A Log-Dagum Weibull Distribution: Properties and Characterization

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Abstract

Developments of new probability models for data analysis are keen interest of importance for all fields. The log-Dagum distribution has a prominent role in the theory and practice of statistics. In this article, a new family of continuous distributions generated from a log Dagum random variable called the log-Dagum Weibull distribution is proposed. The key properties of the proposed distribution are derived. Its density function can be symmetrical, left-skewed, right-skewed and reversed-J shaped and can have increasing, decreasing, bathtub hazard rates shaped. The model parameters are estimated by the method of maximum likelihood and illustrate its importance by means of applications to real data sets.

Keywords: probability distributions; log-dagum distribution; parameter estimation; weibull distribution

1. Introduction

Statistical distributions are extensively used in literature for modelling and forecasting real life phenomena. The recent literature has suggested several ways of extending well-known distributions. There has been an increased interest in defining new classes of univariate continuous distributions by introducing one or more additional shape parameter(s) to the baseline distribution. This induction of parameter(s) has been proved useful in exploring tail properties and also for improving the goodness-of-fit of the generator family. The well-known families are: the beta-G [9], Kumaraswamy-G [6], McDonald-G [3], Gamma-X [2],Gamma-G (type 1) [18], Gamma-G (type 2) [15], Gamma-G (type 3) [17],Log-Gamma-G [4],Logistic-G [16],Exponentiated Generalized-G [7], Transformed-transformer [2], Exponentiated T-X [2], Weibull-G [5], etc.

The proposed new distribution generalizes the log-degum Weibull distributions. Some Structural properties of this distribution are obtained and estimation the parameters via the Method of maximum likelihood presented.

This paper is organized as follows. In section 2; we present the generalized distribution including the corresponding probability density functions (pdf), survival function hazard

functions shape of hazard function and concavity. In section 3; Rth moments, L- moments quantile function and order statistics are presented. Section 4 contains the Shannon entropy and Renyi entropy. Section 5 Bonferroni and Lorenz curves. Section 6 is concerned with characterization via Hazard function, reverse hazard function and truncated moments of distribution. Estimation of model parameters is presented in section 7. Evaluation Measures and Practical Data Examples of the proposed model to real data are given in section 8, followed by concluding remarks.

Let $r(t)$ be the probability density function (pdf) of a random variable $T \in [a; b]$ for $-\infty \leq a < b < \infty$ and let $W[G(x)]$ be a function of the cumulative distribution function (cdf) of a random variable X such that $W[G(x)]$ satisfies the following conditions:

$(i)\ W[G(x)] \in [a; b];$

$(ii)\ W[G(x)]\ is\ differentiable\ and\ monotonically\ non-decreasing, and$ (1)

$(iii)\ W[G(x)] \to a\ as\ x \to -\infty\ and\ W[G(x)] \in b\ as\ x \to \infty$

Recently, Alzaatreh et al. (2013) defined the *T-X family* of distributions by

$$F(x) = \int_0^{W[G(x)]} r(t)dt \qquad (2)$$

Where $W[G(x)]$ satisfies the condition (1). The pdf corresponding to (2) is given by

$$f(x) = \left\{ \frac{d}{dx} W[G(x)] \right\} r\{W[G(x)]\} \qquad (3)$$

In Table 1, we provide the $W[G(x)]$ functions for some members of the T-X family of distributions.

Table 1: Different $W[G(x)]$ functions for special models of the T-X family

S.No.	$W[G(x)]$	Range of T	Members of T-X family
1	$G(x)$	[0,1]	Beta-G (Eugene et al., 2002) Kw-G type 1 (Cordeiro and de Castero, 2011) Mc-G (Alexander et al., 2012) Exp-G (Kw-G type 2) (Cordeiro et al., 2013)
2	$-log\,[G(x)]$	[0,∞]	Gamma-G Type-2 (Ristić and Balakrishnan, 2012) Log-Gamma-G Type-2 (Amini et al., 2012)
3	- log [1- $G(x)$]	[0,∞]	Gamma-G Type-1 (Zografos and Balakrishnan, 2009) Log-Gamma-G Type-1 (Amini et al., 2012) Weibull-X (Alzaatreh et al., 2013) Gamma-X (Alzaatreh et al., 2014)
4	- log [1 - $G^{\alpha}(x)$]	[0,∞]	Exponentiated T-X (Alzaghal et al., 2013)
5	log [- log [1 - $G(x)$]	[−∞, ∞]	Logistic-X

6	$log[\frac{G(x)}{1-G(x)}]$	$[-\infty, \infty]$	Logistic-*G (Torabi and Montazeri, 2014)proposed*
7	$\frac{G(x)}{1-G(x)}$	$[0,\infty]$	Gamma-G Type-3 (Torabi and Montazeri, 2012) Weibull-G

The aim of this paper is to propose a new family of continuous distributions, called the *log Dagum Weibull distribution*, and to study some of its mathematical properties.

2. The Log Dagum Weibull Distribution

A random variable T has the log Dagum distribution with shape parameter $\beta > 0$ and $\lambda > 0$ if it's cumulative distribution function (cdf) is given by

$$\pi(t) = (1 + e^{-\lambda x})^{-\beta} \qquad t \in R, \quad \beta > 0, \quad \lambda > 0 \tag{4}$$

and its corresponding probability density function (pdf) can be expressed as

$$r(t) = \beta\lambda\, e^{-\lambda x}(1 + e^{-\lambda x})^{-\beta-1} \qquad t \in R, \quad \beta > 0, \quad \lambda > 0 \tag{5}$$

Let $G(x)$ and $\vec{G}(x) = 1 - G(x)$ be the baseline cdf and survival function (sf) by replacing $W[G(x)]$ by $log(\frac{G(x)}{1-G(x)})$ and $r(t)$ with (5) in equation (2), we define the cdf of the Log Dagum-X family by

$$F(x) = \left[1 + \left(\frac{G(x)}{1-G(x)}\right)^{-\lambda}\right]^{-\beta} \tag{6}$$

The Log Dagum family pdf is expressed as

$$f(x) = \left[1 + \left(\frac{G(x)}{1-G(x)}\right)^{-\lambda}\right]^{-\beta-1} \left(\frac{G(x)}{1-G(x)}\right)^{-\lambda-1} \frac{\lambda\,\beta\,g(x)}{[1-G(x)]^2} \tag{7}$$

Henceforth, we denote by X a random variable having density function (7). The basic motivations for using the Log Dagum-x family in practice are to construct heavy tailed distributions that are not longer-tailed for modelling real data, to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped, to define special models with all types of the hazard rate function (hrf), to provide consistently better fits than other generated models under the same baseline distribution. The fact is well-demonstrated by fitting the log Degum Weibull distribution to two real data sets. However, we expect that there are other contexts in which the LX special models can produce worse fits than other generated distributions. Clearly, the results indicate that the new family is a very competitive class to other widely known generators with one extra shape parameter.

The corresponding cumulative density function (cdf) probability density function (pdf), hazard function (hrf) and survival function are given as

$$F_{LDW}(x\,,\lambda,\Theta,\beta)\;=(1+(e^{\theta x^{\beta}}-1)^{-\lambda})^{-\beta} \tag{8}$$

[Figure 1 about here.]

Figure 1 gives the plots of the cumulative distribution function of the LDW distribution. The plots of this figure shows that for fixed λ and β and changing θ the curve stretch out insignificantly towards right as θ increases. However, for fixed β and θ and changing λ the curve stretch out towards right significantly as λ increases.

And

$$f_{LDW}(x\,,\lambda,\Theta,\beta)=(1+(e^{\theta x^{\beta}}-1)^{-\lambda})^{-\beta-1}(-1+e^{\theta x^{\beta}})^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\,\beta^{2}x^{\beta-1} \tag{9}$$

[Figure 2 about here.]

Plots of Figure 2 display the density functions of the LDW distribution. Figure 2 portrays that changing λ against the fixed β and θ the density function decreases. but changing β against the fixed λ the nature of the curve towards right as θ increases, however in case of changing θ with fixed β and λ shift the curve towards left.

$$S_{LDW}(x\,,\lambda,\Theta,\beta)=1-(1+(e^{\theta x^{\beta}}-1)^{-\lambda})^{-\beta} \tag{10}$$

[Figure 4 about here.]

The graph of survival function increases for different values of parameters then suddenly starts gradually decreases and converges to zero.

$$h_{LDDW}(x\,,\lambda,\Theta,\beta)=\frac{(1+(e^{\theta x^{\beta}}-1)^{-\lambda})^{-\beta-1}(-1+e^{\theta x^{\beta}})^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}}{1-(1+(e^{\theta x^{\beta}}-1)^{-\lambda})^{-\beta}} \tag{11}$$

[Figure 3 about here.]

