Establishment and characterization of equalities for the ranges of matrices

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Abstract. This note addresses a fundamental problem in matrix theory on establishing and characterizing range equalities for matrix expressions that involve generalized inverses. We first establish a group of necessary and sufficient conditions for the matrix range equality range$(D_1 - C_1A_1^{\dagger}B_1) =$ range$(D_2 - C_2A_2^{\dagger}B_2)$ to hold, where $(\cdot)^{\dagger}$ denotes the Moore–Penrose inverse of matrix. We then give several groups of range equalities with extrusion properties for multiple matrix products associated with two matrices and their conjugate transposes and Moore–Penrose inverses.

Keywords: range, rank, matrix product, Moore–Penrose inverse, reverse order law

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1 Introduction

In this note, let $\mathbb{C}^{m\times n}$ denote the collection of all $m \times n$ complex matrices; A^* denote the conjugate transpose; $r(A)$ denote the rank of A, $\mathscr{R}(A) = \{Ax \,|\, x \in \mathbb{C}^{n\times 1}\}$ denote the range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$; and $[A,\, B]$ denote a columnwise partitioned matrix consisting of two submatrices A and B. A matrix $A \in \mathbb{C}^{m\times n}$ is said to be EP if $\mathscr{R}(A) = \mathscr{R}(A^*)$. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the four Penrose equations

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA. \tag{1.1}$$

Recall that the column and row spaces of matrix are basic concepts in linear algebra, and starting lines from various elementary and advanced researched in matrix theory and applications. One of the regular work on column and row spaces of matrices is to establish and characterize various possible equalities for theses spaces under various assumptions. Here we mention such a specified work on establishing range equalities of matrix expressions that involve matrices and their Moore–Penrose inverses. Assume that $f_1(A_1^{\dagger}, \ldots, A_k^{\dagger})$ and $f_2(B_1^{\dagger}, \ldots, B_l^{\dagger})$ are two matrix expressions, where $A_1, \ldots, A_k$, $B_1, \ldots, B_l$ are given matrices of appropriate sizes. Then a range equality between the two matrix expressions is formulated as

$$\mathscr{R}(f_1(A_1^{\dagger}, \ldots, A_k^{\dagger})) = \mathscr{R}(f_2(B_1^{\dagger}, \ldots, B_l^{\dagger})). \tag{1.2}$$

Here we mention one of the simplest and most important situations in (1.2):

$$\mathscr{R}(D_1 - C_1A_1^{\dagger}B_1) = \mathscr{R}(D_2 - C_2A_2^{\dagger}B_2), \tag{1.3}$$

where $D_i - C_iA_i^{\dagger}B_i$ are usually called the Schur complements of A_i in the block matrices $\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$ in the literature, $i = 1,\, 2$. We next mention two concrete matrix equality problems $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ and $(AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$, which are usually called the reverse order laws for the Moore–Penrose inverses of matrix product, which have also been studied since 1960s and many necessary and sufficient conditions for them to hold were established in the literature. Apparently, the four necessary range conditions for the two equalities to hold are given by

$$\mathscr{R}((AB)^{\dagger}) = \mathscr{R}(B^{\dagger}A^{\dagger}), \quad \mathscr{R}(((AB)^{\dagger})^*) = \mathscr{R}((B^{\dagger}A^{\dagger})^*), \tag{1.4}$$

$$\mathscr{R}((AB)^{\dagger}) = \mathscr{R}(B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}), \quad \mathscr{R}(((AB)^{\dagger})^*) = \mathscr{R}((B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger})^*). \tag{1.5}$$

In order to properly use these range equalities, we should know or establish various rules to simplify or remove the Moore–Penrose inverses in the range equalities of these matrix expressions. The aim of his article is to establish a general calculation process of characterizing the range equality in (1.3) by means of various known algebraic methods in matrix theory. In addition, we propose and prove several groups of fundamental and interesting range equalities with extrusion expressions for multiple matrix products associated with two matrices and their conjugate transposes and Moore–Penrose inverses.

We start with recalling some necessary preliminary results on ranks and generalized inverses of matrices that are used as tools for later sections. The following two lemmas are well known in linear algebra to describe relationships between matrices.

