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Article

Some remarks on fuzzy Hilbert spaces

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Abstract: The aim of this paper is to determine a suitable definition for the concept of fuzzy Hilbert space. In order to achieve this, we firstly focused on various approaches from the already-existent literature. Then we considered another approach to the notion of fuzzy inner product and analysed its properties.

Keywords: fuzzy Hilbert space; fuzzy inner product; fuzzy norm

MSC: 46A16, 46S40

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1. Introduction and preliminaries

The research papers of A.K. Katsaras [15],[16] laid the foundations of the fuzzy functional analysis. Moreover, he was the first one who introduced the concept of a fuzzy norm. This concept has amassed great interest among mathematicians. Thus, in 1992, C. Felbin [10]] introduced a new idea of a fuzzy norm in a linear space by associating a real fuzzy number to each element of the linear space. In 2003, T. Bag and S.K. Samanta [4] put forward a new concept of a fuzzy norm, which was a fuzzy set on $X \times \mathbb{R}$. New fuzzy norm concepts were later introduced by R. Saadati and S.M. Vaezpour [26], C. Alegre and S.T. Romaguera [2], R. Ameri [3], I. Goleţ [18], A.K. Mirmostafaee [20]. In this paper we use the definition introduced by S. Nădăban and I. Dzitac [25].

Definition 1.1. *[25] Let X be a vector space over a field $\mathbb{K}$ and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty]$ is called a fuzzy norm on X if it satisfies:*

(N1) $N(x, 0) = 0, (\forall)x \in X$;
(N2) $[N(x, t) = 1, (\forall)t > 0]$ *iff* $x = 0$;
(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall)x \in X, (\forall)t \geq 0, (\forall)\lambda \in \mathbb{K}^*$;
(N4) $N(x + y, t + s) \geq N(x, t) * N(y, t), (\forall)x, y \in X, (\forall)t, s \geq 0$;
(N5) $(\forall)x \in X, N(x, \cdot)$ *is left continuous and* $\lim\limits_{t\to\infty} N(x, t) = 1$.

The triplet $(X, N, *)$ will be called fuzzy normed linear space (briefly FNLS).

Although there are many research papers focused on the concept of a fuzzy norm and its diverse applications, there are few papers which study the concept of a fuzzy inner product. So, R. Biswas in [5] defined the fuzzy inner product of elements in a linear space and two years later J.K. Kohli and R. Kumar altered the Biswas's definition of inner product space [17]. In fact, they showed that the definition of a fuzzy inner product space in terms of the conjugate of a vector is redundant and that those definitions are only restricted to the real linear spaces. They also introduced the fuzzy co-inner product spaces and the fuzzy co-norm functions in their paper. Two years later, in 1995, Eui-Whan Cho, Young-Key Kim and Chae-Seob Shin introduced and defined in [6] a fuzzy semi-inner-product

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space and investigated some properties of this fuzzy semi inner product space, those definitions are not restricted to the real linear spaces.

In 2008, P. Majumdar and S.K. Samanta [19] succeeded in taking the first leap forward towards finding a reliable definition of a fuzzy inner product space. According to them, a fuzzy inner product space (FIP-space) is a pair (X, P), where X is a linear space over $\mathbb{C}$ and P is a fuzzy set in $X \times X \times \mathbb{C}$ s.t.

(FIP1) For $s, t \in \mathbb{C}$, $P(x+y, z, |t|+|s|) \geq \min\{P(x,z,|t|), P(y,z,|s|)\}$;
(FIP2) For $s, t \in \mathbb{C}$, $P(x,y,|st|) \geq \min\{P(x,x,|s|^2), P(y,y,|t|^2)\}$;
(FIP3) For $t \in \mathbb{C}$, $P(x,y,t) = P(y,x,\bar{t})$;
(FIP4) $P(\alpha x, y, t) = P\left(x, y, \frac{t}{|\alpha|}\right), t \in \mathbb{C}, \alpha \in \mathbb{C}^*$;
(FIP5) $P(x,x,t) = 0, (\forall) t \in \mathbb{C} \setminus \mathbb{R}^+$;
(FIP6) $[P(x,x,t) = 1, (\forall) t > 0]$ iff $x = 0$;
(FIP7) $P(x,x,\cdot) : \mathbb{R} \to [0,1]$ is a monotonic non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} P(x,x,t) = 1$.

