Generalized out-of-time-order correlator in supersymmetric quantum mechanics using tensor product formalism

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Abstract

In this article we study the presence of chaos in supersymmetric(SUSY) quantum mechanics. For that purpose we present a form of 4-point out of time order correlator(OTOC) for SUSY quantum mechanical systems using tensor product formalism. We calculate the 4-point OTOC for SUSY 1D harmonic oscillator and find that the OTOC is exactly equal to that of 1D bosonic harmonic oscillator system. In similar manner using the eigenstate representation of supersymmetric systems we calculate the generalized higher order out of time order correlator. The higher order OTOC is a more sensitive measure of chaos than the usual 4-point correlator used in literature. Finally, we calculate the generalized 2N-point OTOC for SUSY 1D harmonic oscillator.

I. INTRODUCTION

In the recent times Out of Time Ordered correlator (OTOC) has found numerous applications in various branches of Physics like condensed matter physics[3], quantum many-body systems[11], holographic duality[8,9] to name a few. This is considered to be an indication of the presence of chaos in a system. The exponential growth of OTOC in the initial time indicates the presence of a positive Lyapunov exponent which is a quantity that indicates the amount of chaos in the system. Non-oscillatory quantum OTOC growth reaches the MSS bound in the late time scale[5]. Though a recent work has shown that even non-chaoatic systems like 1-d inverted anharmonic oscillator can give a positive value of Lyapunov exponent in the early times but that is as mentioned in that paper is due to the local unstable maximum present in the form of the potential. On the other hand supersymmetric(SUSY) quantum mechanics provides a way too go beyond the standard model physics by introducing a new symmetry transformation between the bosons and fermions. A recent work has been done on the study of OTOC in SUSY systems using partner Hamiltonian formalism in the eigenstate representation[7]. In this work we choose to work in the eigenstate representation to calculate the OTOC for SUSY systems in tensor product formalism, We also present a way to calculate the generalized 2N-point OTOC by following the same approach. The generalized 2N-point OTOC can be regarded to be a more sensitive measure of randomness present in the system. In a semi-classical picture we can replace the commutator $[x(t), p(0)]$ by Poisson Bracket $i\hbar\{x(t), p(0)\}$, which is equal to $i\hbar\frac{\delta x(t)}{\delta x(0)}$. For a chaotic system with positive Lyapunov exponent λ, we can write $\frac{\delta x(t)}{\delta x(0)} \approx e^{\lambda t}$. Clearly for 2N-point OTOC, $[x(t), p(0)]^N$ will lead us to $e^{N\lambda t}$. So even if the value of Lyapunov exponent is small, it gets multiplied by N and magnifies the effect.

This paper is organised as follows; in section III, we have shown explicitly how 4-point out of time order correlators can be calculated for a supersymmetric system using tensor product formalism. We then give an example of SUSY 1-D harmonic oscillator explicitly. In section IV A, we give a generalized way to calculate 2N-point OTOC for arbitrary N. In IV B, we calculate the 2N-point SUSY OTOC for the same harmonic oscillator system.

II. PRELIMINARIES

In this section we write a brief review of the supersymmetric quantum mechanics and the general form of 4-point OTOC, already present in literature. The idea of a more generalized symmetry between bosons and fermions was brought into the picture to solve a few problems that can not be addressed by the standard model of particle physics. There are mainly two formalisms present in the literature to study these systems viz., partner Hamiltonian formalism and tensor product formalism[1,4,6]. In the tensor product formalism we first define a tensor product Hilbert space $\mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{F}}$ of bosonic and fermionic systems and then we proceed by defining our quantum operators as

$$\mathcal{O}_{\mathcal{S}} \equiv \mathcal{O}_{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{F}} : \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{F}} \rightarrow \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{F}} \tag{1}$$

where $\mathcal{O}_{\mathcal{S}}$, $\mathcal{O}_{\mathcal{B}}$, $\mathcal{O}_{\mathcal{F}}$ are supersymmetric, bosonic and fermionic operators respectively. On the other hand in partner Hamiltonian formalism we define two Hamiltonians namely bosonic and fermionic Hamiltonian and their eigenstates are called partner states. In this article we are interested in tensor product formalism only.