Hazard function is a significant indicator for observing the declining circumstance of a product which ranges from increasing, decreasing, bathtub (BT) shapes. So in this regard Figure 3 speaks out itself and justifies the potential of the model. Moreover, the hazard function plots in Figure 3 also portray the declining circumstance of the product as time increases in terms of impulsive spikes at the end of either increasing or decreasing hazard rate. This implies that the hazard function is sensitive against different combinations of the parameters as time changes,

which seems to be a refine image of non stationary process and hence the hazard curve does not remain stable as times passes. Moreover, Figure 3 displays increasing, decreasing bathtub hazard shapes.

Shape of hazard function:

Shape of the density function can be described analytically .the critical point of the LDW density are the root of the equation

$$\frac{\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-1}e^{\theta x^\beta}x^{\beta-2}(-1+\beta)\theta\lambda\beta^2}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-1}e^{\theta x^\beta}x^{2\beta-2}\theta^2\lambda\beta^3}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+\frac{\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-2}e^{\theta x^\beta}x^{2\beta-2}\theta^2\beta^3(-1-\lambda)\lambda}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}-$$

$$\frac{e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-2}x^{2\beta-2}(-1-\beta)\theta^2\beta^3\lambda^2}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-2\beta-2}x^{2\beta-2}\theta^2\beta^4\lambda^2}{\left(1-\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right)^2}=0$$

There may be more than one root.

Concavity:

The concavity of hazard rate function $h''(x)=0$

$$\frac{\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-1}e^{\theta x^\beta}x^{\beta-3}(-2+\beta)(-1+\beta)\theta\lambda\beta^2}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-1}e^{\theta x^\beta}x^{2\beta-3}(-1+\beta)\theta^2\lambda\beta^3}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-1}e^{\theta x^\beta}x^{2\beta-3}\beta^3(-2+2\beta)\theta^2\lambda}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}$$

$$+\frac{\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-1}e^{\theta x^\beta}x^{3\beta-3}\beta^4\theta^3\lambda}{-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}$$

$$+\frac{e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}x^{2\beta-3}(-1+\beta)\beta^3\theta^2(-1-\lambda)\lambda}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}x^{2\beta-3}\beta^3(-2+2\beta)\theta^2(-1-\lambda)\lambda}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{3e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}x^{3\beta-3}\beta^4\theta^3(-1-\lambda)\lambda}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{e^{3\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-\lambda-3}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}x^{3\beta-3}\beta^4\,\theta^3(-2-\lambda)\lambda(-1-\lambda)}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}-$$

$$\frac{e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-2}x^{2\beta-3}\beta^3\,\theta^2(-1-\beta)(-1+\beta)\lambda^2}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-2\beta-2}x^{2\beta-3}\beta^4\,\theta^2(-1+\beta)\lambda^2}{\left(1-\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right)^2}-$$

$$\frac{e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-2}x^{2\beta-3}(-1-\beta)\beta^3\,(-2+2\beta)\theta^2\lambda^2}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-2}x^{2\beta-3}\beta^4\,(-2+2\beta)\theta^2\lambda^2}{\left(1-\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right)^2}-$$

$$\frac{3e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-2}x^{3\beta-3}\beta^4\,(-1-\beta)\theta^3\lambda^2}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{3e^{2\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-2}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-2\beta-2}x^{3\beta-3}\beta^5\,\theta^3\lambda^2}{\left(1-\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right)^2}-$$

$$\frac{e^{3\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-3}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-2}x^{3\beta-3}\,(-1-\beta)\beta^4(-2-2\lambda)\theta^3\lambda^2}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{e^{3\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-3}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-2\beta-2}x^{3\beta-3}\beta^5\,\theta^3(-2-2\lambda)\lambda^2}{\left(1-\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right)^2}-$$

$$\frac{e^{3\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-3}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-2}x^{3\beta-3}\,(-1-\beta)\beta^4(-1-\lambda)\theta^3\lambda^2}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}+$$

$$\frac{e^{3\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-2\lambda-3}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-2\beta-2}x^{3\beta-3}\beta^5\,\theta^3(-1-\lambda)\lambda^2}{\left(1-\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right)^2}+$$

$$\frac{e^{3\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-3\lambda-3}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-3}x^{3\beta-3}\,(-2-\beta)(-1-\beta)\beta^4\theta^3\lambda^3}{1-(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta}}-$$

$$\frac{e^{3\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-3\lambda-3}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-2\beta-3}x^{3\beta-3}(-2-2\beta)\beta^5\,\theta^3\lambda^3}{\left(1-\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right)^2}-$$

$$\frac{e^{3\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-3\lambda-3}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-2\beta-3}x^{3\beta-3}(-1-\beta)\beta^5\,\theta^3\lambda^3}{\left(1-\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right)^2}+$$

$$\frac{2e^{3\theta x^\beta}\left(-1+e^{\theta x^\beta}\right)^{-3\lambda-3}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{3\beta-3}x^{3\beta-3}\beta^6\,\theta^3\lambda^3}{\left(1-\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right)^3}=0$$

[Figure 5 about here]

for different value of parameters the hazard function is concave up and concave down where the point concavity change is called point of inflection.

3. Some Statistical Properties

In this section, we study some statistical properties of the LDW distribution, including Rth moments, L- moments quantile function and order statistics.

3.1 Moments of LDWD.

Let X is a particularly continuous non-negative random variable with PDF $f(X)$, and then the Rth ordinary moment of the (LDW) distribution is given by:

$$E(x^r) = \int_0^\infty X^r f(X)dX$$

$$E(x^r) = \int_0^\infty X^r \left(1-(1-e^{\theta x^\beta})^{-\lambda}\right)^{-\beta-1}(1-e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}dx$$

$$E(x^r) = \sum_{h=0}^\infty \frac{(-k)_h(-1)^h}{h!}\int_0^\infty X^r\left((1-e^{\theta x^\beta})^{-\lambda}\right)^{-\beta-1}(1-e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}dx$$

$$E(x^r) = \int_0^\infty X^r (1-e^{\theta x^\beta})^{\lambda\beta-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}dx$$

$$Let\ y = e^{\theta x^\beta} \qquad dy = e^{\theta x^\beta}\theta\beta x^{\beta-1}dx \qquad x = \frac{1}{\theta^{\frac{1}{\beta}}}\ln(y)^{\frac{1}{\beta}}$$

$$E(x^r) = \lambda\beta\sum_{h=0}^\infty\frac{(-k)_h(-1)^h}{h!}\int_0^\infty\left(\frac{1}{\theta^{\frac{1}{\beta}}}\ln(y)^{\frac{1}{\beta}}\right)^r(1-y)^{\lambda\beta-1}dy$$

$$E(x^r) = \sum_{h=0}^\infty\frac{(-k)_h(-1)^h}{h!}\sum_{m=0}^{\lambda\beta-1}\frac{(-1)^m}{m!}\int_0^\infty(\ln y)^{\frac{r}{\beta}}y^m dy$$

$$Let\quad lny = -z \qquad y = e^{-z}\quad dy = -e^{-z}dz$$

$$E(x^r) = \sum_{h=0}^{\infty} \frac{(-k)_h (-1)^h}{h!} \sum_{m=0}^{\lambda\beta-1} \frac{(-1)^m}{m!} \int_0^{\infty} (-z)^{\frac{r}{\beta}} e^{-zm}(-e^{-z})dz$$

$$E(x^r) = \lambda\beta \frac{1}{\theta^{\frac{r}{\beta}}} \sum_{h=0}^{\infty} \frac{(-k)_h (-1)^h}{h!} \sum_{m=0}^{\lambda\beta-1} \frac{(-1)^m}{m!} \int_0^{\infty} (z)^{\frac{r}{\beta}} e^{-z(m+1)}dz$$

$$Let \quad z(m+1) = g \qquad z = \frac{g}{(m+1)} \quad dy = \frac{dg}{(m+1)}$$

$$E(x^r) \quad = \lambda\beta \frac{1}{\theta^{\frac{r}{\beta}}} \sum_{h=0}^{\infty} \frac{(-k)_h (-1)^h}{h!} \sum_{m=0}^{\lambda\beta-1} \frac{(-1)^m}{m!} \int_0^{\infty} (\frac{g}{(m+1)})^{\frac{r}{\beta}} e^{-g} \frac{dg}{(m+1)}$$

$$E(x^r) = \lambda\beta \frac{1}{\theta^{\frac{r}{\beta}}} \sum_{h=0}^{\infty} \frac{(-k)_h (-1)^h}{h!} \sum_{m=0}^{\lambda\beta-1} \frac{(-1)^m}{m!} \frac{1}{(m+1)^{\frac{r}{\beta}+1}} \Gamma(\frac{r}{\beta} + 1)$$

