon > 0$ in such way that, for every $n > 0$ there are open coverings $\mathcal{B}_n$ by $2\epsilon^n$-balls such that $\sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)$ is uniformly comparable with $\sum_{P\in\wp_n} \mathrm{Vol}\,(P)$, independently of n. The upper bound $d\bar{\beta}$ appears when we see that the sums $\sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^\beta$

are uniformly bounded from above by a term θ^n with $0 < \theta < 1$, for every $\beta > \bar{\beta}$, getting thus an upper bound for the upper box-counting dimension. The parameter $\bar{\beta}$ is the solution to a suitable trascendental equation involving λ_0, ϵ and α is the solution to the Bowen equation (2). To prove this we need to use that the sums $\sum_{P\in\wp_n} \mathrm{Vol}\,(P)^\alpha$ are uniformly bounded beyond zero and infinity. One would like to use the same type of arguments to find a lower bound to the Hausdorff dimension using the solution to the Bowen equation and coverings $\mathcal{B}_n$. However, the balls in $\mathcal{B}_n$ covers the whole n-th generation pieces $P \in \wp_n$ meanwhile Λ is just small fraction of these sets. That is, the coverings $\mathcal{B}_n$ are too large to estimate de Hausdorff measure. This lead us to extract a suitable minimal covering $\mathcal{B}_n(\Lambda)$ of Λ from the $\mathcal{B}_n$. As we are using the volume we need to quantify the fraction of $\mathrm{Vol}\,(P)$ which is covered this minimal covering. Here is where condition (3) enters the argument and a bounded measurable function $\Sigma = \Sigma(x)$ is defined as the asymptotic rate of decay of these fractions with least upper bound $\overline{\Sigma}$. Then, we fix a constant $\underline{\beta}$ which satisfies a trascendental equation only involving the solution to the modified Bowen equation $\underline{\alpha}$, the maximal rate of volume expansion λ_1 and the characteristic scale ϵ and choose $\delta > 0$ such that $\mathcal{H}_{d\underline{\beta},\delta}(\Lambda) < +\infty$. This is possible since, otherwise $d\underline{\beta} \leq \dim_{\mathcal{H}}(\Lambda)$ and we had finished the proof. Then, we use the trascendental equation defining $\underline{\beta}$ to cancel δ in the lower bound of the sums $\sum_{B\in\mathcal{B}_0} \mathrm{Vol}\,(B)^{\underline{\beta}}$, which are then uniformly bounded away from zero, bounded from below by a positive constant $C(\underline{\beta}) > 0$, for **every finite δ-covering** $\mathcal{B}_0 = \{B^0_k\}$ of Λ. This lower bound $C(\underline{\beta}) > 0$ only depends on $\underline{\beta}$, $\underline{\alpha}$, λ_1, ϵ and the geometry of the ambient manifold. Finally, as $\mathrm{Vol}\,(B)$ and $\mathrm{diam}(B)^d$ are uniformly comparable we conclude that the Hausdorff measure $\mathcal{H}_{d\beta,\delta}(\Lambda) > 0$ is positive and therefore $\mathcal{H}_{d\underline{\beta}}(\Lambda) > 0$, so proving that $\mathcal{H}_{d\beta}(\Lambda) = +\infty$ for every $\beta < \underline{\beta}$. This finish the proof.

4. Proof of theorem 1: the upper bound

We first prove our upper bound. Let us recall the set up: let $g = \{g_i : i = 1, \cdots, s\}$ be finitely many contractions $g_i : N \to N$ from an embedded d-cube $N \subset M$ with positive volume into N. We suppose in addition that the boundary is piecewise smooth formed by finitely many codimension one embedded submanifolds. In particular, $\dim_B(\partial N) = d - 1$. We also use the following volume estimates: there are two constants $0 < A_0 < A_1$, only depending on the Riemannian structure, such that

$$A_0 r^d \leq \mathrm{Vol}\,(B(x,r)) \leq A_1 r^d \quad \text{for every} \quad x \in N\ r > 0 \tag{4}$$

Moreover, we get from (4), that, for every $C > 0$,

$$\frac{A_0 C^d}{A_1}\mathrm{Vol}\,(B(x,r)) \leq \mathrm{Vol}\,(B(x,Cr)) \leq \frac{A_1 C^d}{A_0}\mathrm{Vol}\,(B(x,r)) \tag{5}$$

The following lemma introduces the characteristic scale $0 < \epsilon < 1$.

Lemma 4.1. *There exists $0 < \epsilon < 1$ and a constant $C_0 > 0$, such that, for every n-generation atom $P \in \wp_n$, there exists a covering $\mathcal{B}_n(P)$ by $2\epsilon^n$-balls B such that,*

$$\sum_{B\in\mathcal{B}} Vol(B) \leq C_0\, Vol(P). \tag{6}$$

Proof. We will choose $0 < \epsilon < 1$ after establishing some conditions necessary to satisfy the required property in the statement of the lemma. We first recall that

$$\dim_B(\partial N) = \lim_{\rho\to 0^+} \frac{\log \mathcal{N}(\partial N, \rho)}{\log(1/\rho)} = d - 1$$

is the box-counting dimension of the border and choose $\delta > 0$ such that $d_0 := d - 1 + \delta < d$, $\rho_0 > 0$ and $D_0 > 0$ such that, for every $0 < \rho < \rho_0$, the border ∂R can be covered by at most

$D_0\rho^{-d_0}$ balls of radius ρ. Moreover, we choose $0<\rho_0<1$ sufficiently small such that Λ is contained in the complement of every covering of the border ∂N by ρ_0-balls

$$\Lambda \subset N - \bigcup_i B(z_i,\rho_0).$$

This is possible by the border condition on the construction. Then, let $n\ge 1$. Our first condition to define ϵ is the following:

$$\|Dg\|^{-n}2\epsilon^n < \rho_0 \quad \text{for sufficiently large } n.$$

This can be achieved if

$$0<\|Dg\|^{-1}\epsilon<1 \quad \text{and} \quad n\ge N_0=N_0(\rho_0),$$

for a suitable $N_0>0$. Then, ∂N can be covered by no more than $D_0(\|Dg\|^{-n}2\epsilon^n)^{-d_0}$ balls of radius $r=\|Dg\|^{-n}2\epsilon^n$ if $n\ge N_0$. If $1\le n\le N_0$ we choose a larger $D_0>0$ such that:

$$\mathcal{N}(\partial N,\|Dg\|^{-n}2\epsilon^n)\le D_0(\|Dg\|^{-n}2\epsilon^n)^{-d_0} \quad \text{for} \quad n\ge 1$$

Let $\{B(x_i,\|Dg\|^{-n}2\epsilon^n)\}$ be such covering and suppose it is minimal. Moreover, we can suppose that the set of balls $\{B(x_i,\|Dg\|^{-n}\epsilon^n)\}$ is maximal with pairwise disjoint interior. Then, the open set

$$W_n=\bigcup_i B(g_P^n(x_i),2\epsilon^n) \quad \text{contains } \partial P \text{ and has volume}$$

$$\mathrm{Vol}\,(W_n)\le D_0A_1(\|Dg\|^{-n}2\epsilon^n)^{-d_0}(2\epsilon^n)^d.$$

due to (4), where g_P^n is the branch of the IFS generating the n-atom $P\in\wp_n$. Moreover, we may suppose that,

$$\Lambda\cap P\subset R-W_n(P).$$

for sufficiently large n, since $\Lambda\subset N-\bigcup_i B(x_i,\rho_0)$.

Now we choose a maximal family of closed balls $\overline{B}(z_j,\epsilon^n)$ contained in $P-W_n$ with pairwise disjoint interiors. Then, $\{B(z_j,2\epsilon^n)\}$ covers $P-W_n$. By (5),

$$\sum_j \mathrm{Vol}\,(B(z_j,2\epsilon^n))\le \frac{D_0A_12^d}{A_0}\sum_j \mathrm{Vol}\,(B(z_j,\epsilon^n))\le \frac{A_12^d}{A_0}\mathrm{Vol}\,(P),$$

since $\overline{B}(z_j,\epsilon^n)\subset P-W_n$ have pairwise disjoint interiors.