P will be called the fuzzy inner product on X.

We can already observe a serious problem in regard to finding a new reliable definition for the fuzzy inner product. The classical inequality Cauchy-Schwartz cannot be obtained by applying the other axioms and thus had to be introduced itself as an axiom (axiom (FIP2)).

In 2009, M. Goudarzi, S.M. Vaezpour and R. Saadati [13] introduced the concept of intuitionistic fuzzy inner product space. In this context, the Cauchy-Schwartz inequality, the Pythagorean Theorem and some convergence theorems were established.

Definition 1.2. *[13] A fuzzy inner product space (FIP-space) is a triplet $(X; P; *)$, where X is a real linear space, $*$ is a continuous t-norm and P is a fuzzy set on $X^2 \times \mathbb{R}$ satisfying the following conditions for every $x; y; z \in X$ and $t \in \mathbb{R}$.*

(FIP1) $P(x,y,0) = 0$;
(FIP2) $P(x,y,t) = P(y,x,t)$;
(FIP3) $P(x,x,t) = H(t), \forall t \in \mathbb{R}$ *iff* $x = 0$, *where* $H(t) = \begin{cases} 1, & if\ t > 0 \\ 0, & if\ t \leq 0 \end{cases}$;
(FIP4) For any real number α, $P(\alpha x, y, t) = \begin{cases} P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0; \\ 1 - P\left(x, y, \frac{t}{-\alpha}\right), & if\ \alpha < 0 \end{cases}$
(FIP5) $\sup_{s+r=t} (P(x,z,s) * P(y,z,r)) = P(x+y, z, t)$;
(FIP6) $P(x,y,\cdot) : \mathbb{R} \to [0,1]$ *is continuous on* $\mathbb{R} \setminus \{0\}$;
(FIP7) $\lim_{t\to\infty} P(x,y,t) = 1$.

In the same year, M. Goudarzi and S.M. Vaezpour [12] alter the definition of the fuzzy inner product space and prove several interesting results which take place in each fuzzy inner product space. More specifically, they introduced the notion of a fuzzy Hilbert space and deduce a fuzzy version of Riesz representation theorem.

According to M. Goudarzi and S.M. Vaezpour, a fuzzy inner product space (FIP - space) is a triplet $(X, P, *)$, where X is a real linear space, $*$ is a continuous t-norm and P is a fuzzy set in $X \times X \times \mathbb{R}$ s.t. the following conditions hold for every $x, y, z \in X$ and $s, t, r \in \mathbb{R}$.

(FI-1) $P(x,x,0) = 0$ and $P(x,x,t) > 0$, $(\forall) t > 0$;
(FI-2) $P(x,x,t) \neq H(t)$ for same $t \in \mathbb{R}$ iff $x \neq 0$;
(FI-3) $P(x,y,t) = P(y,x,t)$;
(FI-4) For any real number α, $P(\alpha x, y, t) = \begin{cases} P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0; \\ 1 - P\left(x, y, \frac{t}{-\alpha}\right), & if\ \alpha < 0 \end{cases}$

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(FI-5) $\sup_{s+r=t} \left(P(x,z,s) * P(y,z,r)\right) = P(x+y,z,t)$;
(FI-6) $P(x,y,\cdot) : \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R} \setminus \{0\}$;
(FI-7) $\lim_{t\to\infty} P(x,y,t) = 1$.

These definitions were extremely important in later studies. They would have been perfect provided that we considered the complex linear spaces.

In 2013, S. Mukherjee and T.Bag [22] amends the definition put forward by M.Goudarzi and S.M. Vaezpour by discarding the (FI-6) condition and by enacting minor changes to the (FI-4) and (FI-5) conditions.