3.2 L-Moments:

$$B_k = E(x(F(x)^k))$$

Where

$$F(x) = (1 + (e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta}$$

$$B_k = E(x(1 + (e^{\theta x^\beta} - 1)^{-\lambda})^{-k\beta}$$

$$B_k = \int_0^{\infty} x \left(1 - \left(1 - e^{\theta x^\beta}\right)^{-\lambda}\right)^{-\beta k} f(x)dx$$

$$B_k == \int_0^{\infty} x \left(1 - \left(1 - e^{\theta x^\beta}\right)^{-\lambda}\right)^{-\beta k} ((1 - (1 - e^{\theta x^\beta})^{-\lambda})^{-\beta-1} (1 - e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta\lambda\beta^2 x^{\beta-1} dx$$

$$= \int_0^{\infty} x \left(1 - \left(1 - e^{\theta x^\beta}\right)^{-\lambda}\right)^{-\beta(k+1)-1} (1 - e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta\lambda\beta^2 x^{\beta-1} dx$$

$$= \sum_{h=0}^{\infty} \frac{(-k)_h (-1)^h}{h!} \int_0^{\infty} x(1 - e^{\theta x^\beta})^{\lambda\beta(k+1)-1} e^{\theta x^\beta} \theta\lambda\beta^2 x^{\beta-1} dx$$

Let $\quad e^{\theta x^\beta} = y, \quad e^{\theta x^\beta} \beta\theta x^{\beta-1}dx = dy, \quad x = \frac{1}{\theta^{\frac{1}{\beta}}}(lny)^{\frac{1}{\beta}}dy$

When $\quad x \to 0, \quad y \to 1 \quad$ and $\qquad\qquad x \to \infty, \quad y \to \infty$

$$B_k = \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{h=0}^{\infty} \frac{(-k)_h (-1)^h}{h!} \int_0^{\infty} (lny)^{\frac{1}{\beta}} (1 - y)^{\lambda\beta(k+1)-1} \, dy$$

Expanding $(1 - y)^{\lambda\beta(k+1)-1}$ using binomial expension

$$B_k = \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{h=0}^{\infty} \frac{(-k)_h (-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{(\lambda\beta(k+1)-1)} \binom{\lambda\beta(k+1)-1}{m} (-1)^{2m} \int_1^{\infty} (lny)^{\frac{1}{\beta}} (y)^m \, dy$$

Let $lny = -z \qquad\qquad y = e^{-z} \qquad\qquad dy = -e^{-z}dz$

When $y \to 1, \qquad z \to 0 \qquad\qquad$ when $y \to \infty, \qquad z \to \infty$

$$= \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{h=0}^{\infty} \frac{(-k)_h (-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{(\lambda\beta(k+1)-1)} \binom{\lambda\beta(k+1)-1}{m} (-1)^{2m} \int_1^{\infty} (lny)^{\frac{1}{\beta}} (y)^m \, dy$$

$$= \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{h=0}^{\infty} \frac{(-k)_h(-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{(\lambda\beta(k+1)-1)} \binom{\lambda\beta(k+1)-1}{m} (-1)^{2m} \int_0^{\infty} e^{-zm} \, z^{\frac{1}{\beta}} e^{-z} dz$$

$$= \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{h=0}^{\infty} \frac{(-k)_h(-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{(\lambda\beta(k+1)-1)} \binom{\lambda\beta(k+1)-1}{m} (-1)^{2m} \int_0^{\infty} e^{-z(m+1)} \, z^{\frac{1}{\beta}} dz$$

Let $z(m+1) = g$ $\frac{g}{z(m+1)} = z$ $\frac{dg}{z(m+1)} = dz$

When $z \to 0$, $g \to 0$ When $z \to \infty$, $g \to \infty$

$$= \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{h=0}^{\infty} \frac{(-k)_h(-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{(\lambda\beta(k+1)-1)} \binom{\lambda\beta(k+1)-1}{m} (-1)^{2m} (\frac{1}{(m+1)^{\frac{1}{\beta}+1}} \int_0^{\infty} e^{-g} \, g^{\frac{1}{\beta}} dg$$

$$B_k == \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{h=0}^{\infty} \frac{(-k)_h(-1)^h}{h!} e^{m+1} \sum_{k=0}^{\infty} \sum_{m=0}^{(\lambda\beta(k+1)-1)} \binom{\lambda\beta(k+1)-1}{m} (-1)^{2m} (\frac{\Gamma(\frac{1}{\beta}+1)}{(m+1)^{\frac{1}{\beta}+1}}$$

3.3 Quantile function

The quantile function is another way of describing a probability distribution. It can also be called the inverse cdf. It can be used to generate random samples for probability distributions and thereby can serve as an alternative to the pdf. In general, it is given as :

$$G(x) = u$$

$$(1 + (e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta} = u$$

$$1 + (e^{\theta x^\beta} - 1)^{-\lambda} = u^{-1/\beta}$$

$$(e^{\theta x^\beta} - 1)^{-\lambda} = u^{-1/\beta} - 1$$

$$e^{\theta x^\beta} - 1 = \left(u^{-\frac{1}{\beta}} - 1 \right)^{-\frac{1}{\lambda}}$$

$$e^{\theta x^\beta} = \left(u^{-\frac{1}{\beta}} - 1 \right)^{-1/\lambda} + 1$$

$$x = \left[\frac{1}{\theta} \log \left(u^{-\frac{1}{\beta}} - 1 \right)^{-\frac{1}{\lambda}} \right]^{1/\beta}$$

3.4 Ordered Statistic:

The pdf of the jth order statistic for a random sample of size n from a distribution function $F(x)$ and an associated pdf $f(x)$ is given by:

$$f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j}f(x)$$

where $f(x)$ and $F(x)$ are the pdf and cdf of the LDWD, respectively. The pdf of the jth order statistics for a random sample of size n from the LDW distribution is, however, given as follows

$$f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!}\left[\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right]^{j-1}\left[1-[(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda})^{-\beta}\right]^{n-j}$$

$$(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta-1}(-1+e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}$$

So, the pdf of minimum order statistics is obtained by substituting $j=1$ we have:

$$f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!}\left[1-[(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda})^{-\beta}\right]^{n-1}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}$$

$$(-1+e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}$$

While the corresponding pdf of maximum order statistics is obtained by making the substitution of $j=n$ in above equation

$$f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!}\left[\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right]^{n-1}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}$$

$$(-1+e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}$$

4. Entropies

Entropy is the measure of uncertainty. It is actually a concept of physics.

4.1 Shannon Entropy:

$$s = -\int_0^\infty f(x)\ln f(x)dx$$

$$f(x) = (1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta-1}(-1+e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\,\beta^2 x^{\beta-1}$$

$$\ln f(x) = -(\beta+1)\ln(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda})-(\lambda+1)\ln\left(-1+e^{\theta x^\beta}\right)+\theta x^\beta + \ln(\theta\lambda\,\beta^2)+(\beta-1)\ln x$$

$$\ln f(x) = \lambda(\beta+1)\ln(e^{\theta x^\beta}-1)-(\lambda+1)\ln\left(-1+e^{\theta x^\beta}\right)+\theta x^\beta + \ln(\theta\lambda\,\beta^2)+(\beta-1)\ln x$$

$$S=(1-\lambda\beta)\int_0^\infty f(x)\ln\left(e^{\theta x^\beta}-1\right)dx - \theta\int_0^\infty f(x)x^\beta dx - \ln(\theta\lambda\,\beta^2)\int_0^\infty f(x)\,dx -$$

$$(\beta-1)\int_0^\infty f(x)\ln x\,dx$$

Where $\int_0^\infty f(x)\,dx = 1$, $\quad \int_0^\infty f(x)\ln x\,dx = E(\ln x)$, $\quad \int_0^\infty f(x)x^\beta dx = E(x^\beta)$

$$\ln\left(e^{\theta x^\beta}-1\right) = \sum_{k=0}^\infty \frac{(-\theta)^k x^{k\beta}}{k!}$$

$$S = (1-\lambda\beta)\sum_{k=0}^\infty \frac{(-\theta)^k \mu'_{k\beta}}{-k!} - \theta E(x^\beta) - \ln(\theta\lambda\,\beta^2)-(\beta-1)\,E(\ln x)$$