The usual 4-point OTOC is defined as

$$C_T(t_1, t_2) = -\langle [x(t_1), p(t_2)]^2 \rangle \tag{2}$$

where $\langle \mathcal{O} \rangle$ is the thermal average of the operator $\mathcal{O}$. In the eigenstate representation we can rewrite Eq.2 as

$$C_T(t_1, t_2) = \frac{1}{Z} \sum_n e^{-\beta E_n} c_n(t_1, t_2). \tag{3}$$

Here Z is the partition function of the system defined as $\langle e^{-\beta H} \rangle$ with H being the Hamiltonian of the system and E_n its eigenvalues. For quantum chaotic systems initially the thermal OTOC might grow as $\approx \hbar^2 e^{2\lambda t}$ but unlike the classical case it will saturate after t_E time scale(*), usually denoted as the Eherenfest time scale of the system. We would also like to mention another relevant time scale for this study of chaos after which we can express the expectation value of the 4-point out of time order correlators as a product of expectation value of two pairs of operators at different times with some negligible higher order corrections as $\langle \mathcal{O}_1(t_1)\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\mathcal{O}_2(t_2)\rangle \sim \langle \mathcal{O}_1(t_1)\mathcal{O}_1(t_1)\rangle\langle \mathcal{O}_2(t_2)\mathcal{O}_2(t_2)\rangle + \text{h.o.}(e^{-|t_1 - t_2|/t_d})$. This time scale is known as dissipation time scale.

III. GENERALIZED CALCULATION FOR 4-POINT OTOC IN TENSOR PRODUCT FORMALISM

In order to do our study of supersymmetric OTOC we at first build the supersymmetric Hilbert space as $\mathcal{H}_{\mathcal{B}}\otimes\mathcal{H}_{\mathcal{F}}$ and for a general supersymmetric system we write the Hamiltonian of the form

$$\mathbb{H} =\mathbb{H}_B \otimes \mathbb{I}_{\mathbb{F}} + \mathbb{I}_{\mathbb{B}} \otimes \mathbb{H}_F \tag{4}$$

where $\mathbb{H}_B$ and $\mathbb{H}_F$ are the bosonic and fermionic Hamiltonians respectively. For these systems the eigen states can be written as the tensor product of bosonic eigenstate and fermionic eigenstate. We express them as

$$|n_B, n_F\rangle = |n_B\rangle \otimes |n_F\rangle. \tag{5}$$

Now from Eq.2 and we can write the thermal OTOC in the form of Eq.3 where,

$$c_n(t_1, t_2) =\langle n_B, n_F|[x(t_1), p(t_2)]^2|n_B, n_F\rangle \tag{6}$$

and

$$Z = \sum_n e^{-\beta E_n}. \tag{7}$$

Here, β is the inverse temperature. We can write the microcanonical OTOC, c_n as,

$$\begin{aligned} c_n(t_1, t_2) =& \sum_m \langle n_B, n_F|[x(t_1), p(t_2)]|m_B, m_F\rangle\langle m_B, m_F|[x(t_1), p(t_2)]|n_B, n_F\rangle \\ =& \sum_m b_{nm}(t_1, t_2) b^*_{nm}(t_1, t_2) \end{aligned} \tag{8}$$

where

$$b_{nm}(t_1, t_2) = -i\langle n_B, n_F|[x(t_1), p(t_2)]|m_B, m_F\rangle. \tag{9}$$

Using the notation of Eq.1 the position and momentum operators can be expressed as,

$$x =x_B \otimes \mathbb{I}_{\mathbb{F}} + \mathbb{I}_{\mathbb{B}} \otimes x_F \tag{10}$$

$$x =\frac{1}{\sqrt{2\omega}}\left((a + a^\dagger) \otimes \mathbb{I}_{\mathbb{F}} + \mathbb{I}_{\mathbb{B}} \otimes (c + c^\dagger)\right) \tag{11}$$

and

$$p = i\sqrt{\frac{\omega}{2}}\left((a^\dagger - a)\otimes \mathbb{I}_{\mathbb{F}} + \mathbb{I}_{\mathbb{B}}\otimes (c^\dagger - c)\right). \tag{12}$$

Here $a^\dagger$, a, $c^\dagger$, c are bosonic and fermionic creation and annihilation operators respectively. We set $\omega = 1$ for rest of the calculations. Now, From Eq.9 we have,

$$\begin{aligned} b_{nm}(t_1,t_2) =& - i\langle n_B, n_F|[x(t_1),p(t_2)]|m_B,m_F\rangle \\ =& \sum_{k_B,k_F}\{-i\langle n_B,n_F|x(t_1)|k_B,k_F\rangle\langle k_B,k_F|p(t_2)|m_B,m_F\rangle \\ & + i\langle n_B,n_F|p(t_2)|k_B,k_F\rangle\langle k_B,k_F|x(t_1)|m_B,m_F\rangle\}. \end{aligned} \tag{13}$$