Let $\mathcal{B}_n(P)=\{B(g_P^n(x_i),2\epsilon^n)\}\cup\{B(z_j,2\epsilon^n)\}$. Then, we get

$$\begin{aligned}\sum_{B\in\mathcal{B}}\mathrm{Vol}\,(B) &\le D_0A_1(\|Dg\|^{-n}2\epsilon^n)^{-d_0}(2\epsilon^n)^d+\frac{A_12^d}{A_0}\mathrm{Vol}\,(g^n(R))\\ &= D_0A_12^{d-d_0}(\|Dg\|^{d_0}\epsilon^{d-d_0})^n+\frac{A_12^d}{A_0}\mathrm{Vol}\,(P).\end{aligned}$$

On the other hand,

$$\|\underline{Jg}\|^n\le\frac{\mathrm{Vol}\,(P)}{\mathrm{Vol}\,(N)}.$$

Indeed, $\mathrm{Vol}\,(P)\ge\inf_{x\in N}|Jg_P^n(x)|\mathrm{Vol}\,(N)\ge\|\underline{Jg}\|^n\mathrm{Vol}\,(N)$. If $\epsilon>0$ is chosen, such that:

$$\|Dg\|^{d_0}\epsilon^{d-d_0}\le\|\underline{Jg}\| \tag{7}$$

then, one would have,

$$\sum_{B\in\mathcal{B}}\mathrm{Vol}\,(B)\le\left(\frac{A_12^{d-d_0}}{\mathrm{Vol}\,(N)}+\frac{A_12^d}{A_0}\right)\mathrm{Vol}\,(P),$$

and therefore,

$$C_0 = \frac{D_0 A_1 2^{d-d_0}}{\text{Vol}\,(N)} + \frac{A_1 2^d}{A_0}$$

would define a constant with the required property. The solution to the inequality (7) is:

$$0 < \epsilon \le \left(\frac{\|\underline{Jg}\|}{\|Dg\|^{d_0}} \right)^{\frac{1}{d-d_0}}$$

Now, notice that $\|\underline{Jg}\| \le \|Dg\|^d$. Therefore:

$$\left(\frac{\|\underline{Jg}\|}{\|Dg\|^{d_0}} \right)^{\frac{1}{d-d_0}} = \left(\frac{\|\underline{Jg}\|}{\|Dg\|^{d-1+\delta}} \right)^{\frac{1}{1-\delta}} \le (\|Dg\|^{1-\delta})^{\frac{1}{1-\delta}} = \|Dg\|.$$

Thus, we choose $\epsilon = \epsilon(\delta)$:

$$\epsilon = (1-\delta) \left(\frac{\|\underline{Jg}\|}{\|Dg\|^{d_0}} \right)^{\frac{1}{d-d_0}} < \|Dg\|,$$

for any $0 < \delta < 1$. □

Our next lemma needs the following well known

Bounded distortion property

There exists a constant $C > 0$ such that, for every $n > 0$ and for every $x, y \in P$ and every $P \in \wp_n$,

$$\left| \frac{Jf^n(x)}{Jf^n(y)} - 1 \right| \le Cd(f^n(x), f^n(y)) \tag{8}$$

Lemma 4.2. *There exist constants $C_1 > 1$ and $0 < \lambda_0 < \lambda_1$ such that, for every $x \in \Lambda$:*

$$C_1^{-1} e^{-n\lambda_1} \le Vol(\wp_n(x)) \le C_1 e^{-n\lambda_0} \quad \forall\ n > 0. \tag{9}$$

Proof. As $f^n(P) = N$, for every $P \in \wp_n$ then, by the bounded distortion property, there exists a constant $C_1 > 1$, only depending on the volume distortion of f and the volume of N, such that

$$C_1^{-1} \le \text{Vol}\,(\wp_n(x)) Jf^n(x) \le C_1,$$

for every $x \in \Lambda$ and $n > 0$. Then,

$$\sup_{x\in\Lambda} [Jf^n(x)]^{-1} \le \left[\inf_{x\in\Lambda} Jf^n(x) \right]^{-1} \le \left[\inf_{x\in\Lambda} Jf(x) \right]^{-n}$$

Similarly so,

$$\inf_{x\in\Lambda} [Jf^n(x)]^{-1} \ge \left[\sup_{x\in\Lambda} Jf^n(x) \right]^{-1} \ge \left[\sup_{x\in\Lambda} Jf(x) \right]^{-n}$$

Therefore,for any given $0 < \rho < \inf_{x\in\Lambda} Jf(x)$ the constants:

$$\lambda_0 = \log \left(\inf_{x\in\Lambda} Jf(x) - \rho \right) \tag{10}$$

$$\lambda_1 = \log \left(\sup_{x\in\Lambda} Jf(x) + \rho \right). \tag{11}$$

still satisfy (9). □

Lemma 4.3. *Let $s = \alpha$ be the solution to the Bowen equation $P(f|\Lambda, -s\log(Jf)) = 0$. Then,*

$$0 < \inf_{n>0} \sum_{P\in\wp_n} Vol(P)^\alpha \leq \sup_{n>0} \sum_{P\in\wp_n} Vol(P)^\alpha < +\infty. \tag{12}$$

Proof. Let μ_α be the equilibrium state for the Hölder continuous potential $\phi = -\alpha\log(Jf)$. This is a Gibbs measure. Then by the bounded distortion property and Gibbs property, we can find a constant $C_2 > 1$ such that:

$$C_2^{-1} \leq \frac{\mu_\alpha(\wp_n(x))}{\mathrm{Vol}\,(\wp_n(x))^\alpha} \leq C_2$$

for every $x \in \Lambda$ and every $n > 0$. Then (12) follows inmediatly by an standard argument, since $\wp_n$ is a decomposition into disjoint pieces at the n-stage of the construction. □

Lemma 4.4. *Let $0 < \alpha \leq 1$ be the solution to the Bowen equation and $0 < \bar\beta < 1$ be the unique solution to the equation*

$$e^{\lambda_1(1-\alpha)}\epsilon^{d(1-\bar\beta)} = 1 \tag{13}$$

Then,

$$\dim_{\mathcal{H}}(\Lambda) \leq \underline{\dim_B}(\Lambda) \leq \overline{\dim_B}(\Lambda) \leq d\bar\beta. \tag{14}$$

Proof. By lemma 4.1 there exists a uniform constant $C_0 > 0$ and a positive $\epsilon > 0$ such that for every $n > 0$ and for every atom of generation n, $P \in \wp_n$, there exits a finite covering $\mathcal{B}_n(P)$ by $2\epsilon^n$-balls $B_i = B(x_i, 2\epsilon^n))$ such that $\sum_{B\in\mathcal{B}_n(P)} \mathrm{Vol}\,(B) \leq C_0\mathrm{Vol}\,(P)$. We define $\mathcal{B}_n = \{B : B \in \mathcal{B}_n(P),\ P \in \wp_n\}$ the open covering of Λ by these $2\epsilon^n$-balls.

We will show that if $\beta > \bar\beta$ then

$$\sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^\beta \to 0^+ \quad \text{exponentially, as} \quad n \to +\infty. \tag{15}$$

Actually, we will prove that, if $\beta > \bar\beta$ then there exists $C > 0$ and $0 < \theta = \theta(\beta) < 1$ such that

$$\sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^\beta \leq C\theta^n \quad \text{for every} \quad n \geq 1.$$

This would prove our assertion since $A_0\mathrm{diam}(B)^d \leq \mathrm{Vol}\,(B)$ for every $B \in \mathcal{B}_n$ and therefore,

$$A_0\mathcal{N}(\Lambda, 2\epsilon^n)(2\epsilon^n)^{d\beta} \leq A_0\#\mathcal{B}_n(2\epsilon^n)^{d\beta} \leq \sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^\beta \leq C\theta^n,$$

which would imply that

$$\liminf_{\rho\to 0^+} \frac{\mathcal{N}(\Lambda,\rho)}{\log(1/\rho)} \leq \limsup_{\rho\to 0^+} \frac{\mathcal{N}(\Lambda,\rho)}{\log(1/\rho)} < d\beta$$

for every $\beta > \bar\beta$ so proving (14). So, let $\beta > 0$. Then:

$$\sum_{B\in\mathcal{B}(P)} \mathrm{Vol}\,(B)^\beta \leq \sum_{B\in\mathcal{B}(P)} \mathrm{Vol}\,(B) \max_{B\in\mathcal{B}(P)} \mathrm{Vol}\,(B)^{\beta-1}.$$

Hence,

$$\sum_{B\in\mathcal{B}(P)} \mathrm{Vol}\,(B)^\beta \leq C_0\mathrm{Vol}\,(P)[A_0 2^d\epsilon^{nd}]^{\beta-1}.$$

Therefore,

$$\sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^\beta \leq C_0[A_0 2^d]^{\beta-1}[\epsilon^{d(\beta-1)}]^n \sum_{P\in\wp_n} \mathrm{Vol}\,(P).$$