Definition 1.3. *Let X be a linear space over $\mathbb{R}$. A fuzzy set P in $X \times X \times \mathbb{R}$ is called fuzzy real inner product on X if $(\forall)x,y,z \in X$ and $t \in \mathbb{R}$, the following conditions hold:*

(FI-1) $P(x,x,0) = 0$, $(\forall)t < 0$;
(FI-2) $\left[P(x,x,t) = 1, (\forall)t > 0\right]$ *iff* $x = 0$;
(FI-3) $P(x,y,t) = P(y,x,t)$;
(FI-4) $P(\alpha x,y,t) = \begin{cases} P\left(x,y,\frac{t}{\alpha}\right), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0; \\ 1 - P\left(x,y,\frac{t}{\alpha}\right), & if\ \alpha < 0 \end{cases}$
(FI-5) $P(x+y,z,t+s) \geq \min\{P(x,z,t), P(y,z,s)\}$;
(FI-6) $\lim_{t\to\infty} P(x,y,t) = 1$.

The pair (X,P) is called fuzzy real inner space.

In 2010, A. Hasankhani, A. Nazari and M. Saheli [14] introduced a new concept of a fuzzy Hilbert space. This concept is entirely different from the previous ones as this fuzzy inner product generates a new fuzzy norm of type Felbin. In order to present their definition, we firstly need to define some concepts.

Definition 1.4. *[9] A fuzzy set in $\mathbb{R}$, namely a mapping $x : \mathbb{R} \to [0,1]$, with the following properties:*

(1) x is convex, i.e. $x(t) \geq \min\{x(s), x(r)\}$ for $s \leq t \leq r$;
(2) x is normal, i.e. $(\exists)t_0 \in \mathbb{R} : x(t_0) = 1$;
(3) x is upper semicontinuous, i.e. $(\forall)t \in \mathbb{R}, (\forall)\alpha \in [0,1] : x(t) < \alpha, (\exists)\delta > 0\ s.t. |s-t| < \delta \Rightarrow x(s) < \alpha$

is called fuzzy real number. We denote by $\mathfrak{F}(\mathbb{R})$ the set of all fuzzy real numbers.

Definition 1.5. *[24] The arithmetic operation $+, -, \cdot, /$ on $\mathfrak{F}(\mathbb{R})$ are defined by:*

$$(x+y)(t) = \bigvee_{s\in\mathbb{R}} \min\{x(s), y(t-s)\},\ (\forall)t \in \mathbb{R};$$
$$(x-y)(t) = \bigvee_{s\in\mathbb{R}} \min\{x(s), y(s-t)\},\ (\forall)t \in \mathbb{R};$$
$$(xy)(t) = \bigvee_{s\in\mathbb{R}^*} \min\{x(s), y(t/s)\},\ (\forall)t \in \mathbb{R};$$
$$(x/y)(t) = \bigvee_{s\in\mathbb{R}} \min\{x(ts), y(s)\},\ (\forall)t \in \mathbb{R}.$$

Remark. *Let $x \in \mathfrak{F}(\mathbb{R})$ and $\alpha \in (0,1]$. The α-level sets $[x]_\alpha = \{t \in \mathbb{R} : x(t) \geq \alpha\}$ are closed intervals $\left[x_\alpha^-, x_\alpha^+\right]$.*

Definition 1.6. *[14] Let X be a linear space over $\mathbb{R}$. A fuzzy inner product on X is a mapping $< \cdot,\cdot >: X \times X \to \mathfrak{F}(\mathbb{R})$ s.t. $(\forall)x,y,z \in X, (\forall)r \in \mathbb{R}$, we have:*

(IP1) $< x+y,z > = < x,z > \oplus < y,z >$;

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(IP2) $< rx, y >= \tilde{r} < x, y >$, *where* $\tilde{r} = \begin{cases} 1, & if\ t = r \\ 0, & if\ t \neq r \end{cases}$;
(IP3) $< x, y >=< y, x >$;
(IP4) $< x, x > \geq 0$;
(IP5) $\inf_{\alpha \in (0,1]} < x, x >^{-}_{\alpha} = 0$ *if* $x \neq 0$;
(IP6) $< x, x >= \tilde{0}$ *iff* $x = 0$.

The pair $(X, \langle \cdot, \cdot \rangle)$ *is called fuzzy inner product space.*

The disadvantage of this definition is that only linear spaces over $\mathbb{R}$ can be considered. Another disadvantage is the difficulty of working with real fuzzy numbers.

Further, the following research papers [1], [3], [7], [11], [8], [27], [23] will be considered.

