Peer-reviewed version available at *Mathematics* **2020**, *8*, 255; doi:10.3390/math8020255

Existence results for nonlinear fractional problems with non homogeneous integral boundary conditions

Alberto Cabada [1] and Om Kalthoum Wanassi [2]

[1]Departamento de Estatística, Análise Matemática e Optimización
Instituto de Matemáticas, Facultade de Matemáticas,
Universidade de Santiago de Compostela, Spain.
alberto.cabada@usc.es

[2]Department of Mathematics
University of Monastir, Tunisia.
kalthoum.wannassi@gmail.com

Abstract

This paper deals with the study of the existence and non existence of solutions of a three parameter's family of nonlinear fractional differential equation with mixed-integral boundary value conditions. We consider the α-Riemann-Liouville fractional derivative, with $\alpha \in (1,2]$. In order to deduce the existence and non existence results, we first study the linear equation, by deducing the main properties of the related Green's functions. We obtain the optimal set of parameters where the Green's function has constant sign.
After that, by means of the index theory, the nonlinear boundary value problem is studied. Some examples, at the end of the paper, are showed to illustrate the applicability of the obtained results.

AMS Subject Classifications: 34A08. 34B27. 37C25.
Key Words: Fractional Equations, Green's Functions, Integral Boundary Conditions, Fixed Point Index, Existence and Non-existence.

1 Introduction

During the last decade, Fractional Calculus has been applied to almost every field of science, engineering, and mathematics. Some of the areas where Fractional Calculus has made a profound impact include viscoelasticity and rheology, electrical, engineering, electrochemistry, biology, physics, and control theory. For more details on this theory and its applications, [4, 13, 20, 23, 25].

Integral boundary conditions have various applications in applied fields such as chemical engineering, thermoelasticity, population dynamics. For a detailed description of the integral boundary conditions, we refer the reader to [3].
The existence of solutions of nonlinear boundary value problem coupled with integral boundary conditions in ordinary and fractional cases has been widely

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 January 2020 doi:10.20944/preprints202001.0101.v1

Peer-reviewed version available at Mathematics 2020, 8, 255; doi:10.3390/math8020255

studied by many authors, see for example [6, 12, 16, 24, 27, 28] and the references therein.

In 2009, Ahmad and Nieto [2] obtained some existence results for the following nonlinear fractional integrodifferential equations with integral boundary conditions:

$$\begin{cases} {}^C D^q x(t) = f(t, x(t), (\chi x)(t)), \quad 0 < t < 1, \ 1 < q \leq 2, \\ \alpha x(0) - \beta x'(0) = \int_0^1 q_1(x(s))ds, \\ \alpha x(1) - \beta x'(1) = \int_0^1 q_2(x(s))ds, \end{cases}$$

where $f : [0,1] \times X \times X \longrightarrow X$, for $\gamma : [0,1] \times [0,1] \longrightarrow [0,\infty)$,

$$(\chi x)(t) = \int_0^t \gamma(t,s)x(s)ds,$$

$q_1, q_2 : X \longrightarrow X$ $\alpha \geq 0$, $\beta \geq 0$ are real numbers and X is a Banach space, by employing to Guo-Krasnoselskii fixed point theorem and contraction mapping principle.

In [5], it is studied the following nonlinear problem involving nonlinear integral conditions:

$$\begin{cases} {}^C D^\alpha y(t) = f(t, y(t)), \qquad t \in [0,T], \ 1 < \alpha \leq 2, \\ y(0) - y'(0) = \int_0^T g(s, y(s)ds, \\ y(T) - y'(T) = \int_0^T h(s, y(s))ds. \end{cases}$$

Here, f, g and $h : [0,T] \times E \longrightarrow E$ are given functions satisfying adequate assumptions and E is a Banach space. By means of the technique associated with measures of non compactness and the fixed point theorem of Monch type, it is proved the existence of solutions of the problem.