We now define the 1st term of b_{nm} as

$$\mathcal{B}^{(1)}_{nm} = -i\sum_{k_B,k_F}\langle n_B,n_F|x(t_1)|k_B,k_F\rangle\langle k_B,k_F|p(t_2)|m_B,m_F\rangle \tag{14}$$

Using the Heisenberg picture representation of the operators $x(t_1)$ and $p(t_2)$ we rewrite this equation as,

$$\mathcal{B}^{(1)}_{nm} = -i\sum_{k_B,k_F} e^{i(n_B+n_F-k_B-k_F)t_1}\ e^{i(k_B+k_F-m_B-m_F)t_2}\langle n_B,n_F|x|k_B,k_F\rangle\langle k_B,k_F|p|m_B,m_F\rangle. \tag{15}$$

We now use Eq.11 to calculate the expectation value of x as

$$\begin{aligned} x_{nk} =& \langle n_B,n_F|x|k_B,k_F\rangle \\ =& \frac{1}{\sqrt{2}}\left[\langle n_B|(a+a^\dagger)|k_B\rangle\langle n_F|k_F\rangle + \langle n_B|k_B\rangle\langle n_F|(c+c^\dagger)|k_F\rangle\right] \\ =& \frac{1}{\sqrt{2}}\left[\sqrt{n_B+1}\delta_{n_B+1,k_B}\delta_{n_F,k_F} + \sqrt{n_B}\delta_{n_B-1,k_B}\delta_{n_F,k_F} + \delta_{n_B,k_B}\delta_{n_F+1,k_F} + \delta_{n_B,k_B}\delta_{n_F-1,k_F}\right] \end{aligned} \tag{16}$$

and similarly using Eq.12 we have,

$$\begin{aligned} p_{km} =& \langle k_B,k_F|p|m_B,m_F\rangle \\ =& i\sqrt{\frac{1}{2}}\left[\langle k_B|a^\dagger - a|m_B\rangle\langle k_F|m_F\rangle + \langle k_B|m_B\rangle\langle k_F|c^\dagger - c|m_F\rangle\right] \\ =& i\sqrt{\frac{1}{2}}\left[\sqrt{m_B+1}\delta_{m_B+1,k_B}\delta_{k_F,m_F} - \sqrt{m_B}\delta_{m_B-1,k_B}\delta_{k_F,m_F} + \delta_{k_B,m_B}\delta_{m_F+1,k_F} - \delta_{k_B,m_B}\delta_{m_F-1,k_F}\right] \end{aligned} \tag{17}$$

Substituting Eq.16 and Eq.17 in Eq.15 we get

$$\begin{aligned}\mathcal{B}^{(1)}_{nm} =&\frac{1}{2}\sum_{k_B,k_F} e^{i((n_B+n_F-k_B-k_F)t_1+(k_B+k_F-m_B-m_F)t_2)}[\sqrt{n_B+1}\delta_{n_B+1,k_B}\delta_{n_F,k_F} \\ &+\sqrt{n_B}\delta_{n_B-1,k_B}\delta_{n_F,k_F}+\delta_{n_B,k_B}\delta_{n_F+1,k_F}+\delta_{n_B,k_B}\delta_{n_F-1,k_F}][\sqrt{m_B+1}\delta_{m_B+1,k_B}\delta_{k_F,m_F} \\ &-\sqrt{m_B}\delta_{m_B-1,k_B}\delta_{k_F,m_F}+\delta_{k_B,m_B}\delta_{m_F+1,k_F}-\delta_{k_B,m_B}\delta_{m_F-1,k_F}]\end{aligned} \tag{18}$$

After some straight forward simplification we get

$$\mathcal{B}^{(1)}_{nm} = \frac{1}{2}\left(e^{i(-t_1+t_2)}(n_B+1) - e^{i(t_1-t_2)}(n_B) + e^{i(-t_1+t_2)} - e^{i(t_1-t_2)}\right)\delta_{nm}. \tag{19}$$

Similarly the second term of b_{nm} can be evaluated as

$$\begin{aligned}\mathcal{B}^{(2)}_{nm} =&i\sum_{k_B,k_F}\langle n_B,n_F|p(t_2)|k_B,k_F\rangle\langle k_B,k_F|x(t_1)|m_B,m_F\rangle \\ =&i\sum_{k_B,k_F} e^{i(n_B+n_F-k_B-k_F)t_2}\, e^{i(k_B+k_F-m_B-m_F)t_1}\langle n_B,n_F|p|k_B,k_F\rangle\langle k_B,k_F|x|m_B,m_F\rangle.\end{aligned} \tag{20}$$