In 2016, M. Saheli and S.Khajepour Gelousalar [28] modified the definition of the fuzzy inner product space and proved some properties of the new fuzzy inner product space.

Definition 1.7. *[28] A fuzzy inner product space is a triplet* $(X, P, *)$, *where* X *is a fuzzy set in* $X \times X \times \mathbb{R}$ *satisfying the following conditions for every* $x, y, z \in X$ *and* $t, s \in \mathbb{R}$:

(FI1) $P(x, y, 0) = 0$;
(FI2) $P(x, y, t) = P(y, x, t)$;
(FI3) $[P(x, x, t) = 1, (\forall) t > 0]$ *iff* $x = 0$;
(FI4) $(\forall)\alpha \in \mathbb{R}, t \neq 0, \quad P(\alpha x, y, t) = \begin{cases} P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0; \\ 1 - P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha < 0 \end{cases}$
(FI5) $P(x, z, t) * P(y, z, s) \leq P(x + y, z, t + s), (\forall) t, s > 0$;
(FI6) $\lim_{t \to \infty} P(x, y, t) = 1$.

Also, in 2016, Z. Solimani and B. Daraby [29] slightly altered the definition of a fuzzy scalar product introduced in [14] by changing the (IP2) condition and merging the (IP4) and (IP5). Furthermore, the fuzzy scalar product now takes values in $\mathfrak{F}^*(\mathbb{R}) = \{\eta \in \mathfrak{F}(\mathbb{R}) : \eta(t) = 0 \text{ if } t < 0\}$.

Definition 1.8. *[29] Let* X *be a linear space over* $\mathbb{R}$. *A fuzzy inner product on* X *is a mapping* $< \cdot, \cdot >: X \times X \to \mathfrak{F}^*(\mathbb{R})$ *with the following properties* $(\forall) x, y, z \in X, (\forall) r \in \mathbb{R}$:

(FIP1) $< x + y, z >=< x, z > \oplus < y, z >$;
(FIP2) $< rx, y >= |\tilde{r}| < x, y >$;
(FIP3) $< x, y >=< y, x >$;
(FIP4) $x \neq 0 \Rightarrow < x, x > (t) = 0, (\forall) t < 0$;
(FIP5) $< x, x >= \tilde{0}$ *iff* $x = 0$.

The pair $(X, < \cdot, \cdot >)$ *is called fuzzy inner product space.*

In anul 2017, E. Mostofian, M. Azhini and A. Bodaghi [21] presented two new concepts of fuzzy inner product spaces and investigated some of basic properties of these spaces.

In this paper we introduce a new definition of the fuzzy inner product space starting from P. Majumdar and S.K. Samanta's definition [19]. In fact, we modified the P. Majumdar and S.K. Samanta's definition of inner product space and we introduced and proved some new properties of the fuzzy inner product function.

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2. A new approch for fuzzy inner product space

We will denote by $\mathbb{C}$ the space of complex numbers and we will denote by $\mathbb{R}_+^*$ the set of all strict positive real numbers.

Definition 2.1. *Let H be a linear space over $\mathbb{C}$. A fuzzy set P in $H \times H \times \mathbb{C}$ is called a fuzzy inner product on H if it satisfies:*

(FIP1) $P(x,x,v) = 0, (\forall)x \in H, (\forall)v \in \mathbb{C} \setminus \mathbb{R}_+^*$;
(FIP2) $P(x,x,t) = 1, (\forall)t \in \mathbb{R}_+^*$ *if and only if* $x = 0$;
(FIP3) $P(\alpha x,y,v) = P\left(x,y,\frac{v}{|\alpha|}\right), (\forall)x,y \in H, (\forall)v \in \mathbb{C}, (\forall)\alpha \in \mathbb{C}^*$;
(FIP4) $P(x,y,v) = P(y,x,\overline{v}), (\forall)x,y \in H, (\forall)v \in \mathbb{C}$;
(FIP5) $P(x+y,z,v+w) \geq \min\{P(x,z,v), P(y,z,w)\}, (\forall)x,y,z \in H, (\forall)v,w \in \mathbb{C}$;
(FIP6) $P(x,x,\cdot) : \mathbb{R}_+ \to [0,1], (\forall)x \in H$ *is left continuous and* $\lim_{t\to\infty} P(x,x,t) = 1$;
(FIP7) $P(x,y,st) \geq \min\{P(x,x,s^2), P(y,y,t^2)\}, (\forall)x,y \in H, (\forall)s,t \in \mathbb{R}_+^*$.