In [10], it is considered the following nonlinear fractional differential equations with boundary value conditions

$$\begin{cases} D^\alpha u(t) + g(t)f(t, u(t)) = 0, \qquad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \int_0^1 h(t)u(t)dt, \end{cases}$$

where $1 < \alpha \leq 2$, $g \in C((0,1), [0,\infty))$ and g may be singular at $t = 0$ or/and at $t = 1$, $h \in L^1[0,1]$ and $f \in C[0,1] \times [0,1], [0,\infty))$. The authors derive the Green's function associated to the above problem and sharp estimates on it are established. Thus, by using fixed point theorem in cones, they proved some results on the existence of positive solutions.

In this paper, we will consider the following nonlinear fractional differential equation with non homogeneous integral boundary conditions:

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + f(t, t^{2-\alpha}u(t)) = 0, \qquad t \in I := [0,1], \\ \lim\limits_{t \to 0^+} t^{2-\alpha}u(t) = \mu \int_0^1 u(s)ds, \quad u'(1) = \eta \int_0^1 u(s)ds. \end{cases} \tag{1}$$

Here $\lambda \in \mathbb{R}$, μ, $\eta \geq 0$, D^α, $1 < \alpha \leq 2$, is the Riemann-Liouville fractional derivative and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

We look for solutions $u : I \to \mathbb{R}$ such that function $t^{2-\alpha}\, u(t) \in C^1(I)$. Notice that, as a direct consequence, we deduce that, in particular, $u \in C^1((0,1])$.

 doi:10.20944/preprints202001.0101.v1

Peer-reviewed version available at *Mathematics* **2020**, *8*, 255; doi:10.3390/math8020255

Moreover, it may be discontinuous at $t = 0$.
We are interested in to prove the existence and non existence of solutions of the treated problem. To this end, we will use the classical index theory [1, 11, 29], in the line of the papers [9, 14, 15, 26], where it is used for Ordinary Differential Equations.
The main tool used consists on the construction of the Green's function related to the linear problem

$$\begin{cases} D^\alpha u(t) - \lambda u(t) + y(t) = 0, \quad t \in I, \\ \lim\limits_{t\to 0^+} t^{2-\alpha}u(t) = \mu \int_0^1 u(s)ds, \quad u'(1) = \eta \int_0^1 u(s)ds. \end{cases} \tag{2}$$

Such construction continues the author's work [7], where the homogeneous Mixed boundary conditions $\mu = \eta = 0$ are considered. So, we use the qualitative properties obtained on those reference and study the parameter relationship between α, λ, μ and η that ensure the constant sign of the Green's function related to the linear problem 2. We follow similar arguments to the ones used on [8, 16, 18, 19, 21].

The paper is scheduled as follows: after some introductory results, we study, in Section 3, the related linear equation and deduce suitable properties on the qualitative behavior and constant sign of the related Green's function. Next section is devoted to ensure the existence and nonexistence of solutions of the considered nonlinear boundary value problem. The results follow from index theory. Finally, in last section, some examples are given to point out the applicability of the obtained results.

2 Preliminary Results

In this section, we introduce some notations and definitions that which we need in later.

Definition 1 *([17]) The Riemann-Liouville fractional integral of order $\alpha > 0$ for a measurable function $f : (0, +\infty) \to \mathbb{R}$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \ t > 0,$$

where Γ is the Euler Gamma function, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2 *([17]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a measurable function $f : (0, +\infty) \to \mathbb{R}$ is defined as*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds = (\frac{d}{dt})^n I^{n-\alpha} f(t),$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$. Here $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of the real number α.