On the other hand, by (6) in lemma 4.1,

$$\begin{aligned}\sum_{P\in\wp_n} \mathrm{Vol}\,(P) &\leq \sum_{P\in\wp_n} \mathrm{Vol}\,(P)^{\alpha} \max_{P\in\wp_n} \mathrm{Vol}\,(P)^{1-\alpha} \\ &\leq C_1^{1-\alpha} e^{-n\lambda_0(1-\alpha)} \sum_{P\in\wp_n} \mathrm{Vol}\,(P)^{\alpha}.\end{aligned}$$

for every $n > 0$, by (9). Thus

$$\sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^{\beta} \leq C_2(\beta)[e^{\lambda_0(\alpha-1)}\epsilon^{d(\beta-1)}]^n, \tag{16}$$

where

$$C_2(\beta) = C_1C_0[A_0 2^d]^{\beta-1} \sup_{n>0} \sum_{P\in\wp_n} \mathrm{Vol}\,(P)^{\alpha}.$$

does not depend on $n > 0$. Now, $h(\beta) = e^{\lambda_0(\alpha-1)}\epsilon^{d(\beta-1)}$ is strictly decreasing in β, therefore $e^{\lambda_0(\alpha-1)}\epsilon^{d(\beta-1)} < 1$ for $\beta > \bar{\beta}$, where $\bar{\beta}$ is the unique solution to the equation

$$e^{\lambda_0(\alpha-1)}\epsilon^{d(\bar{\beta}-1)} = e^{\lambda_0(1-\alpha)}\epsilon^{d(1-\bar{\beta})} = 1,$$

so proving that $\sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^{\beta} \to 0^+$, as $n \to +\infty$, bounded by a geometric progression with ratio

$$\theta(\beta) = e^{\lambda_0(\alpha-1)}\epsilon^{d(\beta-1)} < 1.$$

□

5. Proof of theorem 1: the lower bound

To prove the other inequality we start choosing $\zeta > 0$ a positive number and a ζ-covering $\mathcal{B}_0 = \{B_k^0\}$ by open balls. We will suppose, without loss of generality, that $\mathcal{B}_0$ is finite. Then we choose $n > 0$ large enough such that $2\epsilon^n < \lambda(\mathcal{B}_0) < \zeta$, the Lebesgue number of $\mathcal{B}_0$.

By lemma 4.1, for every $P \in \wp_n$ there exists a finite open cover

$$\mathcal{B}_n(P) = \{B(x_i(P), 2\epsilon^n), B(y_j(P), 2\epsilon^n)\}$$

such that $\sum_{B\in\mathcal{B}_n(P)} \mathrm{Vol}\,(B) \leq C_0 \mathrm{Vol}\,(P)$ for a suitable constant $C_0 > 0$. Indeed,

$$\partial P \subset \bigcup_i B(x_i(P), 2\epsilon^n) \quad \text{and} \quad P - W_n(P) \subset \bigcup_j B(y_j(P), 2\epsilon^n)$$

where $W_n(P) = \bigcup_i B(x_i(P), 2\epsilon^n)$ and $B(y_j(P), \epsilon^n) \subset P - W_n(P)$ is a collection of pairwise disjoint open balls, by construction. The covering of each border ∂P comes from a fixed, minimal, covering of ∂N, $\{B(x_i, \|Dg\|^{-n}2\epsilon^n)\}$ by $\|Dg\|^{-n}2\epsilon^n$-balls so that

$$g_P^n(B(x_i, \|Dg\|^{-n}2\epsilon^n)) \subset B(x_i(P), 2\epsilon^n) \quad \text{and} \quad x_i(P) = g_P^n(x_i),$$

We can suppose that the covering $\{B(x_i(P), 2\epsilon^n)\}$ of the border ∂P is minimal and that the elements of the **reduced covering**

$$\mathcal{B}_n^*(P) = \{B(x_i(P), \epsilon^n), B(y_j(P), \epsilon^n)\}$$

are pairwise disjoint. For this we choose $\{B(x_i, \|Dg\|^{-n}\epsilon^n)\}$ a maximal family of pairwise disjoint open $\|Dg\|^{-n}\epsilon^n$-balls covering ∂N. Therefore $\{B(x_i, \|Dg\|^{-n}2\epsilon^n)\}$ covers the border. Then we choose a minimal subcovering from $\{B(x_i(P), 2\epsilon^n)\}$ of the border ∂P. We denote

$$\mathcal{B}_n = \{B \in \mathcal{B}_n(P) : P \in \wp_n\}$$

this covering of the n-stage of the geometric construction $\bigcup \wp_n$ by the $2\epsilon^n$-balls. Now we use the Lebesgue number property to arrange the open balls $B \in \mathcal{B}_n$ as follows:

$$\begin{array}{rcl} \mathcal{B}_n(B_1^0) & := & \{B \in \mathcal{B}_n : B \subset B_1^0\} \\ \mathcal{B}_n(B_2^0) & := & \{B \in \mathcal{B}_n - \mathcal{B}_n(B_1^0) : B \subset B_2^0\} \\ & \vdots & \\ \mathcal{B}_n(B_l^0) & := & \{B \in \mathcal{B}_n - \bigcup_{l'<l} \mathcal{B}_n(B_{l'}^0) : B \subset B_l^0\} \end{array}$$

For each $B_k^0 \in \mathcal{B}_0$ the family $\mathcal{B}_n(B_k^0)$ is made of collections of open balls in $\mathcal{B}_n(P)$, where $P \cap B_k^0 \neq \emptyset$. However, eventhough for every P the ϵ^n-balls of the reduced covering $\mathcal{B}_n^*(P)$ are disjoint it might happen that, for different atoms $P, Q \subset B_k^0$ there are overlapping balls $B \cap B' \neq \emptyset$ with $B \in \mathcal{B}_n^*(P)$ and $B' \in \mathcal{B}_n^*(Q)$. In order to control those overlappings we argue as follows.

Lemma 5.1. *There exists a constant $D > 1$ with the following property: for every $B_k^0 \in \mathcal{B}_0$ there exists a collection of disjoint ϵ^n-balls $B_i \in \mathcal{B}_n^*(B_k^0)$ such that*

$$\sum_{B \in \mathcal{B}_n(B_k^0)} Vol(B) \leq \frac{A_1 D}{A_0} \sum_i Vol(B_i). \tag{17}$$

Remark 5.1. The constant $D > 1$ only depends on the topological dimension of the border of the atoms generating Λ and is, therefore, independent of the ζ-covering $\mathcal{B}_0$.

Proof. For any given $P \in \wp_n$, the ϵ^n-balls $B(y_j(P), \epsilon^n)$ are disjoint, by definition. Moreover, by the open condition, for every $Q \neq P$, the corresponding interior balls $B(y_j(P), \epsilon^n)$ and $B(y_k(Q), \epsilon^n)$ are pairwise disjoint, so the unique overlappings between elements in $\mathcal{B}_n^*(P)$ for different atoms $P \in \wp_n$ can occur near the border. As $\partial \mathcal{B}_n = \{B(x_i(P), 2\epsilon^n) : P \in \wp_n, i > 0\}$ is a minimal $2\epsilon^n$-covering of the border of the n-th stage in the construction, $\partial \wp_n = \bigcup_{P \in \wp_n} \partial P$, we can find a constant $D > 1$, only depending on the topological dimension of $\partial \wp_n$ such that there at most D-overlappings: for every i,

$$\#\{(j, Q) : B(x_i(P), \epsilon^n) \cap B(x_j(Q), \epsilon^n) \neq \emptyset\} \leq D.$$

Then, we begin with some $B_1 = B(x_i(P), \epsilon^n) \in \mathcal{B}_n^*(B_k^0)$. It is straightforward to prove that, for all the overlapping balls:

$$\bigcup_{(j,Q)} B(x_j(Q), \epsilon^n) \subset 4B_1,$$

where $tB(x, r) := B(x, tr)$, and

$$\sum_{(j,Q)} \text{Vol}\,(B(x_j(Q), \epsilon^n)) \leq \frac{A_1 D}{A_0} \text{Vol}\,(B_1).$$

Then we remove all those overlappings balls $B(x_j(Q), \epsilon^n)$ for $Q \neq P$ and choose $B_2 \in \partial \mathcal{B}_n$ out of the set of overlappings associated to B_1:

$$B_2 \in \partial \mathcal{B}_n - \{B(x_j(Q), 2\epsilon^n) : B_1 \cap B(x_j(Q), \epsilon^n) \neq \emptyset\}.$$

Then we remove the B_2 overlappings and proceed inductively until we get a disjoint collection $\{B_i\}$ with the required properties. $\square$

Lemma 5.2.

$$\sum_{B \in \mathcal{B}_n} Vol(B)^\beta \leq C_4(\beta) \sum_{B_k^0 \in \mathcal{B}_0} Vol(B_k^0)^\beta, \tag{18}$$

where

$$C_4(\beta) = \frac{2^{d(\beta-1)} A_1 D}{A_0} [\epsilon^{nd(\beta-1)} \zeta^{d(1-\beta)}].$$

Proof. By lemma 5.1

$$\sum_{B\in\mathcal{B}_n(B_k^0)} \mathrm{Vol}\,(B) \leq \frac{A_1 D}{A_0} \sum_i \mathrm{Vol}\,(B_i) \leq \frac{A_1 D}{A_0} \mathrm{Vol}\,(B_k^0).$$