Substituting the expressions of the expectation values of p and x we obtain

$$\begin{aligned}\mathcal{B}^{(2)}_{nm} =& -\frac{1}{2}\sum_{k_B,k_F} e^{i((n_B+n_F-k_B-k_F)t_2+(k_B+k_F-m_B-m_F)t_1)}[\sqrt{n_B}\delta_{n_B-1,k_B}\delta_{n_F,k_F}-\sqrt{n_B+1}\delta_{n_B+1,k_B}\delta_{n_F,k_F} \\ &+\delta_{n_B,k_B}\delta_{n_F-1,k_F}-\delta_{n_B,k_B}\delta_{n_F+1,k_F}][\sqrt{m_B+1}\delta_{m_B+1,k_B}\delta_{k_F,m_F}+\sqrt{m_B}\delta_{m_B-1,k_B}\delta_{k_F,m_F} \\ &+\delta_{k_B,m_B}\delta_{m_F+1,k_F}+\delta_{k_B,m_B}\delta_{m_F-1,k_F}]\end{aligned}$$

After doing the multiplication and taking the summation over k_B and k_F we get:

$$\mathcal{B}^{(2)}_{nm} = \frac{1}{2}\left(e^{i(t_1-t_2)}(n_B+1) - e^{i(-t_1+t_2)}(n_B) - e^{i(-t_1+t_2)} + e^{i(t_1-t_2)}\right)\delta_{mn}. \tag{21}$$

Finally adding the two terms $\mathcal{B}^{(1)}_{nm}$ and $\mathcal{B}^{(2)}_{nm}$ we get

$$b_{nm} = \cos(t_1-t_2)\delta_{nm} \tag{22}$$

Thus the microcanonical OTOC is given by

$$\begin{aligned}c_n(t_1,t_2) =&\sum_m b_{nm}(t_1,t_2)b^*_{nm}(t_1,t_2) \\ =&\cos^2(t_1-t_2)\end{aligned} \tag{23}$$

Substituting $c_n(t_1, t_2)$ in the expression of thermal OTOC we obtain

$$\begin{aligned} C_T(t_1, t_2) =& \frac{1}{Z} \sum_n e^{-\beta E_n} c_n(t_1, t_2) \\ =& \frac{1}{Z} \sum_n e^{-\beta E_n} cos^2(t_1 - t_2) \\ =& \cos^2(t_1 - t_2) \end{aligned} \tag{24}$$

By comparing Eq.23 and Eq.24 with the result obtained by Hasimoto[12] , we conclude that the forms of four point microcanonical and the thermal OTOC are same for SUSY and non-SUSY 1D harmonic oscillator.

IV. 2N-POINT OTOC IN TENSOR PRODUCT FORMALISM

A. General form of 2N point OTOC

Now we generalize our result for the 2N-point OTOC and define it as

$$C_T(t_1, t_2) = -\langle [x(t_1), p(t_2)]^N \rangle \tag{25}$$

In the eigenstate representation it takes the form,

$$C_T(t_1, t_2) = \frac{1}{Z} \sum_n e^{-\beta E_n} c_n(t_1, t_2) \tag{26}$$

where,

$$c_n(t_1, t_2) = \langle n_B, n_F | [x(t_1), p(t_2)]^N | n_B, n_F \rangle. \tag{27}$$

Now we can write c_n as,

$$c_n(t_1, t_2) = \langle n_B, n_F | [x(t_1), p(t_2)]^N | n_B, n_F \rangle \tag{28}$$

$$\begin{aligned} =& \sum_{k_1 .., k_{(N-1)}} \langle n_B, n_F | [x(t_1), p(t_2)] | k_{1B}, k_{1F} \rangle \langle k_{1B}, k_{1F} | [x(t_1), p(t_2)] | k_{2B}, k_{2F} \rangle .. \\ & .. \langle k_{(N-1)B}, k_{(N-1)F} | [x(t_1), p(t_2)] | n_B, n_F \rangle \end{aligned}$$

$$= \sum_{k_1 .., k_{(N-1)}} d_{nk_1} d_{k_1 k_2} ... d_{k_{N-1} n} \tag{29}$$

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Here,

$$
\begin{aligned}
d_{nm} &= \langle n_B, n_F | [x(t_1), p(t_2)] | m_B, m_F \rangle \\
&= \langle n_B, n_F | x(t_1) p(t_2) | m_B, m_F \rangle - \langle n_B, n_F | p(t_2) x(t_1) | m_B, m_F \rangle \\
&= \sum_k \langle n_B, n_F | x(t_1) | k_B, k_F \rangle \langle k_B, k_F | p(t_2) | m_B, m_F \rangle - \\
&\qquad \langle n_B, n_F | p(t_2) | k_B, k_F \rangle \langle k_B, k_F | x(t_1) | m_B, m_F \rangle
\end{aligned} \tag{30}
$$