Let $C(I)$ be the Banach space of all continuous functions defined on I endowed with the norm $\|f\| =: \max\{|f(t)| : t \in I\}$.
Define for $t \in I$, $f_\gamma(t) = t^\gamma f(t)$. Let $C_\gamma(I)$, $\gamma \geq 0$ be the space of all functions f such that $f_\gamma \in C(I)$. It is well known that $C_\gamma(I)$ is a Banach space endowed with the norm

$$\|f\|_\gamma =: \max\{t^\gamma |f(t)| : t \in I\}.$$

Peer-reviewed version available at Mathematics 2020, 8, 255; doi:10.3390/math8020255

3 Linear Problem

This section is devoted to the study of the linear problem (2). It is not difficult to verify that, provided $E_{\alpha,\alpha-1}(\lambda) \neq 0$,

$$v_1(t) = \Gamma(\alpha-1)\left(t^{\alpha-2}E_{\alpha,\alpha-1}(\lambda t^\alpha) - \frac{E_{\alpha,\alpha-2}(\lambda)}{E_{\alpha,\alpha-1}(\lambda)}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\right) \tag{3}$$

is the unique solution of the problem

$$\begin{cases} D^\alpha v_1(t) - \lambda v_1(t) = 0, \quad t \in I, \\ \lim\limits_{t\to 0^+} t^{2-\alpha}v_1(t) = 1, \quad v_1'(1) = 0, \end{cases} \tag{4}$$

and

$$v_2(t) = \frac{t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{E_{\alpha,\alpha-1}(\lambda)}. \tag{5}$$

the unique one of

$$\begin{cases} D^\alpha v_2(t) - \lambda v_2(t) = 0, \quad t \in I, \\ \lim\limits_{t\to 0^+} t^{2-\alpha}v_2(t) = 0, \quad v_2'(1) = 1. \end{cases} \tag{6}$$

Moreover, as it is showed in [7, Theorem 6], if $E_{\alpha,\alpha-1}(\lambda) \neq 0$, the unique solution of problem

$$\begin{cases} D^\alpha v(t) - \lambda v(t) + y(t) = 0, \quad t \in I, \\ \lim\limits_{t\to 0^+} t^{2-\alpha}v(t) = v'(1) = 0, \end{cases}$$

follows the expression

$$v(t) = \int_0^1 G_1(t,s)y(s)ds,$$

with

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)E_{\alpha,\alpha-1}(\lambda(1-s)^\alpha)}{(1-s)^{2-\alpha}E_{\alpha,\alpha-1}(\lambda)} - (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha), & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)E_{\alpha,\alpha-1}(\lambda(1-s)^\alpha)}{(1-s)^{2-\alpha}E_{\alpha,\alpha-1}(\lambda)}, & 0 \le t < s < 1. \end{cases} \tag{7}$$

In order to characterize the uniqueness of solutions of Problem (2), we denote

$$\theta = \int_0^1 v_1(t)dt \quad \text{and} \quad \sigma = \int_0^1 v_2(t)dt.$$

Theorem 3 *Let $y \in C(0,1] \cap L^\infty(0,1)$, $1 < \alpha \le 2$, $\mu, \eta \ge 0$ and $\lambda \in \mathbb{R}$ be such that $E_{\alpha,\alpha-1}(\lambda) \neq 0$ and $1 - \mu\theta - \eta\sigma \neq 0$. Then problem (2) has a unique solution $u \in C^1_{2-\alpha}(I)$, given by*

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = G_1(t,s) + \frac{(\mu v_1(t) + \eta v_2(t))}{(1-\mu\theta-\eta\sigma)}\left(\int_0^1 G_1(r,s)dr\right), \tag{8}$$

with G_1, v_1, v_2 given in (3), (5) and (8) respectively.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 January 2020 doi:10.20944/preprints202001.0101.v1

Peer-reviewed version available at *Mathematics* **2020**, *8*, 255; doi:10.3390/math8020255

Table 1: Some values of θ and σ for $1 < \alpha \leq 2$.