Hence,

$$\begin{aligned}
\sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^\beta &\leq \max_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^{\beta-1} \sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B) \\
&\leq [A_1 2^d \epsilon^{nd}]^{\beta-1} \frac{A_1 D}{A_0} \sum_{B_k^0\in\mathcal{B}_0} \mathrm{Vol}\,(B_k^0) \\
&\leq [A_1 2^d \epsilon^{nd}]^{\beta-1} \frac{A_1 D}{A_0} \max_{B_k^0\in\mathcal{B}_0} \mathrm{Vol}\,(B_k^0)^{1-\beta} \sum_{B_k^0\in\mathcal{B}_0} \mathrm{Vol}\,(B_k^0)^\beta \\
&\leq [A_1 2^d \epsilon^{nd}]^{\beta-1} \frac{A_1 D}{A_0} [A_1 \zeta^d]^{1-\beta} \sum_{B_k^0\in\mathcal{B}_0} \mathrm{Vol}\,(B_k^0)^\beta \\
&\leq C_4(\beta) \sum_{B_k^0\in\mathcal{B}_0} \mathrm{Vol}\,(B_k^0)^\beta.
\end{aligned}$$

□

Now, we extract an efficient covering of Λ from $\mathcal{B}_n$. For this we choose a minimal covering

$$\mathcal{B}_n(\Lambda\cap P) = \{B \in \mathcal{B}_n : B\cap\Lambda\cap P \neq \emptyset\}.$$

and define

$$\mathcal{B}_n(\Lambda) = \bigcup_{P\in\wp_n} \mathcal{B}_n(\Lambda\cap P).$$

This is a minimal covering of Λ, since the n-atoms $P\in\wp_n$ are pairwise disjoint. We then define:

$$\rho(P) = \frac{\sum_{B\in\mathcal{B}_n(\Lambda\cap P)} \mathrm{Vol}\,(B)}{\mathrm{Vol}\,(P)} \tag{19}$$

This is the fraction of P which is covered by the $2\epsilon^n$-balls in $\mathcal{B}_n(\Lambda\cap P)$.

Lemma 5.3. *There exists a bounded measurable non negative function,*

$$\overline{\Sigma}(x) = \limsup_{n\to+\infty} -\frac{\log\rho(\wp_n(x))}{n}, \quad \forall\, x\in\Lambda.$$

Moreover, for sufficiently small $\delta > 0$,

$$-\infty < \log\left(\frac{\inf_{x\in\Lambda} Jf(x)}{\epsilon^{-(d(\Lambda)+\delta-d)}}\right) \leq \Sigma(x) \leq \log\left(\frac{\sup_{x\in\Lambda} Jf(x)}{(\|Df\|\epsilon)^{-(d_B(\Lambda)-\delta)}\epsilon^d}\right) < +\infty$$

for every $x\in\Lambda$

To prove lemma 5.3 we need first to estimate $\#\mathcal{B}_n(\Lambda\cap P)$. We start proving the following

Lemma 5.4. *Let* $\|Df\| = \sup_{x\in\bigcup_i N_i}\|Df(x)\|$. *Then, there exists* $\delta_0 > 0$ *and a constant* $D_1 > 1$ *such that, for every* $0<\delta<\delta_0$,

$$D_1^{-1}(\|Df\|^n 2\epsilon^n)^{-(d_B(\Lambda)-\delta)} \leq \#\mathcal{B}_n(\Lambda\cap\wp_n(x))), \tag{20}$$

for every $x\in\Lambda$ *and* $n>0$, *where* $\epsilon=\epsilon(\delta)$.

Proof. We first choose $\delta_0 > 0$ such that $\|Df\|\epsilon < 1$ for every $0 < \delta < \delta_0$. By definition,

$$\epsilon = (1-\delta)\left(\frac{\|\underline{Jg}\|}{\|Dg\|^{d-1+\delta}}\right)^{\frac{1}{1-\delta}},$$

where

$$\|\underline{Jg}\| = \min_i \inf_{x\in N} Jg_i(x) \quad \text{and} \quad \|Dg\| = \max_i \sup_{x\in N} Dg_i(x).$$

We will use the singular values of $Dg_i(x)$, $0 < s_i^d(x) \le \cdots \le s_i^1(x) < 1$ to estimate the product $\|Df\|\epsilon$. Indeed, as

$$Jg_i(x) = s_i^1(x)\cdots s_i^d(x) \qquad \|Dg_i(x)\| = s_i^1(x) \qquad Df(g_i(x)) = (s_i^d(x))^{-1}$$

we are led to estimate:

$$\Delta(x,\delta) = (1-\delta)\left(\frac{s_i^1(x)\cdots s_i^d(x)}{(s_i^1(x))^{d-1+\delta}}\right)^{\frac{1}{1-\delta}} (s_i^d(x))^{-1}.$$

This is a continuous function of $x \in N$ and $\delta \ge 0$. Then, provided that at least one singular value $s_i^k(x) < s_i^1(x)$, which is certainly the case because we are dealing with a non conformal map, then:

$$\Delta(x,0) = \frac{s_i^1(x)\cdots s_i^d(x)}{(s_i^1(x))^{d-1}}(s_i^d(x))^{-1} = \frac{s^2(x)}{s^1(x)}\cdots\frac{s^{d-1}(x)}{s^1(x)} < 1.$$

Thus we can find, by continuity and a compacity argument, $\delta_0 > 0$, a small positive number, such that:

$$\Delta(x,\delta) < 1 \quad \text{for every} \quad 0 < \delta < \delta_0, \quad \forall\, x \in \Lambda.$$

Then, taking supremum and infimum appropiately, we conclude that $\|Df\|\epsilon < 1$, as we claimed.

As $\|Df\|\epsilon < 1$, we can find a constant $D_1 > 1$ such that:

$$D_1^{-1}(\|Df\|^n 2\epsilon^n)^{-(d_B(\Lambda)-\delta)} \le \mathcal{N}(\Lambda, \|Df\|^n 2\epsilon^n) \le D_1(\|Df\|^n 2\epsilon^n)^{-(d_B(\Lambda)+\delta)} \quad \forall\, n > 0,$$

since $(\|Df\|\epsilon)^n \to 0^+$ as $n \to +\infty$. Now, notice that

$$\mathcal{N}(\Lambda, \|Df\|^n 2\epsilon^n) \le \#\mathcal{B}_n(\Lambda \cap \wp_n(x))),$$

since

$$f^n(\Lambda \cap P) = \Lambda \quad \text{and} \quad f^n(B(x_i, 2\epsilon^n)) \subset B(f(x_i), \|Df\|^n 2\epsilon^n),$$

defining an open $\|Df\|^n 2\epsilon^n$-covering of Λ with the same cardinality of $\#\mathcal{B}_n(\Lambda \cap \wp_n(x))$. Here we use the fact that $f : \Lambda \to \Lambda$ es topologically exact, that is, for every open set $U \subset \Lambda$ there exists $n > 0$ such that $f^n(U) = \Lambda$, since $f|\Lambda$ es topologically conjugated to the full unilateral shift. Hence:

$$D_1^{-1}(\|Df\|^n 2\epsilon^n)^{-(d_B(\Lambda)-\delta)} \le \#\mathcal{B}_n(\Lambda \cap \wp_n(x))),$$

as we wanted to prove. □

Proof of lemma 5.3. We are looking for upper and lower bounds for

$$\rho(\wp_n(x)) = \frac{\sum_{B\in\mathcal{B}_n(\Lambda\cap\wp_n(x))} \mathrm{Vol}\,(B)}{\mathrm{Vol}\,(\wp_n(x))}$$

For this we start recalling that there exists $C_2 > 1$ such that, for every $x \in \Lambda$,

$$C_2^{-1} \le \mathrm{Vol}\,(\wp_n(x)) Jf^n(x) \le C_2 \quad \forall\, x \in \Lambda,\ n > 0. \tag{21}$$

This, together with (20) provides a lower bound:

$$\rho(\wp_n(x)) \ge \frac{D_1^{-1}(\|Df\|^n 2\epsilon^n)^{-(d_B(\Lambda)-\delta)}(2\epsilon^n)^d}{C_2[\sup_{x\in\Lambda} Jf(x)]^n} = C_3 D_3^n$$

where

$$C_3 = \frac{D_1^{-1} 2^{-(d_B(\Lambda)-\delta-d)}}{C_2} \quad \text{and} \quad D_3 = \frac{(\|Df\|\epsilon)^{-(d_B(\Lambda)-\delta)}\epsilon^d}{\sup_{x\in\Lambda} Jf(x)}.$$

Now, notice that

$$\#\mathcal{B}_n(\Lambda \cap \wp_n(x)) \le \mathcal{N}(\Lambda \cap \wp_n(x), \epsilon^n) \le \mathcal{N}(\Lambda, \epsilon^n).$$

Recall that, by construction $\mathcal{B}_n(P)$ is made of open balls $B_i(2\epsilon^n)$ covering the border ∂P, so forming an open neighborhood $W_n(P)$ of ∂P and balls $B_j(2\epsilon^n)$ covering $P - W_n(P)$ which were choosen in such way that $\{B_j(\epsilon^n)\}$ is a packing formed by a maximal family of open ϵ^n-balls with pairwise disjoint interiors. Now, by construction, $\Lambda \cap P \subset P - W_n(P)$, for every $P \in \wp_n$ and $n > 0$, as we saw in the proof of lemma 4.1. Therefore, $\mathcal{B}_n(\Lambda \cap P)$ is formed exclusively by sets in the class $\{B_j(2\epsilon^n)\}$. In particular, the $B_j(\epsilon^n) \in \mathcal{B}_n(\Lambda \cap P)$ are pairwise disjoint and hence $\#\mathcal{B}_n(\Lambda \cap \wp_n(x)) \le \mathcal{N}(\Lambda \cap \wp_n(x), \epsilon^n)$, as we claimed.