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Lemma 1.1. *Let* $A \in \mathbb{C}^{m\times n}$. *Then the rank equality* $r(A) = r(A^*)$ *holds.*

Lemma 1.2. *The matrix rank inequality* $r(XAY) \le r(A)$ *always holds. If the two matrix equalities* $X_1AY_1 = B$ *and* $A = X_2BY_2$ *hold, then the rank equality* $r(A) = r(B)$ *holds.*

The following lemma regarding Moore–Penrose inverses associated with a matrix and its operations can be found in [1,2].

Lemma 1.3. *Let* $A \in \mathbb{C}^{m\times n}$. *Then the following equalities hold*

$$(A^\dagger)^* = (A^*)^\dagger, \quad (A^\dagger)^\dagger = A, \tag{1.6}$$
$$A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^* = A^*(A^*AA^*)^\dagger A^*, \tag{1.7}$$
$$(A^*)^\dagger A^* = (AA^\dagger)^* = AA^\dagger, \quad A^*(A^*)^\dagger = (A^\dagger A)^* = A^\dagger A, \tag{1.8}$$
$$(AA^*)^\dagger = (A^\dagger)^*A^\dagger, \quad (A^*A)^\dagger = A^\dagger(A^\dagger)^*, \quad (AA^*A)^\dagger = A^\dagger(A^\dagger)^*A^\dagger. \tag{1.9}$$

Lemma 1.4. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, $P \in \mathbb{C}^{p\times m}$, *and* $Q \in \mathbb{C}^{q\times n}$. *Then the following results hold*

$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \ \ \textit{and} \ \ r(A) = r(B) \Rightarrow \mathscr{R}(A) = \mathscr{R}(B), \tag{1.10}$$
$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \Rightarrow \mathscr{R}(PA) \subseteq \mathscr{R}(PB), \tag{1.11}$$
$$\mathscr{R}(A) = \mathscr{R}(B) \Rightarrow \mathscr{R}(PA) = \mathscr{R}(PB). \tag{1.12}$$
$$\mathscr{R}(A) = \mathscr{R}(B) \Leftrightarrow r[A,\, B] = r(A) = r(B) \Leftrightarrow AA^\dagger = BB^\dagger, \tag{1.13}$$
$$\mathscr{R}(AQ^\dagger Q) = \mathscr{R}(AQ^\dagger) = \mathscr{R}(AQ^*Q) = \mathscr{R}(AQ^*). \tag{1.14}$$

We need to use the following well-known equalities for calculating ranks of the Schur complement $D - CA^\dagger B$ (cf. [3,4]).

Lemma 1.5. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, *and* $D \in \mathbb{C}^{l\times k}$. *Then*

$$r(D - CA^\dagger B) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A). \tag{1.15}$$

In particular, if $\mathscr{R}(A) \supseteq \mathscr{R}(B)$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(C^*)$, *then*

$$r(D - CA^\dagger B) = r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A). \tag{1.16}$$

2 Main results

In this section, we show how to derive elementary necessary and sufficient conditions for the range equality in (1.3) to hold using the basic matrix rank formulas in Lemma 1.5.

Theorem 2.1. *Let* $A_i \in \mathbb{C}^{m_i\times n_i}$, $B_i \in \mathbb{C}^{m_i\times k_i}$, $C_i \in \mathbb{C}^{l\times n_i}$, *and* $D_i \in \mathbb{C}^{l\times k_i}$, $i = 1, 2$. *Then the following five statements are equivalent*:

(i) $\mathscr{R}(D_1 - C_1A_1^\dagger B_1) = \mathscr{R}(D_2 - C_2A_2^\dagger B_2)$.

(ii) $r[D_1 - C_1A_1^\dagger B_1,\, D_2 - C_2A_2^\dagger B_2] = r(D_1 - C_1A_1^\dagger B_1) = r(D_2 - C_2A_2^\dagger B_2)$.

(iii) $$r\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 & 0 \\ 0 & A_2^*A_2A_2^* & 0 & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D_1 & D_2 \end{bmatrix} = r\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 \\ 0 & A_2^*A_2A_2^* & 0 \\ C_1A_1^* & C_2A_2^* & D_1 \end{bmatrix} = r\begin{bmatrix} A_1^*A_1A_1^* & 0 & 0 \\ 0 & A_2^*A_2A_2^* & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D_2 \end{bmatrix}.$$