α	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
λ	-5	-4.5	-4	-3.5	-3	-2.5	-2	-1.5	-1	-0.5
θ	12.647	5.62054	2.83501	1.47577	0.657725	-0.0336574	-1.21724	54.6033	2.13745	1.20846
σ	-4.52049	-1.94567	-0.99204	-0.688973	-0.638201	-0.777128	-1.40827	38.7227	1.22996	0.630732

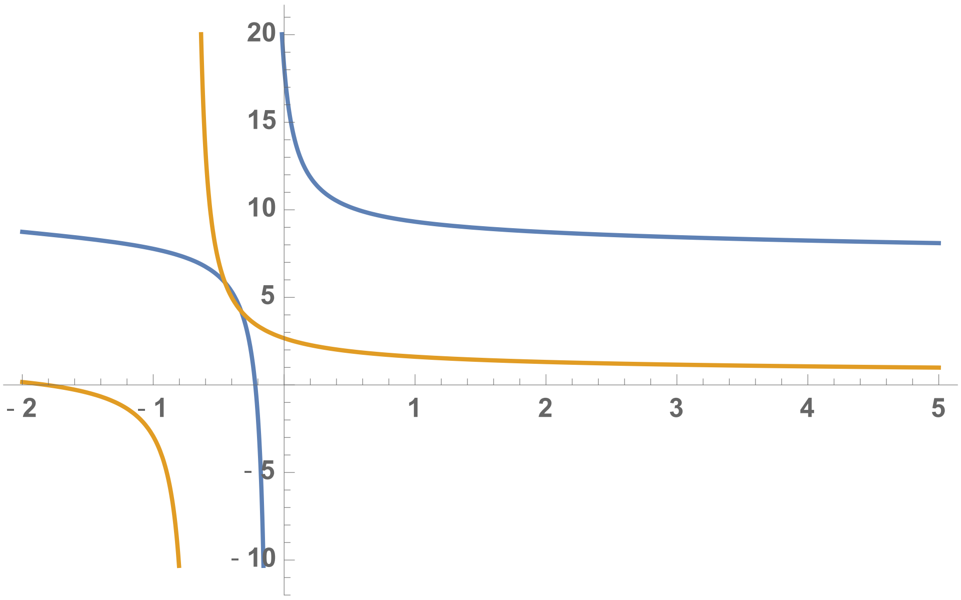

Figure 1: Graph of θ for $\alpha = 1.1$ (blue) and $\alpha = 1.5$ (orange)

Proof. Arguing in a similar way as in [7, Theorem 6], we deduce that

$$u(t) = \int_0^1 G_1(t,s)y(s)ds + \mu\left(\int_0^1 u(s)ds\right)v_1(t) + \eta\left(\int_0^1 u(s)ds\right)v_2(t). \quad (9)$$

Let us denote $\int_0^1 u(s)ds = A$. Then, from the previous equality, we deduce that

$$A = \int_0^1 u(t)dt = \int_0^1 \left(\int_0^1 G_1(t,s)y(s)ds\right)dt + A(\mu\theta + \eta\sigma),$$

that is, since $1 - \mu\theta - \eta\sigma \neq 0$,

$$A = \frac{\int_0^1\left(\int_0^1 G_1(t,s)y(s)ds\right)dt}{(1-\mu\theta-\eta\sigma)}.$$

Replacing A in (9), we obtain the following expression of the function u

$$u(t) = \int_0^1 G_1(t,s)y(s)ds + \frac{\int_0^1\left(\int_0^1 G_1(t,s)y(s)ds\right)dt}{(1-\mu\theta-\eta\sigma)}(\mu v_1(t) + \eta v_2(t)). \quad (10)$$

Peer-reviewed version available at Mathematics 2020, 8, 255; doi:10.3390/math8020255