We can write the first term of d_{nm} as,

$$\mathcal{D}^{(1)}_{nm} = e^{i(E^n - E^k)t_1} e^{i(E^k - E^m)t_2} x_{nk} p_{km} \tag{31}$$

Here, $E^n = E^n_B + E^n_F$ and we define $E^n - E^k = E^{nk}$. Now, substituting p_{km} with $\frac{i}{2} x_{km} E^{km}$, we get,

$$\mathcal{D}^{(1)}_{nm} = \frac{i}{2} e^{iE^{nk}t_1} e^{iE^{km}t_2} x_{nk} x_{km} E^{km} \tag{32}$$

Similarly the second term can be written as

$$\mathcal{D}^{(2)}_{nm} = \frac{i}{2} e^{iE^{nk}t_2} e^{iE^{km}t_1} x_{nk} x_{km} E^{nk} \tag{33}$$

Now, adding equation 32 and 33, we can write d_{nm} as,

$$
\begin{aligned}
d_{nm} &= \langle n_B, n_F | [x(t_1), p(t_2)] | m_B, m_F \rangle \\
&= \sum_k \frac{i}{2} x_{nk} x_{km} \left(e^{iE^{nk}t_1} e^{iE^{km}t_2} E^{km} - e^{iE^{nk}t_2} e^{iE^{km}t_1} E^{nk} \right)
\end{aligned} \tag{34}
$$

We can calculate d_{nm} for any arbitrary n and m. We can put these values to Eq. 29. Now, eq. 27 will give us microcannonical OTOC and Eq 26 will give us the Thermal OTOC.

B. SUSY OTOC in Harmonic Oscillator

Harmonic oscillator potential is very special in SUSY quantum mechanics because of its shape invariant nature. It also has equally spaced energy spectrum and the Heisenberg-Weyl algebra can be realized for such potentials. So, for this potential from Eq.9 and Eq.22 we see that $\langle n_B, n_F | [x(t_1), p(t_2)] | m_B, m_F \rangle$ is independent of the energy levels of the harmonic oscillator and is equal to $\cos(t_1 - t_2)$. Therefore, we have,

$$d_{nm}(t_1, t_2) = \cos(t_1 - t_2) \delta_{nm} \tag{35}$$

Finally, from Eq.29 we get,

$$c_n(t_1, t_2) = \cos^N(t_1 - t_2) \tag{36}$$

Substituting $c_n(t_1, t_2)$ in the Eq.9 and Eq.26 we obtain the expression for 2N-point OTOC,

$$\begin{aligned} C_T(t_1, t_2) =& \frac{1}{Z}\sum_n e^{-\beta E_n} c_n(t_1, t_2) \\ =& \frac{1}{Z}\sum_n e^{-\beta E_n} \cos^N(t_1 - t_2) \\ =& \cos^N(t_1 - t_2) \end{aligned} \tag{37}$$

Because of the unique nature of this potential we get this oscillatory thermal OTOC for any higher power of the commutation relation. However, for some other potentials it might show different behaviour for higher point correlators as it is expected to be a more sensitive measure of chaos present in the system.

V. DISCUSSION AND CONCLUSION

To summarize, we have shown the explicit calculations for determining both the temperature independent microcanonical and temperature dependent thermal 4-point supersymmetric OTOC in tensor product formalism. We have considered the harmonic oscillator system for which the thermal OTOC (thermal average of mircocanonical OTOC) is temperature independent. Then we have studied how the 2N-point SUSY OTOC can be computed for arbitrary N using tensor product formalism. Again we have considered the harmonic oscillator system and have shown that the SUSY 2N-point OTOC goes as $\cos^N(t_1 - t_2)$. However, the general approach to calculate the 2N-point OTOC can be performed for various SUSY systems and can be used to measure randomness with desired sensitivity, which is yet to be done in literature.

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