(iv) $$\mathscr{R}\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 & 0 \\ 0 & A_2^*A_2A_2^* & 0 & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D_1 & D_2 \end{bmatrix} = \mathscr{R}\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 \\ 0 & A_2^*A_2A_2^* & 0 \\ C_1A_1^* & C_2A_2^* & D_1 \end{bmatrix} = \mathscr{R}\begin{bmatrix} A_1^*A_1A_1^* & 0 & 0 \\ 0 & A_2^*A_2A_2^* & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D_2 \end{bmatrix}.$$

(v) $\mathscr{R}\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 \\ 0 & A_2^*A_2A_2^* & 0 \\ C_1A_1^* & C_2A_2^* & D_1 \end{bmatrix} \supseteq \mathscr{R}\begin{bmatrix} 0 \\ A_2^*B_2 \\ D_2 \end{bmatrix}$ *and* $\mathscr{R}\begin{bmatrix} A_1^*A_1A_1^* & 0 & 0 \\ 0 & A_2^*A_2A_2^* & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D_2 \end{bmatrix} \supseteq \mathscr{R}\begin{bmatrix} A_1^*B_1 \\ 0 \\ D_1 \end{bmatrix}$.

In particular, if $\mathscr{R}(A_i) \supseteq \mathscr{R}(B_i)$ *and* $\mathscr{R}(A_i^*) \supseteq \mathscr{R}(C_i^*)$, $i = 1, 2$, *then the following four statements are equivalent:*

(vi) $\mathscr{R}(D_1 - C_1A_1^{\dagger}B_1) = \mathscr{R}(D_2 - C_2A_2^{\dagger}B_2)$.

(vii) $r\begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & C_2 & D_1 & D_2 \end{bmatrix} = r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & 0 \\ C_1 & C_2 & D_1 \end{bmatrix} = r\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D_2 \end{bmatrix}$.

(viii) $\mathscr{R}\begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & C_2 & D_1 & D_2 \end{bmatrix} = \mathscr{R}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & 0 \\ C_1 & C_2 & D_1 \end{bmatrix} = \mathscr{R}\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D_2 \end{bmatrix}$.

(ix) $\mathscr{R}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & 0 \\ C_1 & C_2 & D_1 \end{bmatrix} \supseteq \mathscr{R}\begin{bmatrix} 0 \\ B_2 \\ D_2 \end{bmatrix}$ *and* $\mathscr{R}\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D_2 \end{bmatrix} \supseteq \mathscr{R}\begin{bmatrix} B_1 \\ 0 \\ D_1 \end{bmatrix}$.

Proof. The equivalence of (i) and (ii) follows from (1.13). Next by (1.15),

$$\begin{aligned} r[D_1 - C_1A_1^{\dagger}B_1,\, D_2 - C_2A_2^{\dagger}B_2] &= r([D_1,\, D_2] - C_1A_1^{\dagger}[B_1,\, 0] - C_2A_2^{\dagger}[0,\, B_2]) \\ &= r\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 & 0 \\ 0 & A_2^*A_2A_2^* & 0 & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D_1 & D_2 \end{bmatrix} - r(A_1) - r(A_2), \\ r(D_1 - C_1A_1^{\dagger}B_1) &= r\begin{bmatrix} A_1^*A_1A_1^* & A_1^*B_1 \\ C_1A_1^* & D_1 \end{bmatrix} - r(A_1) \\ &= r\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 \\ 0 & A_2^*A_2A_2^* & 0 \\ C_1A_1^* & C_2A_2^* & D_1 \end{bmatrix} - r(A_1) - r(A_2), \\ r(D_2 - C_2A_2^{\dagger}B_2) &= r\begin{bmatrix} A_2^*A_2A_2^* & A_2^*B_2 \\ C_2A_2^* & D_2 \end{bmatrix} - r(A_2) \\ &= r\begin{bmatrix} A_1^*A_1A_1^* & 0 & 0 \\ 0 & A_2^*A_2A_2^* & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D_2 \end{bmatrix} - r(A_1) - r(A_2). \end{aligned}$$

Substituting these rank equalities into (ii) yields the two rank equalities in (iii). Applying (1.10) to the two rank equalities in (iii) leads to the equivalence of (iii), (iv), and (v). The equivalences of (vi), (vii), (viii), and (ix) follow from (i), (iii), (iv), (v), and (1.16). □

As a special case of Theorem 2.1, we can establish a general rule of characterizing the range equality of Schur complement and its conjugate transpose.