The pair (H,P) will be called fuzzy inner product space.

Example 2.1. *Let H be a linear space over $\mathbb{C}$ and $< \cdot,\cdot >: H \times H \to \mathbb{C}$ be an inner product. Then $P : H \times H \times \mathbb{C} \to [0,1]$,*

$$P(x,y,s) = \begin{cases} \frac{s}{s+|<x,y>|}, & \text{if } s \in \mathbb{R}_+^* \\ 0, & \text{if } s \in \mathbb{C} \setminus \mathbb{R}_+^* \end{cases}$$

is a fuzzy inner product on H.

Let verify now the conditions from the definition.

(FIP1) $P(x,x,v) = 0, (\forall)x \in H, (\forall)v \in \mathbb{C} \setminus \mathbb{R}_+^*$ is is obvious from definition of P.
(FIP2) $P(x,x,t) = 1, (\forall)v \in \mathbb{R}_+^* \Leftrightarrow t + |< x,x >| = t, (\forall)t > 0 \Leftrightarrow |< x,x >| = 0 \Leftrightarrow x = 0.$
(FIP3) $P(\alpha x,y,v) = P\left(x,y,\frac{v}{|\alpha|}\right), (\forall)x,y \in H, (\forall)v \in \mathbb{C}, (\forall)\,\alpha \in \mathbb{C}$ is obvious for $v \in \mathbb{C} \setminus \mathbb{R}_+^*$.
If $v \in \mathbb{R}_+^*$, than

$$P(\alpha x,y,v) = \frac{v}{v + |< \alpha x,y >|} = \frac{v}{v + |\,\alpha\,| \cdot |< x,y >|} = \frac{\frac{v}{|\alpha|}}{\frac{v}{|\alpha|} + |< x,y >|} = P\left(x,y,\frac{v}{|\alpha|}\right).$$

(FIP4) $P(x,y,v) = P(y,x,\overline{v}), (\forall)x,y \in H, (\forall)v \in \mathbb{C}$ is obvious for $v \in \mathbb{C} \setminus \mathbb{R}_+^*$.
If $v \in \mathbb{R}_+^*$, then $v = \overline{v}$ and

$$P(x,y,v) = \frac{v}{v + |< x,y >|} = \frac{\overline{v}}{\overline{v} + |< y,x >|} = P(y,x,\overline{v})$$

(FIP5) $P(x+y,z,v+w) \geq \min\{P(x,z,v), P(y,z,w)\}, (\forall)x,y,z \in H, (\forall)v,w \in \mathbb{C}$.
If at least one of v and w is from $\mathbb{C} \setminus \mathbb{R}_+^*$, then the result is obvious.
If $v,w \in \mathbb{R}_+^*$, let us assume without loss of generality that $P(x,z,v) \leq P(y,z,w)$. Then

$$\frac{v}{v + |< x,z >|} \leq \frac{w}{w + |< y,z >|} \Rightarrow$$

$$\Rightarrow \frac{v + |< x,z >|}{v} \geq \frac{w + |< y,z >|}{w} \Rightarrow$$

$$\Rightarrow 1 + \frac{|< x,z >|}{v} \geq 1 + \frac{|< y,z >|}{w} \Rightarrow$$

$$\Rightarrow \frac{|< x,z >|}{v} \geq \frac{|< y,z >|}{w} \Rightarrow$$

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$$\Rightarrow \frac{w}{v} |< x,z >| \geq |< y,z >| \Rightarrow$$

$$\Rightarrow |< x,z >| + \frac{w}{v} |< x,z >| \geq |< x,z >| + |< y,z >| \Rightarrow$$

$$\Rightarrow \frac{v+w}{v} |< x,z >| \geq |< x+y,z >| \Rightarrow$$

$$\Rightarrow \frac{|< x,z >|}{v} \geq \frac{|< x+y,z >|}{v+w} \Rightarrow$$

$$\Rightarrow \frac{|< x,z >|}{v} + 1 \geq \frac{|< x+y,z >|}{v+w} + 1 \Rightarrow$$