4.2   Renyi Entropy:

$$R = \frac{1}{1-k}\int_0^\infty \left(f(x)\right)^k dx$$

$$R = \frac{1}{1-k}\int_0^\infty \left((1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta-1}(-1+e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\,\beta^2 x^{\beta-1})\right)^k dx$$

$$R = \frac{1}{1-k}\,\theta^k\lambda^k\,\beta^{2k}\int_0^\infty (1-(1-e^{\theta x^\beta})^{-\lambda})^{-k(\beta+1)}(1-e^{\theta x^\beta})^{-k(\lambda+1)}e^{k\theta x^\beta}x^{k(\beta-)1})\,dx$$

$$R = \frac{1}{1-k}\,\theta^k\lambda^k\,\beta^{2k}\sum_{h=0}^\infty \frac{(-k)_h(-1)^k}{h!}\int_0^\infty (1-e^{\theta x^\beta})^{\lambda k(\beta+1))}(1-e^{\theta x^\beta})^{-k(\lambda+1)}e^{k\theta x^\beta}x^{k(\beta-)1}dx$$

Let $e^{\theta x^\beta}=y,\qquad e^{\theta x^\beta}\beta\theta x^{\beta-1}dx=dy,\qquad x=\frac{1}{\theta^{\frac{1}{\beta}}}(lny)^{\frac{1}{\beta}}dy$

When $\qquad x\to 0,\quad y\to 1\quad$ and $\qquad\qquad x\to\infty,\quad y\to\infty$

$$R = \theta^{k-1}\lambda^k\,\beta^{2k-1}\sum_{h=0}^\infty \frac{(-k)_h(-1)^k}{h!}\int_1^\infty (1-y)^{k(\lambda\beta-1)}\,y^{k-1}\,\frac{1}{\theta^{(k-1)\left(1-\frac{1}{\beta}\right)}}(lny)^{\left(1-\frac{1}{\beta}\right)(k-1)}dy$$

$$R = \frac{1}{1-k}\frac{\theta^{k-1}\lambda^k\,\beta^{2k-1}}{\theta^{(k-1)\left(1-\frac{1}{\beta}\right)}}\sum_{h=0}^\infty \frac{(-k)_h(-1)^k}{h!}\sum_{k=0}^\infty \sum_{m=0}^{k(\lambda\beta-1)}(-1)^m\int_1^\infty y^{m+k-1}\,(lny)^{\left(1-\frac{1}{\beta}\right)(k-1)}dy$$

Let $lny=-z\qquad\qquad y=e^{-z}\qquad\qquad dy=-e^{-z}dz$

When $y\to 1,\qquad z\to 0\qquad\qquad$ when $\quad y\to\infty,\qquad z\to\infty$

$$R = \frac{1}{1-k}\frac{\theta^{k-1}\lambda^k\,\beta^{2k-1}}{\theta^{(k-1)\left(1-\frac{1}{\beta}\right)}}\sum_{h=0}^\infty \frac{(-k)_h(-1)^k}{h!}\sum_{k=0}^\infty \sum_{m=0}^{k(\lambda\beta-1)}(-1)^m\int_0^\infty e^{-z(m+k-1)}\,z^{\left(1-\frac{1}{\beta}\right)(k-1)}e^{-z}dz$$

$$= \frac{1}{1-k}\frac{\theta^{k-1}\lambda^k\,\beta^{2k-1}}{\theta^{(k-1)\left(1-\frac{1}{\beta}\right)}}\sum_{h=0}^\infty \frac{(-k)_h(-1)^k}{h!}\sum_{k=0}^\infty \sum_{m=0}^{k(\lambda\beta-1)}(-1)^m\int_0^\infty e^{-z((m+k-1)-1)}\,z^{\left(1-\frac{1}{\beta}\right)(k-1)}dz$$

Let $z((m+k-2))=g\qquad\qquad \frac{g}{z(m+k-2)}=z\qquad\qquad \frac{dg}{z(m+k-2)}=dz$

When $\qquad\qquad z\to 0,\quad g\to 0\qquad$ When $\qquad\qquad z\to\infty,\quad g\to\infty$

$$= \frac{1}{1-k}\frac{\theta^{k-1}\lambda^k\,\beta^{2k-1}}{\theta^{(k-1)\left(1-\frac{1}{\beta}\right)}}\sum_{h=0}^\infty \frac{(-k)_h(-1)^k}{h!}\sum_{k=0}^\infty \sum_{m=0}^{k(\lambda\beta-1)}(-1)^m\frac{1}{(m+k-2)^{\left(1-\frac{1}{\beta}\right)(k-1)}}\int_0^\infty g^{\left(1-\frac{1}{\beta}\right)(k-1)}\,e^{-g}dg$$

$$R = \frac{1}{1-k}\frac{\theta^{k-1}\lambda^k\,\beta^{2k-1}}{\theta^{(k-1)\left(1-\frac{1}{\beta}\right)}}\sum_{h=0}^\infty \frac{(-k)_h(-1)^k}{h!}\sum_{k=0}^\infty \sum_{m=0}^{k(\lambda\beta-1)}(-1)^m\frac{\Gamma\left(1-\frac{1}{\beta}\right)(k-1)}{(m+k-2)^{\left(1-\frac{1}{\beta}\right)(k-1)}}$$

5.   Bonferoni and Lorenz Curves

     In 1905, Max O. Lorenz represented a model for inequality of wealth distribution and C.E. Bonferroni in 1930 proposed a measure of income inequality.

Both are used in financial mathematics to check equal distribution of wealth.

Bonferroni and Lorenz curves are defined as follows:

$$E(p) = \frac{1}{p\mu} \int_0^q x\, f(x)\, dx$$

and

$$L(p) = \frac{1}{\mu} \int_0^q x\, f(x)\, dx$$

respectively, where $p, q \in (0,1)$.

$$E(p) = \frac{1}{p\mu} \int_0^q x\, (1 - (e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta-1} (-1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta \lambda \beta^2 x^{\beta-1} dx$$

Let $(1 - e^{\theta x^\beta})^{-\lambda} = y$, $(1 - e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \lambda \beta \theta x^{\beta-1} dx = dy$,

$$x = \frac{1}{\theta^{\frac{1}{\beta}}} \left(1 - y^{-\frac{1}{\lambda}}\right)^{\frac{1}{\beta}}$$

When $x \to 0$, $y \to 0$ and $x \to q$, $y \to (1 - e^{\theta q^\beta})^{-\lambda}$

$$E(p) = \frac{1}{p\mu} \frac{\beta}{\theta^{\frac{1}{\beta}}} \int_0^{(1-e^{\theta q^\beta})^{-\lambda}} ln\,(1 - y^{-\frac{1}{\lambda}})^{\frac{1}{\beta}} (1 - y)^{-\beta} dy$$

$$E(p) = \frac{1}{p\mu} \frac{\beta}{\theta^{\frac{1}{\beta}}} \sum_{n=0}^\infty \frac{(a)_n}{n!} \int_0^{(1-e^{\theta q^\beta})^{-\lambda}} ln\,(1 - y^{-\frac{1}{\lambda}})^{\frac{1}{\beta}} y^n dy$$

Let $ln(1 - y^{-\frac{1}{\lambda}})^{\frac{1}{\beta}} = z$ $1 - y^{-\frac{1}{\lambda}} = e^z$

$y = (1 - e^z)^{-\lambda}$ $d\,y = -\lambda(1 - e^z)^{-\lambda-1} dz$

When $y \to 0$, $z \to 0$, $y \to (1 - e^{\theta q^\beta})^{-\lambda}$ $z \to \theta q^\beta$

$$E(p) = \frac{1}{p\mu} \frac{\beta}{\theta^{\frac{1}{\beta}}} \sum_{n=0}^\infty \frac{(a)_n}{n!} \int_0^{\theta q^\beta} z^{\frac{1}{\beta}} (1 - e^z)^{-\lambda n} \lambda(1 - e^z)^{-\lambda-1} e^z dz$$

$$E(p) = \frac{1}{p\mu} \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{n=0}^\infty \frac{(a)_n}{n!} \int_0^{\theta q^\beta} z^{\frac{1}{\beta}} (1 - e^z)^{-\lambda(n+1)-1} e^z dz$$

Expanding $(1 - e^z)^{-\lambda(n+1)-1}$ by using binomial expansion

$$E(p) = \frac{1}{p\mu} \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{n=0}^\infty \frac{(a)_n}{n!} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \binom{\lambda(n+1)-1}{n} \int_0^{\theta q^\beta} z^{\frac{1}{\beta}} e^{-nz}\, e^z dz$$

$$E(p) = \frac{1}{p\mu} \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{n=0}^\infty \frac{(a)_n}{n!} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \binom{\lambda(n+1)-1}{n} \int_0^{\theta q^\beta} z^{\frac{1}{\beta}} e^{-z(n-1)}\, dz$$

$$E(p) = \frac{1}{p\mu} \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \binom{\lambda(n+1)-1}{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (n-1)^k \int_0^{\theta q^\beta} z^{\frac{1}{\beta}+k} \, dz$$

$$E(p) = \frac{1}{p\mu} \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \binom{\lambda(n+1)-1}{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (n-1)^k \left(\frac{z^{\frac{1}{\beta}+k+1}}{\frac{1}{\beta}+k+1} \right) \Bigg|_0^{\theta q^\beta}$$

$$E(p) = \frac{1}{p\mu} \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \binom{\lambda(n+1)-1}{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (n-1)^k \frac{(\theta q^\beta)^{\frac{1}{\beta}+k+1}}{\frac{1}{\beta}+k+1}$$

$$E(p) = \frac{1}{p\mu} \left[\lambda\beta\theta^{k+1} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \binom{\lambda(n+1)-1}{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (n-1)^k \frac{q^{1+\frac{1}{\beta}+\frac{k}{\beta}}}{\frac{1}{\beta}+k+1} \right]$$

and

$$L(p) = \frac{1}{\mu} \left[\lambda\beta\theta^{k+1} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \binom{\lambda(n+1)-1}{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (n-1)^k \frac{q^{1+\frac{1}{\beta}+\frac{k}{\beta}}}{\frac{1}{\beta}+k+1} \right]$$