Corollary 2.2. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{k\times n}$, *and* $D \in \mathbb{C}^{k\times k}$. *Then the following five statements are equivalent:*

(i) $\mathscr{R}(D - CA^{\dagger}B) = \mathscr{R}((D - CA^{\dagger}B)^*)$, *namely,* $D - CA^{\dagger}B$ *is EP.*

(ii) $r[D - CA^{\dagger}B,\, (D - CA^{\dagger}B)^*] = r(D - CA^{\dagger}B)$.

(iii) $r\begin{bmatrix} A^*AA^* & 0 & A^*B & 0 \\ 0 & AA^*A & 0 & AC^* \\ CA^* & B^*A & D & D^* \end{bmatrix} = r\begin{bmatrix} A^*AA^* & 0 & A^*B \\ 0 & AA^*A & 0 \\ CA^* & B^*A & D \end{bmatrix} = r\begin{bmatrix} A^*AA^* & 0 & 0 \\ 0 & AA^*A & AC^* \\ CA^* & B^*A & D^* \end{bmatrix}$.

(iv) $\mathscr{R}\begin{bmatrix} A^*AA^* & 0 & A^*B & 0 \\ 0 & AA^*A & 0 & AC^* \\ CA^* & B^*A & D & D^* \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^*AA^* & 0 & A^*B \\ 0 & AA^*A & 0 \\ CA^* & B^*A & D \end{bmatrix} = \mathscr{R}\begin{bmatrix} A^*AA^* & 0 & 0 \\ 0 & AA^*A & AC^* \\ CA^* & B^*A & D^* \end{bmatrix}$.

(v) $\mathscr{R}\begin{bmatrix} A^*AA^* & 0 & A^*B \\ 0 & AA^*A & 0 \\ CA^* & B^*A & D \end{bmatrix} \supseteq \mathscr{R}\begin{bmatrix} 0 \\ AC^* \\ D^* \end{bmatrix}$ *and* $\mathscr{R}\begin{bmatrix} A^*AA^* & 0 & 0 \\ 0 & AA^*A & AC^* \\ CA^* & B^*A & D^* \end{bmatrix} \supseteq \mathscr{R}\begin{bmatrix} A^*B \\ 0 \\ D \end{bmatrix}$.

In particular, if $\mathscr{R}(A) \supseteq \mathscr{R}(B)$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(C^*)$, *then the following four statements are equivalent:*

(vi) $\mathscr{R}(D - CA^\dagger B) = \mathscr{R}((D - CA^\dagger B)^*)$.

(vii) $r\begin{bmatrix} A & 0 & B & 0 \\ 0 & A^* & 0 & C^* \\ C & B^* & D & D^* \end{bmatrix} = r\begin{bmatrix} A & 0 & B \\ 0 & A^* & 0 \\ C & B^* & D \end{bmatrix} = r\begin{bmatrix} A & 0 & 0 \\ 0 & A^* & C^* \\ C & B^* & D^* \end{bmatrix}$.

(viii) $\mathscr{R}\begin{bmatrix} A & 0 & B & 0 \\ 0 & A^* & 0 & C^* \\ C & B^* & D & D^* \end{bmatrix} = \mathscr{R}\begin{bmatrix} A & 0 & B \\ 0 & A^* & 0 \\ C & B^* & D \end{bmatrix} = \mathscr{R}\begin{bmatrix} A & 0 & 0 \\ 0 & A^* & C^* \\ C & B^* & D^* \end{bmatrix}$.

(ix) $\mathscr{R}\begin{bmatrix} A & 0 & B \\ 0 & A^* & 0 \\ C & B^* & D \end{bmatrix} \supseteq \mathscr{R}\begin{bmatrix} 0 \\ C^* \\ D^* \end{bmatrix}$ *and* $\mathscr{R}\begin{bmatrix} A & 0 & 0 \\ 0 & A^* & C^* \\ C & B^* & D^* \end{bmatrix} \supseteq \mathscr{R}\begin{bmatrix} B \\ 0 \\ D \end{bmatrix}$.

The above results clearly show that we can construct certain block matrices that are composed of the given matrices and their products and use the rank and range equalities of the block matrices to characterize (1.3). Since block matrices and ranks of matrices are among the most basic concepts in elementary linear algebra, the results and their derivations are easy to understand and used without much preparation. Apparently, the ranks of the block matrices can be adequately simplified when the given matrices are given in various specified forms, such as, zero matrices, identity matrices, and nonsingular matrices. Theoretically speaking, we can always establish some equivalent facts about the range equality on the left-hand side of (1.2) by repeatedly applying the above procedures.