Let $0 < \delta < \delta_0$, where $\delta_0 > 0$ was defined in lemma 5.4. Then, there exists a constant $D_2 > 1$ such that

$$D_2^{-1}(\epsilon^n)^{-(d(\Lambda)-\delta} \le \mathcal{N}(\Lambda, \epsilon^n) \le D_2(\epsilon^n)^{-(d(\Lambda)+\delta)} \quad \forall\ n > 0.$$

Thus,

$$\rho(\wp_n(x)) \le \frac{D_2(\epsilon^n)^{-(d(\Lambda)+\delta)}(2\epsilon^n)^d}{C_2^{-1}[\inf_{x\in\Lambda} Jf(x)]^n} = \frac{C_2 D_2 2^d (\epsilon^n)^{-(d(\Lambda)+\delta-d)}}{[\inf_{x\in\Lambda} Jf(x)]^n} = C_4 D_4^n$$

where

$$C_4 = C_2 D_2 2^d \quad \text{and} \quad D_4 = \frac{\epsilon^{-(d(\Lambda)+\delta-d)}}{\inf_{x\in\Lambda} Jf(x)}$$

Hence,

$$-C_4 - n\log D_4 \le -\log(\rho(\wp_n(x)) \le -C_3 - n\log D_3, \qquad x \in \Lambda,\ n > 0,$$

and then

$$\overline{\Sigma}(x) = \limsup_{n\to+\infty} -\frac{\log(\rho(\wp_n(x))}{n}$$

is a bounded Borel measurable non negative function defined over Λ. Moreover,

$$\log\left(\frac{\inf_{x\in\Lambda} Jf(x)}{\epsilon^{-(d(\Lambda)+\delta-d)}}\right) \le \overline{\Sigma}(x) \le \log\left(\frac{\sup_{x\in\Lambda} Jf(x)}{(\|Df\|\epsilon)^{-(d_B(\Lambda)-\delta)}\epsilon^d}\right)$$

□

We notice that the lower and upper bounds for $\Sigma = \Sigma(x)$ depends on several parameters as δ, ϵ, the largest volume and distance expansion produced by iteration of f. The next lemma shows that δ can be chosen small enough in such way that the lower bound of Σ is positive.

Lemma 5.5. *There exists $0 < \delta_1 < \delta_0$ such that:*

$$\log\left(\frac{\inf_{x\in\Lambda} Jf(x)}{\epsilon^{-(d(\Lambda)+\delta-d)}}\right) > 0 \quad \textit{for every} \quad 0 < \delta < \delta_1$$

Proof. We want to bound

$$\frac{\inf_{x\in\Lambda} Jf(x)}{\epsilon^{-(d(\Lambda)+\delta-d)}} = \inf_{x\in\Lambda} Jf(x)\epsilon^{d(\Lambda)+\delta-d}.$$

Let $0 < s_i^d(x) \le \cdots \le s_i^1(x) < 1$ be the singular values of $Dg_i(x)$. As

$$\epsilon = (1-\delta)\left(\frac{s_i^1(x)\cdots s_i^d(x)}{(s_i^1(x))^{d-1+\delta}}\right)^{\frac{1}{1-\delta}}$$

and

$$Jf(x) = (s_i^1(x)\cdots s_i^d(x))^{-1} \quad \text{for every} \quad x \in N_i,$$

then:

$$Jf(x)\epsilon^{d(\Lambda)+\delta-d} = (1-\delta)(s_i^1(x)\cdots s_i^d(x))^{\tau_1(\delta)}(s_i^1(x))^{\tau_2(\delta)} = \Delta(x,\delta)$$

where:

$$\tau_1(\delta) = \frac{d(\Lambda)+\delta-d}{1-\delta} = \frac{d(\Lambda)+2\delta-d-1}{1-\delta}$$

and

$$\tau_2(\delta) = -\frac{(d-1+\delta)(d_B(\Lambda)+\delta-d)}{1-\delta}.$$

For every $x\in\Lambda$, $\Delta(x,\delta)$ is a continuous function of $\delta\geq 0$. Evaluating at $\delta=0$ we get:

$$\Delta(x,0) = (s_i^1(x)\cdots s_i^d(x))^{d(\Lambda)-d-1}(s_i^1(x))^{-(d-1)(d_B(\Lambda)-d)}$$

as $d_B(\Lambda)<d$ and $d\geq 2$, $d(\Lambda)-d-1<0$ so that:

$$(s_i^1(x)\cdots s_i^d(x))^{d(\Lambda)-d-1} \geq (s_i^1(x))^{d(d(\Lambda)-d-1)},$$

thus

$$\Delta(x,0) \geq (s_i^1(x))^{d(d(\Lambda)-d-1)}(s_i^1(x))^{(d-1)(d-d_B(\Lambda))}.$$

Now,

$$d(d(\Lambda)-d-1)+(d-1)(d-d_B(\Lambda)) = -2d,$$

so we get that

$$\Delta(x,0) \geq (s_i^1(x))^{-2d} > 1.$$

Hence, by continuity and a compacity argument, we find $0<\delta_1<\delta_0$ such that:

$$\Delta(x,\delta) > 1 \quad \text{for every} \quad 0<\delta<\delta_1, \quad \forall\, x\in\Lambda.$$

so proving that,

$$\log\left(\frac{\inf_{x\in\Lambda} Jf(x)}{\epsilon^{-(d(\Lambda)+\delta-d)}}\right) > 0,$$

for every $0<\delta<\delta_1$ □

Corollary 5.1. *For every* $0<\delta<\delta_1$*:*

$$0 < \inf_{x\in\Lambda}\Sigma(x) \leq \sup_{x\in\Lambda}\Sigma(x) < +\infty$$

We fix $0<\delta<\delta_1$ once for all. This fixes $\epsilon=\epsilon(\delta)$

Lemma 5.6. *Let* $\overline{\Sigma}=\sup_{x\in\Lambda}\Sigma(x)>0$ *be the supremum of* $\Sigma=\Sigma(x)$ *and* $s=\underline{\alpha}$ *be the solution to the modified Bowen equation*

$$P(f|\Lambda, -s\log(e^{\overline{\Sigma}}Jf)) = 0. \tag{22}$$

Then, for every $\beta>0$*,*

$$\sum_{B\in\mathcal{B}_n(\Lambda)} Vol(B)^\beta \geq C_5(\beta)[e^{\lambda_1(\underline{\alpha}-1)}\epsilon^{d(\beta-1)}]^n, \tag{23}$$

where $\mathcal{B}_n(\Lambda)$ *is a minimal covering of* Λ *by the open* $2\epsilon^n$*-balls in* $\mathcal{B}_n$ *and*

$$C_5(\beta) = [A_1 2^d]^{\beta-1}\inf_{n>0}\sum_{P\in\wp_n}(\rho(P)\,Vol(P))^{\underline{\alpha}}.$$

is a positive constant independent of $n>0$.