$$\Rightarrow \frac{v + |< x,z >|}{v} \geq \frac{(v+w) + |< x+y,z >|}{v+w} \Rightarrow$$

$$\Rightarrow \frac{v}{v + |< x,z >|} \leq \frac{v+w}{(v+w) + |< x+y,z >|} \Rightarrow$$

$$\Rightarrow P(x,z,v) \leq P(x+y,z,v+w)$$

$$\Rightarrow P(x+y,z,v+w) \geq \min\left\{P(x,z,v), P(y,z,w)\right\}, (\forall)x,y,z \in H, (\forall)v,w \in \mathbb{C}.$$

(FIP6) $P(x,x,\cdot) : \mathbb{R}_+ \to [0,1], (\forall)x \in H$ is left continuous function and $\lim_{t\to\infty} P(x,x,t) = 1$.

$$\lim_{t\to\infty} P(x,x,t) = \lim_{t\to\infty} \frac{t}{t + |< x,x >|} = \lim_{t\to\infty} \frac{t}{t(1 + \frac{|<x,x>|}{t})} = 1.$$

$F(x,x,\cdot)$ is left continuous in $t > 0$ follows from definition.

(FIP7) $P(x,y,st) \geq \min\{P(x,x,s^2), P(y,y,t^2)\}, (\forall)x,y \in H, (\forall)s,t \in \mathbb{R}_+^*$. If at least one of s and t is from $\mathbb{C} \setminus \mathbb{R}_+^*$, then the result is obvious.
If $s,t \in \mathbb{R}_+^*$, let us assume without loss of generality that $P(x,x,s^2) \leq P(y,y,t^2)$. Then

$$\frac{s^2}{s^2 + |< x,x >|} \leq \frac{t^2}{t^2 + |< y,y >|} \Leftrightarrow$$

$$t^2 |< x,x >| \geq s^2 |< y,y >| .$$

Thus by Cauchy–Schwartz inequality we obtain

$$s |< x,y >| \leq \sqrt{|< x,x >|} \cdot s\sqrt{|< y,y >|} \leq \sqrt{|< x,x >|} \cdot t\sqrt{|< x,x >|} = t |< x,x >| \Rightarrow$$

$$\Rightarrow s^2 |< x,y >| \leq st |< x,x >| \Rightarrow$$

$$\Rightarrow s^3 t + s^2 |< x,y >| \leq s^3 t + st |< x,x >| \Rightarrow$$

$$\Rightarrow s^2 (st + |< x,y >|) \leq st(s^2 + |< x,x >|) \Rightarrow$$

$$\frac{s^2}{s^2 + |< x,x >|} \leq \frac{st}{st + |< x,y >|} \Rightarrow$$

$$\Rightarrow P(x,x,s^2) \leq P(x,y,st) \Rightarrow$$

$$\Rightarrow P(x,y,st) \geq \min\left\{P(x,x,s^2), P(y,y,t^2)\right\}, (\forall)x,y \in H, (\forall)s,t \in \mathbb{R}_+^*.$$

Proposition 2.1. *For $x,y \in H$, $v \in \mathbb{C}$ and $\alpha \in \mathbb{C}$ we have*

$$P(x,\alpha y,v) = P\left(x,y,\frac{v}{|\alpha|}\right).$$

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Proof. From (FIP3) and (FIP4) it follows $P(x,\alpha y,v) = P(\alpha y,x,\overline{v}) = P\left(y,x,\frac{\overline{v}}{|\alpha|}\right) = P\left(x,y,\frac{\overline{\overline{v}}}{|\alpha|}\right) = P\left(x,y,\frac{v}{|\alpha|}\right)$.

Proposition 2.2. *For $x \in H$, $v \in \mathbb{R}_+^*$ we have*

$$P(x,0,v) = 1.$$

Proof. From (FIP3) and (FIP6) it follows

$$P(x,0,v) = P(x,0,2nv) = P(x,x-x,nv+nv) \geq \min\{P(x,x,nv),P(x,x,nv)\} =$$
$$= P(x,x,nv) \overset{n\to\infty}{\longrightarrow} 1$$

So $P(x,0,v) = 1$.