3 Miscellaneous matrix range equalities

In the following, we present several groups of range equalities with extrusion expressions for multiple matrix products associated with two matrices and their conjugate transposes and Moore–Penrose inverses.

Theorem 3.1. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$. *Then the following four groups of range equalities hold*

$$\mathscr{R}(B(AB)^\dagger A) = \mathscr{R}(B(AB)^* A) = \mathscr{R}(BB^*A^*A), \tag{3.1}$$

$$\mathscr{R}(A^\dagger(B^\dagger A^\dagger)^\dagger B^\dagger) = \mathscr{R}(A^\dagger(B^\dagger A^\dagger)^* B^\dagger) = \mathscr{R}((A^*A)^\dagger(BB^*)^\dagger), \tag{3.2}$$

$$\begin{aligned} \mathscr{R}((B(AB)^\dagger A)^\dagger) &= \mathscr{R}((B(AB)^\dagger A)^*) = \mathscr{R}((B(AB)^* A)^\dagger) \\ &= \mathscr{R}((B(AB)^* A)^*) = \mathscr{R}(A^*ABB^*), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathscr{R}((A^\dagger(B^\dagger A^\dagger)^\dagger B^\dagger)^\dagger) &= \mathscr{R}((A^\dagger(B^\dagger A^\dagger)^\dagger B^\dagger)^*) = \mathscr{R}((A^\dagger(B^\dagger A^\dagger)^* B^\dagger)^\dagger) \\ &= \mathscr{R}((A^\dagger(B^\dagger A^\dagger)^* B^\dagger)^*) = \mathscr{R}((BB^*)^\dagger(A^*A)^\dagger). \end{aligned} \tag{3.4}$$

Proof. The second range equality is evident by noticing that $(AB)^* = B^*A^*$. To show the first range equality, we first obtain by Lemma 1.2 and the definition of the Moore–Penrose inverse the following two rank equalities

$$r(B(AB)^\dagger A) \leq r((AB)^\dagger) = r(AB), \tag{3.5}$$

$$r(B(AB)^\dagger A) \geq r(AB(AB)^\dagger AB) = r(AB), \tag{3.6}$$

both of which imply

$$r(B(AB)^\dagger A) = r(AB). \tag{3.7}$$

Also by (1.14),

$$\mathscr{R}(B(AB)^\dagger A) \supseteq \mathscr{R}(B(AB)^\dagger AB) = \mathscr{R}(B(AB)^*) \supseteq \mathscr{R}(B(AB)^* A), \tag{3.8}$$

Combining (3.7) and (3.8) with (1.10) leads to (3.1). Replacing A and B in (3.1) with $B^\dagger$ and $A^\dagger$, respectively, yields (3.2). Applying (1.14), (3.1), and (3.2) to $(B(AB)^\dagger A)^\dagger$ and $(A^\dagger(B^\dagger A^\dagger)^\dagger B^\dagger)^\dagger$ leads to the range equalities in (3.3) and (3.4), respectively. □

Obviously, the right-hand sides of (3.1)–(3.4) display certain extrusion patterns in contrast with the left-hand sides. So that we name (3.1)–(3.4) matrix range extrusion equalities. Given the above, we would believe intuitively that there are many possible variations and extensions of these range extrusion equalities associated with various mixed matrix products. Here we present several of groups range extrusion equalities as follows.

Theorem 3.2. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then the following range equalities hold*

$$\begin{aligned}&\mathscr{R}(A(B(AB)^{\dagger}A)^{\dagger}B) = \mathscr{R}(A(B(AB)^{\dagger}A)^{*}B)\\ &= \mathscr{R}(A(B(AB)^{*}A)^{\dagger}B) = \mathscr{R}(A(B(AB)^{*}A)^{*}B)\\ &= \mathscr{R}(AA^{*}ABB^{*}B);\end{aligned}\tag{3.9}$$