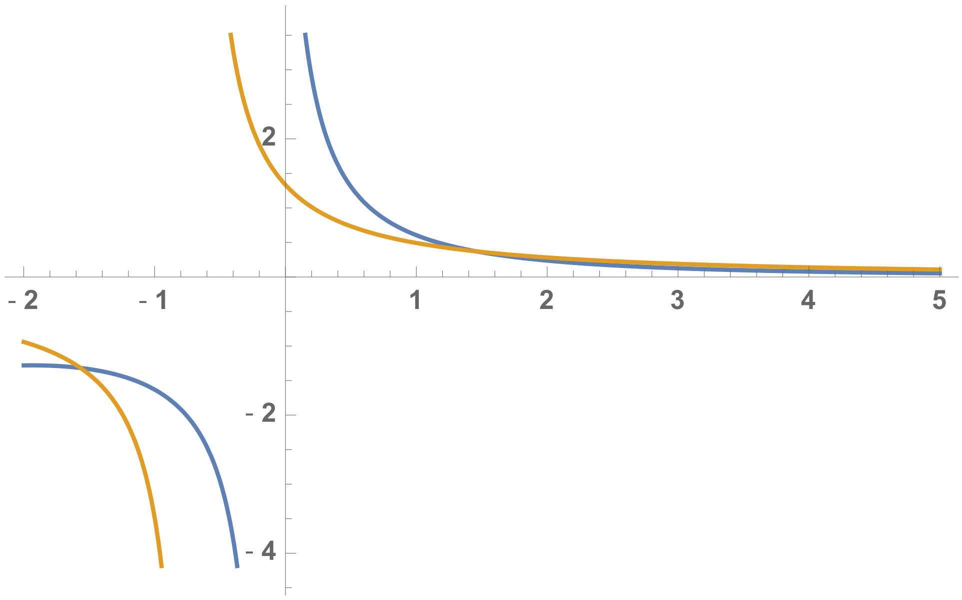

Figure 2: Graph of σ for $\alpha = 1.1$ (blue)and $\alpha = 1.5$ (orange)

According to Fubini's Theorem, we have

$$\begin{aligned} u(t) &= \int_0^1 \left(G_1(t,s) + \frac{\mu v_1(t) + \eta v_2(t)}{(1-\mu\theta-\eta\sigma)} \int_0^1 G_1(t,s)dt \right) y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds, \end{aligned}$$

and the result is concluded. ■

In our approach, we need the following properties of $G_1(t,s)$ proved in [7, Lemmas 8, 9].

Lemma 4 *Let G_1 be the Green's function given in (7) and λ_1^* be the first negative zero of $E_{\alpha,\alpha-1}(\lambda) = 0$. Then for $1 < \alpha \leq 2$, it is satisfied that*

$$G_1(t,s) > 0 \quad for\ all\ t,s \in (0,1)\ if\ and\ only\ if \quad \lambda > \lambda_1^*.$$

Lemma 5 *Let G_1 be the Green's function given in (7), $1 < \alpha \leq 2$ and $\lambda > \lambda_1^*$. Then there exists a positive constant M and a continuous function m such that $m(t) > 0$ on $(0,1]$ and $m(0) = 0$, for which the following inequalities are fulfilled:*

$$m(t) \leq \frac{t^{2-\alpha}G_1(t,s)}{s(1-s)^{\alpha-2}} \leq M, \quad for\ all\ t,s \in (0,1). \tag{11}$$

Next, we prove the following properties for the Green's function $G(t,s)$. To this end, in the table 1, by means of numerical approach, we give some values of θ and σ for $1 < \alpha \leq 2$ for which we ensure that the Green's function $G(t,s)$ has a constant sign.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 January 2020 doi:10.20944/preprints202001.0101.v1

Peer-reviewed version available at *Mathematics* **2020**, *8*, 255; doi:10.3390/math8020255

Lemma 6 *Let G be the Green's function related to problem (2) and λ_1^* be the first negative zero of $E_{\alpha,\alpha-1}(\lambda) = 0$. Then for $(1-\mu\theta-\eta\sigma) > 0$ and $1 < \alpha \leq 2$, the following properties hold:*

1. *G is a continuous function on $(0,1] \times [0,1)$.*
2. *If $\lambda > \lambda_1^*$ then $G(t,s) > 0$ for all $t, s \in (0,1)$*
3. *Consider the function $m(t)$ and the positive contant M, introduced in Lemma 5. Then the following inequality holds:*

$$m(t) \leq \frac{t^{2-\alpha}G(t,s)}{s(1-s)^{\alpha-2}} \leq M', \quad \textit{for all } \ t, s \in (0,1), \tag{12}$$

with

$$M' = M\left(1 + \frac{L}{(1-\mu\theta-\eta\sigma)(\alpha-1)}\right)$$

and

$$L = \mu\,\|v_1\|_{2-\alpha} + \eta\,\|v_2\|_{2-\alpha}.$$

Proof.