Proof. Let $\mathcal{B}_n(\Lambda \cap P)$ be the minimal covering of $\Lambda \cap P$ previously defined in lemma lemma 20. Then, by definition,

$$\sum_{B \in \mathcal{B}_n(\Lambda \cap P)} \mathrm{Vol}\,(B) \geq \rho(P)\mathrm{Vol}\,(P).$$

Multiplying and dividing each term of the sum by $\mathrm{Vol}\,(B)^{\beta}$ and using again (5) we get that

$$[A_1 2^d]^{(1-\beta)} \epsilon^{nd(1-\beta)} \sum_{B \in \mathcal{B}_n(\Lambda \cap P)} \mathrm{Vol}\,(B)^{\beta} \geq \rho(P)\mathrm{Vol}\,(P).$$

Therefore,

$$[A_1 2^d]^{(1-\beta)} \epsilon^{nd(1-\beta)} \sum_{B \in \mathcal{B}_n(\Lambda)} \mathrm{Vol}\,(B)^{\beta} \geq \sum_{P \in \wp_n} \rho(P)\mathrm{Vol}\,(P).$$

Now, multiplying and dividing each term of the sum $\sum_{P \in \wp_n} \rho(P)\mathrm{Vol}\,(P)$ by the factor $(\rho(P)\mathrm{Vol}\,(P))^{\underline{\alpha}}$ and using the volume bounds (9) we get that

$$\sum_{P \in \wp_n} \rho(P)\mathrm{Vol}\,(P) \geq [e^{-\lambda_1 n(1-\underline{\alpha})}] \sum_{P \in \wp_n} (\rho(P)\mathrm{Vol}\,(P))^{\underline{\alpha}}.$$

We thus get easily (23). We use thermodynamic formalism to prove that:

$$\inf_{n>0} \sum_{P \in \wp_n} (\rho(P)\mathrm{Vol}\,(P))^{\underline{\alpha}} \;>\; 0. \tag{24}$$

By lemma 5.3,

$$\rho(\wp_n(x)) \geq e^{-n\overline{\Sigma}} \quad \forall\, x \in \Lambda,\; n \geq 1,$$

where $\overline{\Sigma} \geq 0$ is the supremum of $\Sigma(x)$, the asymptotic fraction of the volume of P which is covered by $2\epsilon^n$-balls of the minimal covering $\mathcal{B}_n(\Lambda \cap P)$. Let $\mu_{\underline{\alpha}}$ be the equilibrium state for the Hölder continuous potential $\phi = -\underline{\alpha} \log(e^{\overline{\Sigma}} Jf)$. Then, for $\mu_{\underline{\alpha}}$-a.e. $x \in \Lambda$:

$$\begin{aligned}
(\rho(\wp_n(x))\mathrm{Vol}\,(\wp_n(x)))^{\underline{\alpha}} &\geq (e^{-n\overline{\Sigma}}\mathrm{Vol}\,(\wp_n(x)))^{\underline{\alpha}} \\
&\geq C_1^{-\underline{\alpha}} (e^{-n\overline{\Sigma}} [Jf^n(x)]^{-1})^{\underline{\alpha}} \\
&= C_1^{-\underline{\alpha}} \left(e^{n\overline{\Sigma}} \prod_{k=0}^{n-1} Jf(f^k(x)) \right)^{-\underline{\alpha}} \\
&= C_1^{-\underline{\alpha}} \exp\left(\sum_{k=0}^{n-1} \phi(f^k(x)) \right) \\
&\geq (C_1^{\underline{\alpha}} G)^{-1} \mu_{\underline{\alpha}}(\wp_n(x)),
\end{aligned}$$

by the Gibbs property (1). Thus,

$$\sum_{P \in \wp_n} (\rho(P)\mathrm{Vol}\,(P))^{\underline{\alpha}} \geq C_1^{-\underline{\alpha}} G_1^{-1} > 0$$

for every $n > 0$, so proving (24). □

Corollary 5.2. *Let $\zeta > 0$ be a small positive number and $\mathcal{B}_0$ a ζ-covering of Λ. Then, for every $\beta > 0$ there exists a constant $C_6(\beta) > 0$ such that:*

$$\sum_{B_k^0 \in \mathcal{B}_0} Vol\,(B_k^0)^{\beta} \geq C_6(\beta)[e^{n\lambda_1(\underline{\alpha}-1)} \zeta^{d(\beta-1)}] \tag{25}$$

for every $n > 0$ such that $2\epsilon^n < \lambda(\mathcal{B}_0) < \zeta$, where $\lambda(\mathcal{B}_0)$ is the Lebesgue number of $\mathcal{B}_0$. The constant $C_6(\beta)$ neither depends on the covering, nor on ζ or $n > 0$.

Proof. By (18),

$$\sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^{\beta} \le C_4(\beta) \sum_{B_k^0\in\mathcal{B}_0} \mathrm{Vol}\,(B_k^0)^{\beta},$$

where

$$C_4(\beta) = \frac{2^{d(\beta-1)}A_1 D}{A_0}[\epsilon^{nd(\beta-1)}\zeta^{d(1-\beta)}].$$

and by (23)

$$\sum_{B\in\mathcal{B}_n(\Lambda)} \mathrm{Vol}\,(B)^{\beta} \ge C_5(\beta)[e^{\lambda_1(\underline{\alpha}-1)}\epsilon^{d(\beta-1)}]^n,$$

where

$$C_5(\beta) = [A_1 2^d]^{\beta-1} \inf_{n>0} \sum_{P\in\wp_n} (\rho(P)\mathrm{Vol}\,(P))^{\underline{\alpha}}.$$

Then:

$$\begin{aligned}
\sum_{B_k^0\in\mathcal{B}_0} \mathrm{Vol}\,(B_k^0)^{\beta} &\ge \frac{1}{C_4(\beta)} \sum_{B\in\mathcal{B}_n} \mathrm{Vol}\,(B)^{\beta} \ge \frac{1}{C_4(\beta)} \sum_{B\in\mathcal{B}_n(\Lambda)} \mathrm{Vol}\,(B)^{\beta} \\
&\ge \frac{C_5(\beta)}{C_4(\beta)}[e^{\lambda_1(\underline{\alpha}-1)}\epsilon^{d(\beta-1)}]^n = C_6(\beta)\frac{[e^{\lambda_1(\alpha-1)}\epsilon^{d(\beta-1)}]^n}{\epsilon^{nd(\beta-1)}\zeta^{d(1-\beta)}} \\
&= C_6(\beta)[e^{n\lambda_1(\alpha-1)}\zeta^{d(\beta-1)}].
\end{aligned}$$

where

$$C_6(\beta) = \frac{A_0 C_5(\beta)}{2^{d(\beta-1)}A_1 D} = \frac{A_0[A_1 2^d]^{\beta-1}}{2^{d(\beta-1)}A_1 D} \inf_{n>0} \sum_{P\in\wp_n} (\rho(P)\mathrm{Vol}\,(P))^{\underline{\alpha}} > 0$$

□

Conclusion of the proof of theorem 1:

Let $\underline{\beta}$ be the unique solution to the equation:

$$e^{\lambda_1(\underline{\alpha}-1)}\epsilon^{d(\underline{\beta}-1)} = 1, \tag{26}$$

equivalently, $\lambda_1(\underline{\alpha}-1) + d(\underline{\beta}-1)\log\epsilon = 0$ Choose $\zeta > 0$ a small positive number such that $\mathcal{H}_{d\underline{\beta},\zeta}(\Lambda) < +\infty$. We can do this for, otherwise, $\mathcal{H}_{d\underline{\beta}}(\Lambda) = +\infty$ so that $d\underline{\beta} \le \dim_{\mathcal{H}}(\Lambda)$, concluding the proof. Let $\mathcal{B}_0$ be any finite ζ-covering of Λ and fix

$$n = \left[\frac{\log(\lambda(\mathcal{B}_0)/2)}{\log\epsilon}\right] + 1$$

where $[x]$ is the integer part of x, that is,

$$n < \frac{\log(\lambda(\mathcal{B}_0)/2)}{\log\epsilon} + 1 \quad \text{and then} \quad n\lambda_1(\underline{\alpha}-1) > \left(\frac{\log(\lambda(\mathcal{B}_0)/2)}{\log\epsilon} + 1\right)\lambda_1(\underline{\alpha}-1),$$

since $\lambda_1(\underline{\alpha}-1)<0$. Observe that the lower bound in (25) depends on $n=n(\mathcal{B}_0)$ and ζ. We cancel the dependence on $\mathcal{B}_0$ as follows. As $2\epsilon^n<\lambda(\mathcal{B}_0)<\zeta$ we have, by (25), that