the following range equalities hold

$$\begin{aligned}&\mathscr{R}((A(B(AB)^{\dagger}A)^{\dagger}B)^{\dagger}) = \mathscr{R}((A(B(AB)^{\dagger}A)^{*}B)^{*})\\ &= \mathscr{R}((A(B(AB)^{*}A)^{\dagger}B)^{*}) = \mathscr{R}((A(B(AB)^{\dagger}A)^{\dagger}B)^{*})\\ &= \mathscr{R}((A(B(AB)^{*}A)^{*}B)^{\dagger}) = \mathscr{R}((A(B(AB)^{\dagger}A)^{*}B)^{\dagger})\\ &= \mathscr{R}((A(B(AB)^{*}A)^{\dagger}B)^{\dagger}) = \mathscr{R}((A(B(AB)^{*}A)^{*}B)^{*})\\ &= \mathscr{R}(B^{*}BB^{*}A^{*}AA^{*});\end{aligned}\tag{3.10}$$

the following range equalities hold

$$\begin{aligned}&\mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger}) = \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{*}A^{\dagger})\\ &= \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger}) = \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})\\ &= \mathscr{R}((BB^{*}B)^{\dagger}(AA^{*}A)^{\dagger});\end{aligned}\tag{3.11}$$

and the following range equalities hold

$$\begin{aligned}&\mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}) = \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{*}A^{\dagger})^{*})\\ &= \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger})^{*}) = \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})^{*})\\ &= \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{\dagger}) = \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{*}A^{\dagger})^{\dagger})\\ &= \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}) = \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{*})\\ &= \mathscr{R}((A^{*}AA^{*})^{\dagger}(B^{*}BB^{*})^{\dagger}).\end{aligned}\tag{3.12}$$

Theorem 3.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then the following range equalities hold*

$$\begin{aligned}&\mathscr{R}(B(A(B(AB)^{\dagger}A)^{\dagger}B)^{\dagger}A)^{\dagger}) = \mathscr{R}(B(A(B(AB)^{*}A)^{*}B)^{*}A)^{\dagger})\\ &= \mathscr{R}(B(A(B(AB)^{*}A)^{*}B)^{\dagger}A)^{\dagger}) = \mathscr{R}(B(A(B(AB)^{*}A)^{*}B)^{*}A)^{*})\\ &= \mathscr{R}((A^{*}A)^{2}(BB^{*})^{2}),\end{aligned}\tag{3.13}$$

and the following range equalities hold

$$\begin{aligned}&\mathscr{R}((A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}) = \mathscr{R}((A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{*}B^{\dagger})^{\dagger})\\ &= \mathscr{R}((A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}) = \mathscr{R}((A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{*}B^{\dagger})^{*})\\ &= \mathscr{R}(((BB^{*})^{2})^{\dagger}((A^{*}A)^{2})^{\dagger}).\end{aligned}\tag{3.14}$$

Theorem 3.4. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and denote $\widehat{A} = AA^{*}AA^{*}A$ and $\widehat{B} = BB^{*}BB^{*}B$. Then the following range equalities hold*

$$\begin{aligned}&\mathscr{R}(A(B(A(B(AB)^{\dagger}A)^{\dagger}B)^{\dagger}A)^{\dagger}B) = \mathscr{R}(A(B(A(B(AB)^{*}A)^{*}B)^{*}A)^{\dagger}B)\\ &= \mathscr{R}(A(B(A(B(AB)^{*}A)^{*}B)^{\dagger}A)^{\dagger}B) = \mathscr{R}(A(B(A(B(AB)^{*}A)^{\dagger}B)^{\dagger}A)^{\dagger}B)\\ &= \mathscr{R}(A(B(A(B(AB)^{*}A)^{*}B)^{*}A)^{*}B) = \mathscr{R}(\widehat{A}\widehat{B}),\end{aligned}\tag{3.15}$$

and the following range equalities hold

$$\begin{aligned}&\mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger}) = \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger})\\ &= \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger}) = \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})\\ &= \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger}) = \mathscr{R}(\widehat{B}^{\dagger}\widehat{A}^{\dagger}).\end{aligned}\tag{3.16}$$

Apparently, the above result can further be simplified when A and B are given in some specified forms. For example, if $AA^*A = A$ and $BB^*B = B$, namely, A and B are two partial isometries, then the right-hand sides of (3.10)–(3.16) can be reduced to $\mathscr{R}(AB)$, $\mathscr{R}(B^*A^*)$, $\mathscr{R}(A^*ABB^*)$, $\mathscr{R}(BB^*A^*A)$, respectively.

It is worth pointing out that the most predictable application of the above results is to establish various equivalence relationships between matrix range equalities that involve the products of matrices and their Moore–Penrose inverses. As examples, we give some basic range equalities associated with (1.4) and (1.5).