6. Characterization

In order to develop a stochastic function for a certain problem, it is necessary to know whether function fulfils the theory of specific underlying probability distribution, it is required to study characterizations of specific probability distribution. Different characterization techniques have developed. Glanzel (1987, 1988 and 1990), Hamedani . (1993, 2002, 2011 and 2015), Ahsanullah and Hamedani (2007, 2012), Ahsanullah et al. (2013), Shakil et al. (2014), and Merovci et al. (2016) have worked on characterization.

6.1 Characterization based on Hazard function

Definition: Let x be a continuous random variable with pdf $f(x)$ if and only if the hazard function $h(x)$ of a twice differentiable function F, satisfies equation

$$\frac{d}{dx}[lnf(x)] = \frac{h'(x)}{h(x)} - h(x)$$

For random variable X having LDW distribution with hazard rate function we obtain the following equation

$$h'(x)(1 - (1 + (e^{\theta x^\beta} - 1)^{-\lambda})^{-\beta-1} - h(x)(1 + (-1 + e^{\theta x^\beta})^{-\lambda})^{-\beta-1}(-1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta\lambda\beta^2 x^{\beta-1} =$$

$$\frac{d}{dx}[\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1} (-1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta\lambda\beta^2 x^{\beta-1}]$$

$$\frac{d}{dx}\left[h(x)(-1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda})^{-\beta-1}\right] = \frac{d}{dx}\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1} (-1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta\lambda\beta^2 x^{\beta-1}$$

$$h(x) = \frac{\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1} (-1 + e^{\theta x^\beta})^{-\lambda-1} e^{\theta x^\beta} \theta\lambda\beta^2 x^{\beta-1}}{(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda})^{-\beta}}$$

After manipulation, integrating and simplifying, we obtain as

$$F(x) = \left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta}$$

This is the cdf of LDW distribution.

6.2 Characterization based on reversed hazard Function:

$$\frac{d}{dx}[lnf(x)] = \frac{\varphi'(x)}{\varphi(x)} - \varphi(x)$$

For random variable X having LDW distribution with hazard rate function we obtain the following equation

$$\varphi'(x)\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1} + \varphi(x)\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1}(-1 + e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1} = \frac{d}{dx}\Big[\Big(1 +$$

$$\left(e^{\theta x^\beta} - 1\right)^{-\lambda}\Big)^{-\beta-1}(-1 + e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}\Big]$$

$$\frac{d}{dx}\left[\varphi(x)\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1}\right] = \frac{d}{dx}\Big[\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1}(-1 + e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}$$

$$\varphi(x) = \frac{\left[\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1}\left(-1 + e^{\theta x^\beta}\right)^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}\right]}{\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1}}$$

After manipulation, integrating and simplifying, we obtain as

$$F(x) = \left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1}$$

6.3 Characterization through Ratio of Truncated Moments

In this section, we characterize WD distribution using Theorem 1 (Glanzel; 1987) on the basis of simple relationship between two functions of X. Theorem 1 is given in appendix A.

Preposition

Suppose that random variable $X{:}\Omega \to (0, \infty)$ is continuous. Let

$$h_1(x) = \frac{1}{\lambda\beta}\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{\beta+1}\left(e^{\theta x^\beta} - 1\right)^{\lambda+1} \text{and} \qquad h_2(x) = \frac{2e^{\theta x^\beta}}{\left(1 + \left(e^{\theta x^\beta} - 1\right)^{-\lambda}\right)^{-\beta-1}\left(e^{\theta x^\beta} - 1\right)^{-\lambda-1}}$$

$,x > 0$

The pdf of X is (9) if and only if $p(x)$ has the form $p(x) = e^{-\theta x^\beta}, \ x > 0$

Proof:

For random variable X having LDW distribution with pdf (9) and cdf (8), we proceeds as

$$\left(1 - f(X)\right)E(h_1(x)/X \geq x) = e^{-\theta x^\beta}$$

$$\left(1 - f(X)\right)E(h_2(x)/X \geq x) = e^{-2\theta x^\beta}$$

$$p(x) = e^{-\theta x^\beta} \quad \text{and} \quad p'(x) = e^{-\theta x^\beta}\theta\beta x^{\beta-1}$$

$$s'(t) = \frac{p'(t)h_2(t)}{p(t)h_2(t)-h_1(t)} = -2\theta\beta x^{\beta-1} \quad \text{and} \quad s(t) = -2\theta x^\beta$$

Therefore in the light of Theorem 1, X has pdf (9)

7. Maximum Likelihood Estimation

Several approaches for parameter estimation have been proposed in the literature but the maximum likelihood method is the most commonly employed.

Let $X_1, X_2, \ldots, X_m$ be a random sample of size n of the LDW distribution then the total log-likelihood (LL) function is given

$$L(\lambda,\theta,\beta) = \beta^{2n}\prod_{i=1}^{n}\left(1 + \left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}\lambda^{n}$$

$$\prod_{i=1}^{n}(-1+e^{\theta x^\beta})^{-\lambda-1}\theta^n\prod_{i=1}^{n}e^{\theta x^\beta}\prod_{i=1}^{n}x^{\beta-1} \quad ,$$

$$L(\lambda,\theta,\beta) = 2n\text{Log}[\beta] + n\text{Log}[\theta] + n\text{Log}[\lambda] + \theta\sum_{I=1}^{n}x_I^\beta + (\beta-1)\sum_{I=1}^{n}\text{Log}[x_I]$$

$$-(\lambda+1)\sum_{I=1}^{n}\text{Log}\left[e^{x_I^\beta\theta}-1\right] + \lambda(\beta+1)\sum_{I=1}^{n}\text{Log}(1+\left(e^{x_I^\beta\theta}-1\right)),$$

The First derivatives of the log-likelihood function are given as follow

$$\frac{\delta L(\lambda,\theta,\beta)}{\delta\beta} = \frac{2n}{\beta} + \theta\beta\sum_{I=1}^{n}x_I^{\beta-1} + \sum_{I=1}^{n}\text{Log}[x_I] - (\lambda+1)\sum_{I=1}^{n}\theta\beta x_I^{\beta-1} + 2\lambda\sum_{I=1}^{n}\beta x_I^{\beta-1}\theta,$$

$$\frac{\delta L(\lambda,\theta,\beta)}{\delta\lambda} = \frac{n}{\lambda} - \sum_{I=1}^{n}\text{Log}[e^{x_I^\beta\theta}-1] + \lambda(\beta+1)\sum_{I=1}^{n}\text{Log}(1+\left(e^{x_I^\beta\theta}-1\right)),$$

$$\frac{\delta L(\lambda,\theta,\beta)}{\delta\theta} = \frac{n}{\theta} + \sum_{I=1}^{n}x_I^\beta - (\lambda+1)\sum_{I=1}^{n}x_I^\beta + \lambda(\beta+1)\sum_{I=1}^{n}x_I^\beta,$$

Equating equations to zero and solving them numerically, one can obtain the estimates of the unknown parameters.

8 Simulation Study

This section deals the simulation study. In proposed model we generated random variables by using CDF of LDWD with four different value of parameters for n=25,50,100,200. Parameters are estimated with method of MLE by using each generated random variable. In statistical study, bias states to the tendency of a measurement process to over or under estimate the value of population parameters. Bias of MLEs can be fundamental. Squared error is a function which obtained from square values of bias. MSE is always constructive. Bias shows the contrasts between estimated values of parameter variation from true value of parameter.

By using the estimated parameters, we calculated Bias and MSE of LDWD and also calculate their average mean which are shown in different graphs. The resulting Behaviour of these Bias and MSE of these estimated parameters are also shown below with the help of graphs. All simulations were done on computational software `Mathematica 8.0.