$$\begin{aligned}
\sum_{B_k^0\in\mathcal{B}_0}\mathrm{Vol}\,(B_k^0)^{\beta} &\geq C_6(\underline{\beta})\exp\left[n\lambda_1(\underline{\alpha}-1)+d(\underline{\beta}-1)\log\zeta\right]\\
&> C_6(\underline{\beta})\exp\left[\left(\frac{\log(\lambda/2)}{\log\epsilon}+1\right)\lambda_1(\underline{\alpha}-1)+d(\underline{\beta}-1)\log\zeta\right]\\
&= C_6(\underline{\beta})\exp\left[\lambda_1(\underline{\alpha}-1)+\left(\frac{\log(\lambda/2)}{\log\epsilon}\right)\lambda_1(\underline{\alpha}-1)+d(\underline{\beta}-1)\log\zeta]\right]\\
&= C_6(\underline{\beta})\exp\left[\lambda_1(\underline{\alpha}-1)+\frac{\log\zeta}{\log\epsilon}\left(\frac{\log(\lambda/2)}{\log\zeta}\lambda_1(\underline{\alpha}-1)+d(\underline{\beta}-1)\log\epsilon\right)\right]\\
&= C_6(\underline{\beta})\exp\left[\lambda_1(\underline{\alpha}-1)+\frac{\log\zeta}{\log\epsilon}\left(\frac{\log(\lambda/2)}{\log\zeta}-1\right)\lambda_1(\underline{\alpha}-1)\right]\\
&= C_6(\underline{\beta})\exp\left[\lambda_1(\underline{\alpha}-1)+\frac{\log(\lambda/2\zeta)}{\log\epsilon}\lambda_1(\underline{\alpha}-1)\right]\\
&> C_6(\underline{\beta})\exp\left[\lambda_1(\underline{\alpha}-1)+\frac{\log 2}{\log\epsilon}\lambda_1(1-\underline{\alpha})\right]\\
&= C_6(\underline{\beta})\exp\left[\frac{\log(2/\epsilon)}{\log\epsilon}\lambda_1(1-\underline{\alpha})\right]=C_7(\underline{\beta}),
\end{aligned}$$

eliminating the dependence on the covering $\mathcal{B}_0$. Here we use that,

$$\begin{aligned}
\frac{\log(\lambda/2)}{\log\zeta}\lambda_1(\underline{\alpha}-1)+d(\underline{\beta}-1)\log\epsilon &= \left(\frac{\log(\lambda/2)}{\log\zeta}-1\right)\lambda_1(\underline{\alpha}-1)+[\lambda_1(\underline{\alpha}-1)+d(\underline{\beta}-1)\log\epsilon]\\
&= \left(\frac{\log(\lambda/2)}{\log\zeta}-1\right)\lambda_1(\underline{\alpha}-1),
\end{aligned}$$

adding and substracting terms and recalling that $\lambda_1(\underline{\alpha}-1)+d(\underline{\beta}-1)\log\epsilon=0$, by (26). We also use that

$$\frac{\log(\lambda/2\zeta)}{\log\epsilon}\lambda_1(\underline{\alpha}-1)=-\frac{\log(\lambda/2\zeta)}{\log\epsilon}\lambda_1(1-\underline{\alpha})>\frac{\log 2}{\log\epsilon}\lambda_1(1-\underline{\alpha}).$$

which follows from $\log(\lambda/2\zeta)<\log(1/2)$ since $\lambda(\mathcal{B}_0)<\zeta$. Thus, for every finite ζ-covering $\mathcal{B}_0$,

$$\sum_{B_k^0\in\mathcal{B}_0}\mathrm{Vol}\,(B_k^0)^{\underline{\beta}}\geq C_7(\underline{\beta})>0.\quad\text{and therefore}\quad\sum_{B_k^0\in\mathcal{B}_0}\mathrm{diam}(B_k^0)^{\underline{\beta}}\geq C_8(\underline{\beta}),$$

where

$$C_8(\underline{\beta})=\frac{2^{d\underline{\beta}}C_7(\underline{\beta})}{A_1^{\underline{\beta}}}>0,\quad\text{since}\quad\mathrm{Vol}\,(B_k^0)^{\bar{\beta}}\leq A_1^{\underline{\beta}}\left(\frac{\mathrm{diam}(B_l^0)}{2}\right)^{d\underline{\beta}}\quad\text{by (4).}$$

The constant $C_8(\underline{\beta})$ only depends on the geometry of the ambient manifold, the rates of volume expansion of f and the rates of contraction of its inverse branches and several other fixed topological data of the construction. As $C_8(\underline{\beta})$ does not depend on the ζ-covering $\mathcal{B}_0$, we conclude that:

$$0<C_8(\underline{\beta})\leq\mathcal{H}_{d\underline{\beta},\zeta}(\Lambda)<+\infty$$

and therefore, $\mathcal{H}_{d\underline{\beta}}(\Lambda)\geq C_8(\underline{\beta})>0$, since $\mathcal{H}_{d\underline{\beta},\zeta}(\Lambda)$ is non-decreasing in $\zeta>0$. Thus, by an standard argument for the Hausdorff measure, we conclude that

$$\mathcal{H}_{d\beta}(\Lambda)=+\infty\quad\text{for every}\quad\beta<\underline{\beta},$$

so proving that $d\underline{\beta}\leq\dim_{\mathcal{H}}(\Lambda)$. QED

6. Discussion

In this section we will test our approach with a popular example appearing in the theory of self-affine subsets. Let $0 < \tau < \beta$ with $\tau, \beta \neq 1/2$, be real numbers and consider the couple of affine contractions:

$$g_0(x, y) = \begin{bmatrix} \beta & 0 \\ 0 & \tau \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad g_1(x, y) = \begin{bmatrix} \beta & 0 \\ 0 & \tau \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1-\beta \\ 1-\tau \end{bmatrix}$$

This generates a limit set Λ starting with two rectangles R_0 and R_1 of width β and height τ attached to the down-left and upper-right corners of unit rectangle, respectively. This is the simplest non conformal self affine limit set. If $\tau < 1/2$ the rectangles are disjoint and the limit is a plane non conformal Cantor set. These Cantor sets are volume-irreducible. Degenerate configurations in which rectangles R_i are situated above a single interval in the OX axis, are excluded. In this case Λ is contained in a single vertical line and it is volume reducible. If $\tau = 1/2$ the rectangles intersect at the border and the construction converges to the graph of a continuous non differentiable function, unless $\beta = 1/2$, in which case the set is conformal and the limit is the diagonal $x = y$, at least with this choice of translations. A remarkable feature of this example is that, for almost every β such that $0 < \tau < 1/2 < \beta$, the projection of Λ onto the OX axis is an absolutely continuous Bernoulli convolution. See [5] and [17]. For this particular choice of parameters, Hausdorff and box-counting dimension counting dimension are equal to

$$\dim_{\mathcal{H}}(\Lambda) = \dim_B(\Lambda) = 1 - \frac{\log(2\beta)}{\log \tau} = \frac{\log(2\beta/\tau)}{\log(1/\tau)}$$

Notice also that the construction does not satisfy our border condition. However, we can separate the rectangles from the border with the same arguments used previously for Bedford-McMullen carpets. Certainly these perturbations modify the number-theoretical properties which are responsable for the remarkable geometry of these examples. However, as we are arguing, our goal is to give bounds for the Hausdorff dimension. We are just using exact formulas as a test case for our inequalities which are robust. In [17] an explicit a formulae for the Hausdorff and box-counting dimension is given:

Explicit formulae

- $0 < \tau < \beta < 1/2$:
$$\dim_B(\Lambda) = \dim_{\mathcal{H}}(\lambda) = \frac{\log 2}{\log(1/\beta)};$$
- $\beta = 2^{-\frac{1}{n}}$, $n\mathbb{N}$:
$$\dim_B(\Lambda) = \dim_{\mathcal{H}}(\lambda) = \frac{\log(2\beta/\tau)}{\log(1/\tau)};$$
- there exists a $0 < \gamma < 1$ such that for almost every $\gamma < \beta < 1$:
$$\dim_B(\Lambda) = \dim_{\mathcal{H}}(\lambda) = \frac{\log(2\beta/\tau)}{\log(1/\tau)};$$
- $\beta > 1/2$:
$$\dim_B(\Lambda) = \frac{\log(2\beta/\tau)}{\log(1/\tau)};$$

See [17]. We will use these formulas to compare the goodness of our bounds. Indeed, as $Dg_i = Dg$ and $Jg_i = Jg$ for $i = 0, 1$ are constant, our formulas can be easily calculated:

$$\epsilon = (1-\delta)\left(\frac{Jg}{\|Dg\|^{d-1+\delta}}\right)^{\frac{1}{1-\delta}} = (1-\delta)\left(\frac{\beta\tau}{\beta^{1+\delta}}\right)^{\frac{1}{1-\delta}} = (1-\delta)(\beta^{-\delta}\tau)^{\frac{1}{1-\delta}},$$

from where:

$$\log \epsilon = \log(1-\delta) + \frac{\log(\beta^{-\delta}\tau)}{1-\delta} = \frac{\log((1-\delta)^{1-\delta}\beta^{-\delta}\tau)}{1-\delta}$$

On the other hand,

$$\lambda_0 = \log(Jf - \rho) = \log((\beta\tau)^{-1} - \rho) \quad \text{and} \quad \lambda_1 = \log(Jf + \rho) = \log((\beta\tau)^{-1} + \rho).$$