Theorem 3.5. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then*

$$\mathscr{R}(B^\dagger A^\dagger) = \mathscr{R}((BB^*B)^\dagger (AA^*A)^\dagger) \Leftrightarrow \mathscr{R}(B^*A^*) = \mathscr{R}(B^*BB^*A^*AA^*), \tag{3.17}$$

$$\mathscr{R}((B^\dagger A^\dagger)^*) = \mathscr{R}(((BB^*B)^\dagger (AA^*A)^\dagger)^*) \Leftrightarrow \mathscr{R}(AB) = \mathscr{R}(AA^*ABB^*B). \tag{3.18}$$

Proof. We first know that

$$r(B^\dagger A^\dagger) = r((BB^*B)^\dagger (AA^*A)^\dagger) = r(B^*BB^*A^*AA^*) = r(AB), \tag{3.19}$$

$$r((B^\dagger A^\dagger)^*) = r(((BB^*B)^\dagger (AA^*A)^\dagger)^*) = r(AA^*ABB^*B) = r(AB). \tag{3.20}$$

We next show by (1.6)–(1.9), (1.11), and (1.12) that

$$\begin{aligned} r[B^\dagger A^\dagger,\ (BB^*B)^\dagger (AA^*A)^\dagger] &= r[B^*BB^*BB^\dagger A^*,\ B^*BB^*B(BB^*B)^\dagger (AA^*A)^*] \\ &= r[B^*BB^*A^*AA^*,\ B^*A^*], \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} r[(B^\dagger A^\dagger)^*,\ ((BB^*B)^\dagger (AA^*A)^\dagger)^*] &= r[AA^*AA^*(A^\dagger)^*B,\ AA^*AA^*(A^*AA^*)^\dagger BB^*B] \\ &= r[AA^*ABB^*B,\ AB]. \end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22) with (3.19) and (3.20) yield (3.17) and (3.18). □

We leave the proofs of the following results to the reader.

Theorem 3.6. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then the following range equalities hold*

$$\mathscr{R}(B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger) = \mathscr{R}(B^\dagger A^\dagger),$$

$$\mathscr{R}(B^\dagger (A^\dagger (B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger)^\dagger B^\dagger)^\dagger A^\dagger) = \mathscr{R}((BB^*B)^\dagger (AA^*A)^\dagger).$$

Theorem 3.7. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then*

$$(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \Leftrightarrow \mathscr{R}((A^\dagger)^*(B^\dagger)^*) = \mathscr{R}(AB) \Leftrightarrow \mathscr{R}(AA^*ABB^*B) = \mathscr{R}(AB),$$

$$(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger \Leftrightarrow \mathscr{R}(B^\dagger A^\dagger) = \mathscr{R}(B^*A^*) \Leftrightarrow \mathscr{R}(B^*BB^*A^*AA^*) = \mathscr{R}(B^*A^*).$$

Theorem 3.8. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then*

$$\mathscr{R}(AB - ABB^\dagger A^\dagger AB) = \mathscr{R}(AB) \Leftrightarrow r(AB - ABB^\dagger A^\dagger AB) = r(AB) \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}.$$

Theorem 3.9. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then*

$$\mathscr{R}(A^\dagger ABB^\dagger) = \mathscr{R}(BB^\dagger A^\dagger A) \Leftrightarrow A^\dagger ABB^\dagger = BB^\dagger A^\dagger A.$$

Finally, we remark that establishment and characterization of matrix range equalities is a fundamental and challenging problem in matrix theory with fruitful results and facts. As ongoing study, people can propose various simple and complicated matrix range equalities from theoretical applied point of view, which we believe will bring deep insight into many classic and new problems in matrix theory.

References

[1] A. Ben-Israel, T.N.E. Greville. Generalized Inverses: Theory and Applications. 2nd ed., Springer, New York, 2003.

[2] S.L. Campbell, C.D. Meyer, Jr. Generalized Inverses of Linear Transformations. Pitman Publishing Limited, 1979; Dover Publications, 1991; SIAM, Philadelphia, 2009.

[3] G. Marsaglia, G.P.H. Styan. Equalities and inequalities for ranks of matrices. Linear Multilinear Algebra 2(1974), 269–292.

[4] Y. Tian. Reverse order laws for the generalized inverses of multiple matrix products. Linear Algebra Appl. 211(1994), 185–200.