Proposition 2.3. *For $y \in H$, $v \in \mathbb{R}_+^*$ we have*

$$P(0,y,v) = 1.$$

Proposition 2.4. $P(x,y,\cdot) : \mathbb{R}_+ \to [0,1]$ *is a monotonic non-decreasing function on* $\mathbb{R}_+$, $(\forall)x,y \in H$.

Proof. Let $s,t \in \mathbb{R}_+$, $s \leq t$. Then $(\exists)p$ such that $t = s+p$ and

$$P(x,y,t) = P(x+0,y,s+p) \geq \min\{P(x,y,s),P(0,y,p)\} = P(x,y,s).$$

Hence $P(x,y,s) \leq P(x,y,t)$ for $s \leq t$.

Corollary 2.1. $P(x,y,st) \geq \min\left\{P(x,y,s^2),P(x,y,t^2)\right\}, (\forall)x,y \in H, (\forall)s,t \in \mathbb{R}_+^*$.

Proof. Let $s,t \in \mathbb{R}_+$, $s \leq t$. Then

$$P(x,y,s^2) \leq P(x,y,st) \leq P(x,y,t^2).$$

Hence $P(x,y,st) \geq \min\left\{P(x,y,s^2),P(x,y,t^2)\right\}$.

Let now $s,t \in \mathbb{R}_+$, $t \leq s$. Then

$$P(x,y,t^2) \leq P(x,y,st) \leq P(x,y,s^2).$$

Hence $P(x,y,st) \geq \min\left\{P(x,y,s^2),P(x,y,t^2)\right\}$.

Proposition 2.5. $P(x,y,v) \geq \min\{P(x,y-z,v),P(x,y+z,v)\}, (\forall)x,y,z \in H, (\forall)v \in \mathbb{C};$.

Proof.

$$P(x,y,v) = P(x,2y,2v) = P(x,y+z+y-z,v+v) \geq \min\{P(x,y+z,v),P(x,y-z,v)\}.$$

Hence $P(x,y,v) \geq \min\{P(x,y-z,v),P(x,y+z,v)\}$.

Theorem 2.1. *If (H,P) be a fuzzy inner product space, then $H : X \times [0,\infty) \to [0,1]$ defined by*

$$N(x,t) = P(x,x,t^2)$$

is a fuzzy norm on X.

Proof. **(N1)** $N(x,0) = P(x,x,0) = 0, (\forall)x \in H$ from **(FIP1)**;
(N2) $[N(x,t) = 1, (\forall)t > 0] \Leftrightarrow [P(x,x,t^2) = 1, (\forall)t > 0] \Leftrightarrow x = 0$ from **(FIP2)**;
(N3) $N(\lambda x, t) = P(\lambda x, \lambda x, t^2) = P\left(x, \lambda x, \frac{t^2}{|\lambda|}\right) = P\left(\lambda x, x, \frac{\bar{t}^2}{|\lambda|}\right) = P\left(\lambda x, x, \frac{t^2}{|\lambda|}\right) = P\left(x, x, \frac{t^2}{|\lambda|^2}\right) =$
$= N\left(x, \frac{t}{|\lambda|}\right), (\forall)t \geq 0, (\forall)\lambda \in \mathbb{K}^*$;
(N4) $N(x+t, t+s) \geq \min\{N(x,t), N(y,s)\}, (\forall)x,y \in H, (\forall)t,s \geq 0$.
If $t = 0$ or $s = 0$ the previous inequality is obvious. We asume that $t,s > 0$.

$$\begin{aligned} N(x+y,t+s) &= P\left(x+y,x+y,(t+s)^2\right) = \\ &= P(x+y,x+y,t^2+s^2+ts+ts) \geq \\ &\geq P(x,x+y,t^2+ts) \wedge P(y,x+y,s^2+ts) \geq \\ &\geq P(x,x,t^2) \wedge P(x,y,ts) \wedge P(y,x,ts) \wedge P(y,y,s^2) = \\ &= P(x,x,t^2) \wedge P(y,y,s^2) = \\ &= \min\{N(x,t), N(y,s)\}; \end{aligned}$$

(N5) From **(FIP6)** it result that $N(x,\cdot)$ is left continuous and $\lim_{t\to\infty} N(x,t) = 1$.

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