The analysis computes the coming values:

- Average bias of the simulated estimates:

$$\frac{1}{1000}\sum_{i=1}^{1000}(\Theta^* - \Theta)$$

- Average mean square error (MSE) of the simulated estimates:

$$\frac{1}{1000}\sum_{i=1}^{1000}(\Theta^* - \Theta)^2$$

The results are reported in Tables 1 and 2 .

[Table 2 about here.]
[Table 3 about here.]

9.   Evaluation Measures and Practical Data Examples

We illustrate the usefulness of the Log Dagum Weibull distribution and compare the results with the WD GD LD EED and NEED distributions by means of four real data sets. , one of which is data of leukaemia-free survival times of 50 patients with Autologous transplant obtained from [11] and the second data set contains Lifetime of 50 devices [13], Third data set

consists of 100 uncensored data on breaking stress of carbon fibres (in Gba) [32], fourth data consist times to failure of eighteen electronic devices [33]

In this section we illustrate the usefulness of the Log Dagum Weibull distribution .We estimate the unknown parameters of LDWD using MLE method and compare the log likelihood with some other distributions including EED, WD,GD,NEED and LD. We will check goodness of fit of our model with some test statistics like AD test, CVM test, K-S test and p-value. All calculations are executed on computational software MATHEMATICA 11.0

Numerical measures

In order to demonstrate the proposed methodology, we consider four different practical data sets described below with their analysis. They represent different level of skewness ranging from negative skewness to positive skewness along with various demonstration of failure rate pattern, like increasing, decreasing, and bathtub shape. Moreover, perfection of competing models is also tested via the Kolmogrov-Simnorov (K S),the Anderson Darling (A^*) and the Cramer-von Misses (W^*) statistics. The mathematical expressions for the statistics are given by

$$KS = \max\left\{\frac{i}{m} - z_i, z_i - \frac{i-1}{m}\right\},$$

$$A^* = \left(\frac{2.25}{m^2} - \frac{0.75}{m} + 1\right)\left\{-1 - \frac{1}{m}\sum_{1}^{m}(2i-1)\ln(z_i(z_{m-i+1}))\right\},$$

and

$$w^* = \sum_{1}^{m}\left(z_i - \frac{2i-1}{2m}\right)^2 + \frac{1}{12m}.$$

where m denotes the number of classes, $z_i = FX(x_i)$, the $x_i's$ being the ordered observations respectively.

Data set 1. The first data set obtained from [12] of leukemia-free survival times of 50 patients with Autologous transplant.Data sets are presented in the following tables : 0.030 , 0.493, 0.855, 1.184, 1.283, 1.480, 1.776, 2.138, 2.500 ,2.763, 2.993 ,3.224 ,3.421, 4.178, 4.441, 5.691, 5.855, 6.941, 6.941, 7.993,   8.882 ,8.882, 9.145, 11.480, 11.513, 12.105 ,12.796 ,12.993 ,13.849, 16.612,17.138, 20.066 ,20.329 ,22.368 ,26.776 ,28.717, 28.717 ,32.928 ,33.783, 34.211,34.770, 39.539 ,41.118, 45.033, 46.053, 46.941, 48.289,57.401, 58.322 ,60.625.

The measures of goodness of fit including the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (A^*), Cram´er–von Mises (W^*) and Kolmogrov-Smirnov (K-S) statistics are computed to compare the fitted models. The statistics A^* and W^* are described in details in [8]. In general, the smaller values of these statistics, better fit to the data. The required computations are carried out in the Mathematica 11.0.

[Table 4 about here]

[Table 5 about here]

Analysis:

Table 4 and 5 represents the results of test statistics and information criterion respectively for data set 1. Minimum test statistics and information criterion shows the goodness of fit of our designed model.

Data set 2: Second data set Lifetime of 50 devices are obtained from [14].

0.1, 0.2, 1, 1, 1 ,1 ,1, 2, 3, 6, 7, 11 ,12 ,18, 18 ,18 ,18, 18, 21 ,32, 36 ,40, 45 ,46, 47, 50, 55 ,60, 63, 63, 67, 67 ,67 ,67, 72 ,75, 79 ,82, 82, 83, 84, 84 ,84, 85, 85, 85 ,85, 85, 86 ,86.

We fit the LDW model and other competitive models such as the Exponentiated exponential distribution (EED) [10], Weibull distribution (WD) [11],   Gamma distribution (GD) [10],NEED Nadarajah Exponentiated exponential distribution and Lomax distribution (LD) to data sets.

[Table 6 about here.]

[Table 7 about here]

Analysis:

Tables 6 represents the results of test statistics for data set 2.Minimum test statistics shows the goodness of fit of our designed model. It can be easily seen that our model has better fitness than other five models

Data set 3: This data set consists of 100 uncensored data on breaking stress of carbon fibres (in Gba), [32].

.0.39,0.81,0.85,0.98,1.08,1.12,1.17,1.18,1.22,1.25,1.36,1.41,1.47,1.57,1.57,1.59,1.59,1.61,1.61,1.69,1.69,1.71,1.73,1.8,1.84,1.84,1.87,1.89,1.92,2,2.03,2.03,2.05,2.12,2.17,2.17,2.17,2.35,2.38,2.41,2.43,2.48,2.48,2.5,2.53,2.55,2.55,2.56,2.59,2.67,2.73,2.74,2.76,2.77,2.79,2.81,2.81,2.82,2.83,2.85,2.87,2.88,2.93,2.95,2.96,2.97,2.97,3.09,3.11,3.11,3.15,3.15,3.19,3.19,3.22,3.22,

3.27,3.28,3.31,3.31,3.33,3.39,3.39,3.51,3.56,3.6,3.65,3.68,3.68,3.68,3.7,3.75,4.2,4.38,4.42,4.7,4.9,4.91,5.08,5.56.

[Table 8 about here.]

[Table 9 about here.]

Analysis:

Table 8 and 9 represents the results of test statistics and information criterion respectively for data set 3. Minimum test statistics and information criterion shows the goodness of fit of our designed model

Data set3: This data consist times to failure of eighteen electronic devices [33] used to show how the proposed distribution can be applied in practice.

5,11,21,31,46,75,98,122,145,165,196,224,245,293,321,330,350,420

[Table 10 about here.]

[Table 11 about here.]

Concluding Remarks:

There has been an increased interest in defining new generated classes of univariate continuous distributions. The extended distributions have attracted several statisticians to develop new models. In this paper we propose the new *log Dagum-X* family of distributions. We study some of its mathematical properties. The maximum likelihood method is employed for estimating the model parameters. One special model, the distribution Weibull is considered and its properties are studied. It is fitted to two real data sets. The proposed special model consistently better fit than other competing models. We hope that the new family and its generated models will attract wider application in several areas such as engineering, survival and lifetime data, hydrology, economy.

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Appendix:

Theorem 1:

Suppose that probability space (Ω, F, P) and interval $[g_{1,}g_{2,}]$ with $g_1 < g_2 (g_{1,} = -\infty, g_{2,} = \infty)$are given. Let continuous random variable $X: \Omega \to [g_1, g_2]$ has distribution function F. Let real functions $g_1 \, and \, g_2$ be continuous on $[g_1, g_2]$ such that $\frac{E(g_1(x)/X \geq x)}{E(g_2(x)/X \geq x)} = p(x)$ is real function in simple form. Assume that $g_1, g_2 \in c([g_1, g]), p(x) \in c^2([g_1, g_2])$ and F is two times continuously differentiable and strictly monotone function on $[g_1, g_2]$: As a final point, assume that the equation $g_2 p(x) = g_1$has no real solution in $[g_1, g_2]$.

then F(x)=$\int_{lnk}^x k \frac{p'(t)}{p(t)g_2(t) - g_1(t)} \exp(-st) \, dt$ is obtained from the function $g_1, g_2, p(t)$ and