The solution α to the Bowen equation is given by $2(Jg)^{\alpha} = 1$. Indeed, as the Jacobian Jf is constant we have

$$-s\sum_{k=0}^{n-1} \log(Jf) = \log[Jf]^{-sn}$$

and thus

$$\sum_{(i_1,\cdots,i_n)\in 2^n} \exp\left(-s\sum_{k=0}^{n-1}\log(Jf)\right) = \sum_{(i_1,\cdots,i_n)\in 2^n} [Jf]^{-sn} = 2^n[Jf]^{-sn} = (2Jg^s)^n.$$

Therefore,

$$\limsup_{n\to+\infty} \frac{1}{n}\log\left(\sum_{(i_1,\cdots,i_n)\in 2^n} \exp\left(-s\sum_{k=0}^{n-1}\log(Jf)\right)\right) = \log(2Jg^s) = 0$$

if and only if $2Jg^s = 1$ so giving:

$$\alpha = -\frac{\log 2}{\log(\beta\tau)} \quad \text{and} \quad 1-\alpha = \frac{\log(2\beta\tau)}{\log(\beta\tau)}.$$

Hence, the upper bound $U(\delta,\rho) = 2\bar{\beta}$ is

$$\begin{aligned} U(\delta,\rho) &= 2 + \frac{\lambda_0(1-\alpha)}{\log(\epsilon)}(1-\alpha) \\ &= 2 + (1-\delta)\frac{\log((\beta\tau)^{-1}-\rho)}{\log((1-\delta)^{1-\delta}\beta^{-\delta}\tau)}\frac{\log(2\beta\tau)}{\log(\beta\tau)}. \end{aligned}$$

which is an increasing function in δ, for every fixed small $\rho > 0$, as one can verify easily. Suppose that $\rho = 0$ and let $\delta \to 0^+$, then we get the limit

$$2 - \frac{\log(2\beta\tau)}{\log\tau} = 1 - \frac{\log(2\beta)}{\log\tau} = -\frac{\log(2\beta/\tau)}{\log\tau} = \frac{\log(2\beta/\tau)}{\log(1/\tau)}.$$

This shows that:

$$\dim_{\mathcal{H}}(\Lambda) \le \dim_B(\Lambda) \le \frac{\log(2\beta/\tau)}{\log(1/\tau)}$$

for every $0 < \tau < \beta < 1$ and $\tau, \beta \neq 1/2$, excluding degenerate configurations leading to volume reducible sets. According to Pollicot-Weiss formulae:

$$\dim_{\mathcal{H}}(\Lambda) = \dim_B(\Lambda) = \frac{\log(2\beta/\tau)}{\log(1/\tau)}$$

for a large set of contraction rates β and τ. Nevertheless, $\dim_{\mathcal{H}}(\Lambda) < \dim_B(\Lambda)$ for some specific choices of β and τ, depending on delicate number-theoretical properties. This is the case if $\tau = 1/2$ and β^{-1} is a Pisot number, according to [19]. This rises the question to find lower bounds for $\dim_{\mathcal{H}}(\Lambda)$. According to theorem 1 a lower bound $L(\delta,\rho,\Sigma) = 2\underline{\beta}$ is given in terms of $\underline{\alpha}$, the solution to a modified Bowen equation with parameter Σ. Doing a similar calculation as we did before for the standard Bowen equation we get that $\underline{\alpha}$ is the solution to

$$2(e^{\Sigma}Jf)^{-\underline{\alpha}} = 1.$$

That is

$$\underline{\alpha} = \frac{\log 2}{\Sigma + \log Jf} = \frac{\log 2}{\Sigma - \log \beta\tau}$$

and then:

$$1 - \underline{\alpha} = \frac{\Sigma - \log(2\beta\tau)}{\Sigma - \log(\beta\tau)}.$$

Therefore our lower bound is:

$$L(\delta, \rho, \Sigma) = 2 + (1-\delta)\left[\frac{\log((\beta\tau)^{-1} + \rho)}{\log((1-\delta)^{1-\delta}\beta^{-\delta}\tau)}\right]\left[\frac{\Sigma - \log(2\beta\tau)}{\Sigma - \log(\beta\tau)}\right],$$

for a not too large Σ. If $\delta = \rho = 0$ then

$$L(\Sigma) = 2 + \left[\frac{\log(\beta\tau)}{\log(\tau)}\right]\left[\frac{\Sigma - \log(2\beta\tau)}{\Sigma - \log(\beta\tau)}\right]$$

which is a strictly decreasing function of the variable Σ. Moreover, if $\Sigma = 0$ we recover the formula

$$L(0) = \frac{\log(2\beta/\tau)}{\log(1/\tau)},$$

which is an upper bound for the Hausdorff dimension. Now, suppose that β^{-1} is a Pisot number and that $\dim_{\mathcal{H}}(\Lambda) < \dim_B(\Lambda)$. If ρ and δ are positive and sufficiently small, then the function $L(\Sigma) = L(\delta, \rho, \Sigma)$ is still strictly decreasing in Σ and it gives a lower bound for the Hausdorff dimension, for some $\underline{\Sigma}$ not too large, with an upper bound which can be estimated from the data problem, explicitly:

$$L(\underline{\Sigma}) < \dim_{\mathcal{H}}(\Lambda) < L(0),$$

where

$$\underline{\Sigma} \leq \log\left(\frac{Jf}{(\|Df\|\epsilon)^{-(d_B(\Lambda)-\delta)}\epsilon^d}\right).$$

Notice that we only need an upper bound for $d_B(\Lambda)$ to estimate how large Σ can be, in order to calculate $L(\underline{\Sigma})$, the lower bound of the Hausdorff dimension.

Lets do a numerical example. Take $\beta = 0.77$ and $\tau = 0.35$. It is almost sure that $\dim_B(\Lambda) = \dim_{\mathcal{H}}(\Lambda) = 1.4113$, according to Pollicot-Weiss formulae. Let us choose $\delta = 0.1$ and $\rho = 0.1$. With these numbers we get the upper bound $U = 1.5131$. As for the lower bound we first compute an upper bound for the adjustable parameter Σ, giving $\overline{\Sigma} = 1.6909$ by estimating the box-counting dimension as 1.5. If we plot $L = L(\Sigma)$ over the interval $[0, \overline{\Sigma}]$, with the above choices for δ and ρ we get the graphic in figure 1. A lower bound for Hausdorff dimension can be given at $\underline{\Sigma} = 0.5$ with value $L = 1.3355$, approximately. A draw back of the method is that we need an estimation of the box-counting dimension to calculate an upper bound for Σ, but this is not too serious as, usually, it is easier to compute $\dim_B(\Lambda)$ than $\dim_{\mathcal{H}}(\Lambda)$. We can even approximate its value numerically if we lack of an exact formula.

Another feature of the present approach is that our bounds do not depend on the position of the initial rectangles, unless the limit degenerate into a volume reducible subset. Many other examples can be examined with these methods, as Bedford-McMullen, Gatzouras-Lalley, Barański and Feng-Wang carpets. See [5]. Althought we do not recover exact formulas for these examples, as in the present case, good enough bounds can be found. This is the subject of a work in progress and will appear elsewhere.

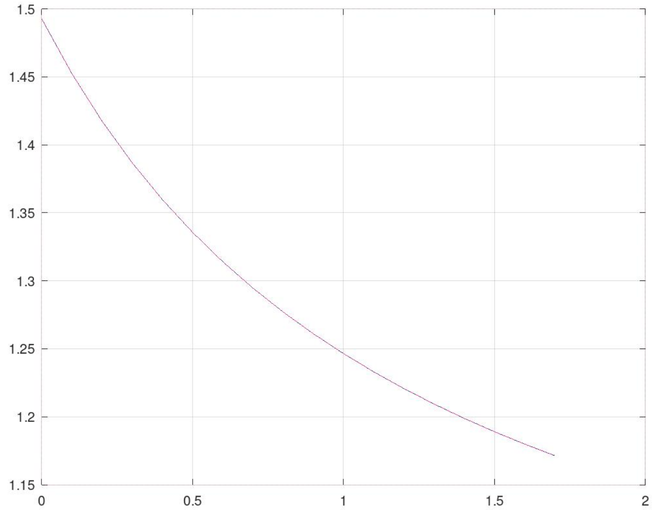

FIGURE 1. Plot of the lower bound $L = L(\Sigma)$ over the interval $\Sigma \in [0, 1.7]$

7. Conclusions

The previous discussion outline the potential application of these methods to the estimation, both from above and from below, of the most important dimensional indicators of a class of hyperbolic repellers. While we are dealing with a special class of dynamically defined Cantor sets, the possibility to approximate more general attractors as limits of suitable sequences of regular geometric constructions opens new perspectives in the study of their geometry, due to the robustness of our estimations. We are confident that, in the future, this approach may provide substancial advances in the understanding of the fractal geometry of these dynamically defined subsets.

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