$s(t)$.where S(t) is obtained from equation $s'(t) = \frac{p'(t)g_2(t)}{p(t)g_2(t) - g_1(t)}$ and k is constant, picked to make $\int_{g_1}^{g_2} df = 1$
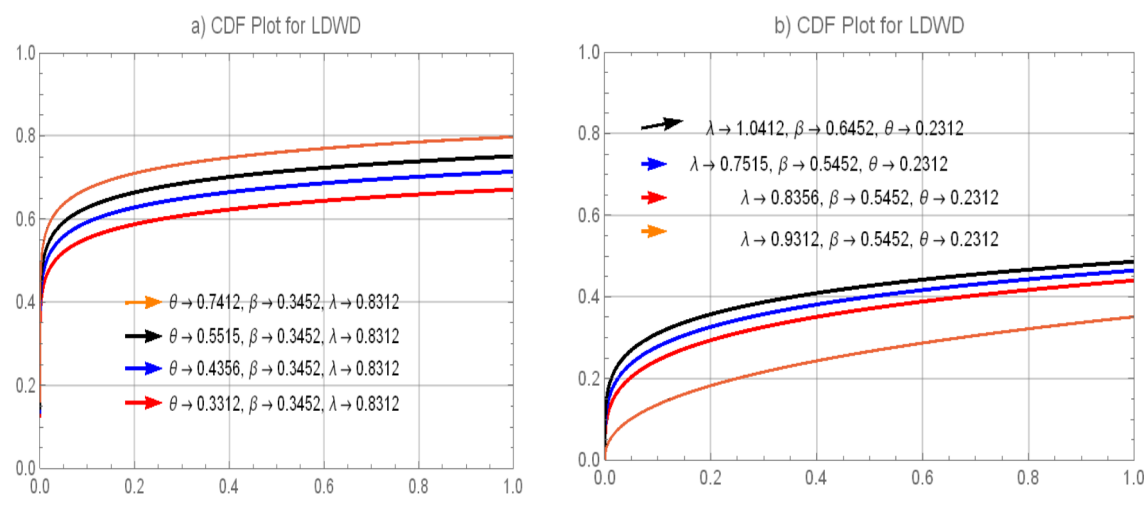
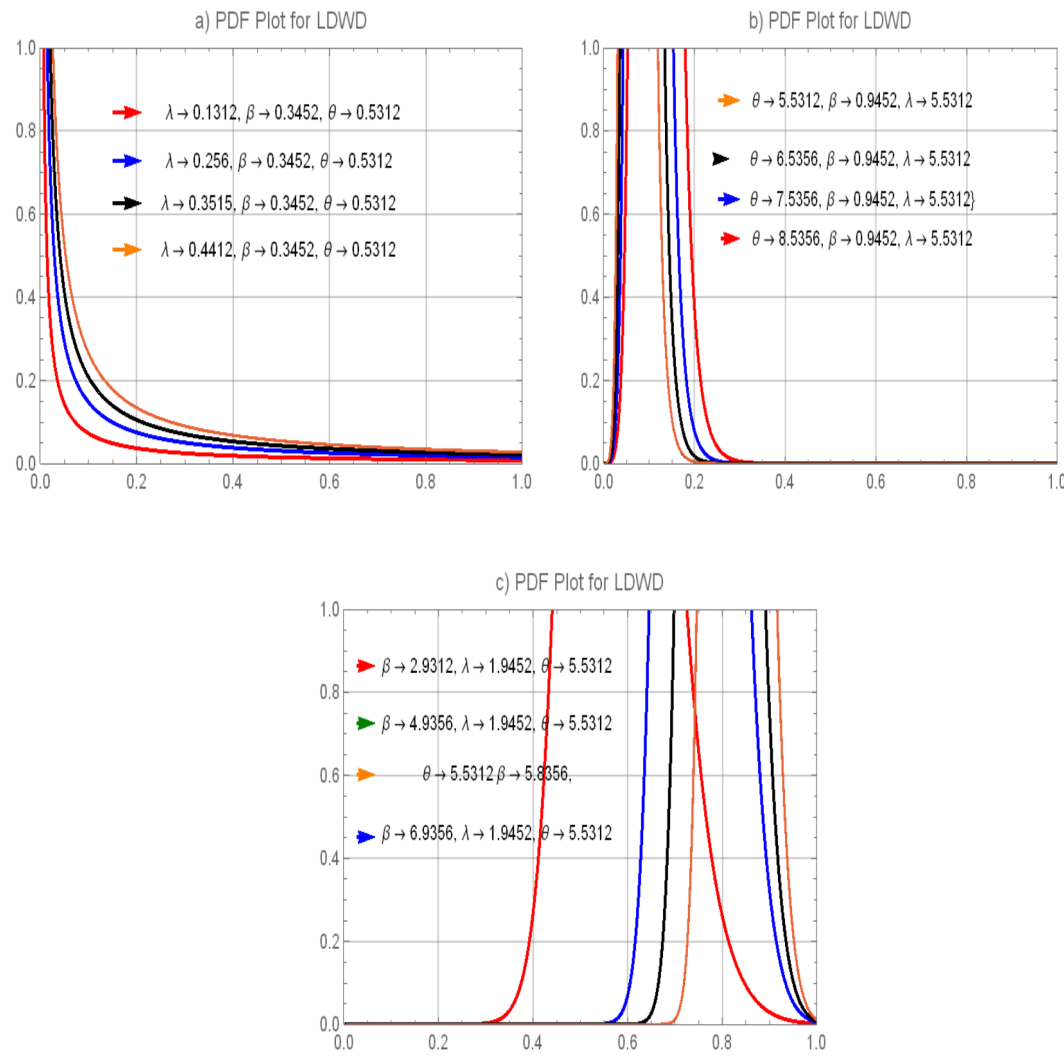Figure 1: Cumulative Distribution plot of LDWD
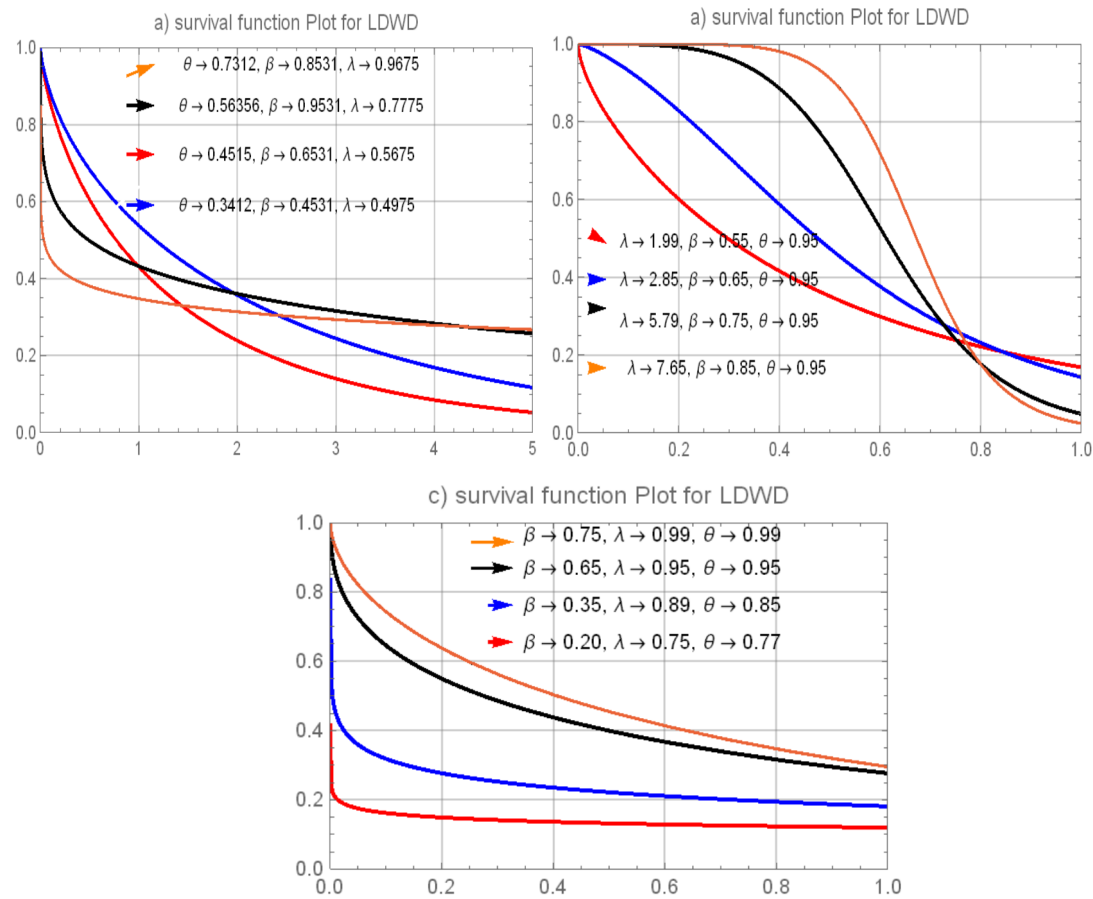

Figure 2: Density plots of LDWD

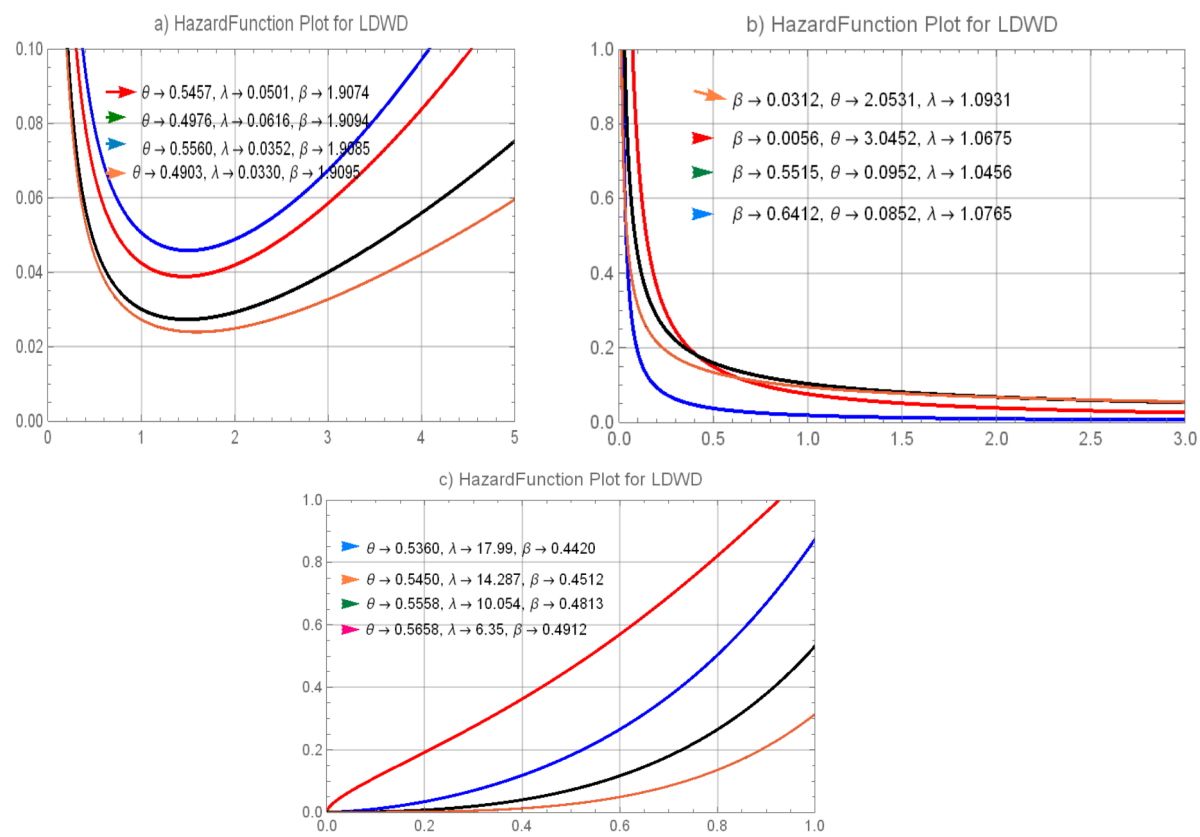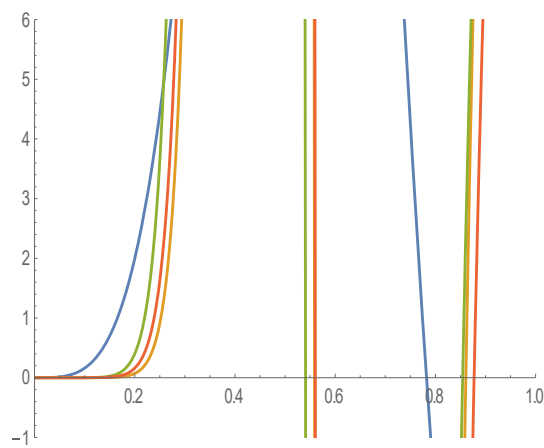Figure 3: Survival plot of LDWD

Figure 4:Hazard plot of LDWD

Figure 5: Shape of Hazard function

Table 2: Average mean of Bias and MSE values for estimators $\widehat{\theta}$, $\hat{\beta}$ and $\hat{\lambda}$ 0f data 1

n	Bias($\hat{\theta}$)	Bias($\hat{\beta}$)	Bias($\hat{\lambda}$)	MSE($\hat{\theta}$)	MSE($\hat{\beta}$)	MSE($\hat{\lambda}$)
25	0.65234	−0.08837	0.56572	0.56572	0.40559	0.86286
50	0.35310	0.23613	0.17950	0.29754	0.52918	0.15562
100	0.27740	0.46278	−0.01169	0.11882	0.72056	0.04713
250`	0.44889	−0.03051	0.07439	0.24297	0.20479	0.03035

Table 3: Average mean of Bias and MSE values for estimators $\hat{\theta}$, $\hat{\beta}$ and $\hat{\lambda}$ of data 2

n	Bias ($\hat{\theta}$)	Bias($\hat{\beta}$)	Bias($\hat{\lambda}$)	MSE($\hat{\theta}$)	MSE($\hat{\beta}$)	MSE($\hat{\lambda}$)
25	1.00745	−0.44500	−0.34131	1.26654	0.25397	0.21862
50	0.57483	−0.23206	−0.39724	0.39676	0.09400	0.19138
100	0.43445	0.41399	−0.68022	0.28879	0.63077	0.50529

Table 4: AD, CVM, The K-S statistics and *p*-values for the data set 1

Distributions	A^*	W^*	K-S	p-value
LDWD	0.403996	0.0651719	0.076948	0.943568
EED	0.362828	0.0483839	0.084435	0.868171
WD	0.411538	0.0562415	0.0868536	0.845013
GD	0.369975	0.0496265	0.0847622	0.86513
LD	2.504843	0.3799524	0.19666206	0.04182
NEED	0.666096	0.0962511	0.0906376	0.805953

Table 5: Information Criteria of Different Distributions for Data 1

Distributions	AIC	AICC	BIC	HQIC	CAIC
LDWD	394.682	395.140	398.235	396.37	395.140
EED	394.954	395.209	398.778	396.41	395.209
WD	395.433	395.689	399.257	396.89	395.689
GD	395.057	395.312	398.881	396.51	395.312
LD	394.783	395.304	400.5187	396.97	395.304
NEED	396.045	396.301	399.869	397.50	396.301

Table:6 The K-S statistics and *p*-values for the data sets 2

Distributions	A^*	W^*	K-S	p-value
LDWD	0.41395	0.06328	0.07134	0.9135
EED	0.36282	0.04838	0.08444	0.8682
WD	0.41153	0.05624	0.08685	0.8450
GD	0.36997	0.04962	0.08476	0.8651
LD	8.09533	1.66869	0.3377	0.00002
NEED	8.11488	1.67229	0.322722	0.00006

Table 7: Information Criteria of Different Distributions for Data 2

Model	AIC	AICC	BIC	CAIC
LDWD	455.064	455.586	460.800	455.586
EED	483.99	484.246	487.814	484.246
WD	486.004	486.259	489.828	486.259
GD	484.38	484.636	488.204	484.636
NEED	516.033	519.857	516.289	516.857
LD	474.0873	474.3427	477.9114	474.3427

Table 8: Information Criteria of Different Distributions for Data 3

Model	AIC	BIC	AICC	HQIC	CAIC
LDWD	288.62	296.43	288.8685	296.4883	296.4883
GD	290.4673	295.6775	290.5909	292.576	290.5909
WD	289.06	296.87	289.3086	292.2217	289.3086
EED	296.36460	301.574	296.4883	298.4733	296.4883
NEED	393.8472	399.0575	393.9709	395.9559	393.9709
LD	474.0873	477.9114	474.3427	475.54356	474.3427

Table 9: The K-S statistics and p-values for the data sets 3

Distributions	A^*	W^*	K-S	p-value
LDWD	0.39666	0.06508	0.0618	0.8395
EED	1.2341	0.2303	0.1077	0.19618
WD	18.9521	3.7772	0.3341	4.02837×10^{-10}
GD	200.5016	32.9885	0.9996	2.22044×10^{-16}
LD	79.3018	17.3623	0.8210	-2.22044×10^{-16}
NEED	16.9307	3.35163	0.3170	3.73137×10^{-9}

Table 10: The K-S statistics and p-values for the data sets 4

Distributions	A^*	W^*	K-S	p-value
LDWD	0.1725	0.02361	0.0840	0.9996
EED	0.4456	0.07077	0.12138	0.9535
WD	0.4609	0.0644	0.1132	0.9752
GD	0.4487	0.06986	0.1206	0.956104
LD	28.2328	5.0981	0.9157	1.5487×10^{-13}
NEED	2.46950	0.4826	0.28141	0.115548

Table 11: Information Criteria of Different Distributions for Data 4

Model	AIC	BIC	HQIC	AICC	CAIC
LDWD	208.2915	210.9626	208.6598	210.0058	210.0057
GD	226.1	227.9	226.9	229.9	229.9
WD	395.433	397.214	395.6789	396.233	396.233
EED	225.2528	227.0335	225.4983	226.05277	226.0527
NEED	237.8595	239.6403	238.10512	238.6596	238.65956
LD	341.4154	343.1962	341.6609	342.2154	342.2154