1. It is obvious from the continuity of G_1, v_1 and v_2.
2. From Lemma 4, $G_1(t,s) > 0$ for all $t, s \in (0,1)$. Moreover, from [7, Lemma 18], we have that v_1 and v_2 are positive on $(0,1]$. And so, Property 2 holds immediately from expression (8).
3. From Lemma 5 and for $t \in (0,1]$ and $s \in (0,1)$, since $(1-\mu\theta-\eta\sigma) > 0$, we have

$$\begin{aligned} t^{2-\alpha}G(t,s) &= t^{2-\alpha}G_1(t,s) + t^{2-\alpha}\frac{(\mu v_1(t) + \eta v_2(t))}{(1-\mu\theta-\eta\sigma)}\left(\int_0^1 G_1(t,s)dt\right) \\ &\geq t^{2-\alpha}G_1(t,s) \\ &\geq s(1-s)^{\alpha-2}m(t). \end{aligned}$$

Now, using again Lemma 5, from equation (8) we obtain

$$\begin{aligned} t^{2-\alpha}G(t,s) &\leq s(1-s)^{\alpha-2}M + \frac{L}{(1-\mu\theta-\eta\sigma)}s(1-s)^{\alpha-2}M\int_0^1 r^{\alpha-2}dr \\ &= s(1-s)^{\alpha-2}M\left(1 + \frac{L}{(1-\mu\theta-\eta\sigma)(\alpha-1)}\right). \end{aligned}$$

■

4 Nonlinear Problem

4.1 Existence of Solutions

This section is concerned with the existence of solutions of the nonlinear fractional differential equation with non homogeneous integral boundary conditions (1). To this end, we will aply the classical index theory.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 January 2020 doi:10.20944/preprints202001.0101.v1

Peer-reviewed version available at Mathematics 2020, 8, 255; doi:10.3390/math8020255

Let K be a cone in a Banach space X. If Ω is a bounded open subset of K, we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K. When D is an open bounded subset of X we write $D_K = D \cap K$, an open subset of K. The following result is well-known in fixed index theory for completely continuous operators T (i.e. continuous and $\overline{T(S)}$ compact for each bounded subset $S \subset K$). See for example [1, 11, ?] for further information.

Lemma 7 *Let D be an open bounded set with $D_K \neq \emptyset$ and $\overline{D_K} \neq K$. Assume that $T : \overline{D_K} \to K$ is a completely continuous operator such that $x \neq Tx$ for $x \in \partial D_K$. Then the fixed point index $i_K(T, D_K)$ has the following properties.*

(1) If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \mu e$ for all $x \in \partial D_K$ and all $\mu > 0$, then $i_K(T, D_K) = 0$.

(2) If $\gamma x \neq Tx$ for all $x \in \partial D_K$ and all $\gamma \geq 1$, then $i_K(T, D_K) = 1$.

(3) Let D^1 be open in X such that $\overline{D^1} \subset D_K$. Then

$$i_K(T, D_K) = i_K(T, D^1) + i_K(T, D_K \backslash \overline{D^1}).$$

(4) If $i_K(T, D_K) \neq 0$ then there exists $u \in D_K$ such that $u = Tu$.

We assume the following regularity for the nonlinear part of the equation:

(H_1) $f : [0, 1] \times [0, \infty) \longrightarrow [0, \infty)$ is a continuous function.

Next, we define the operator $T : C_{2-\alpha}[0, 1] \to C_{2-\alpha}[0, 1]$ by

$$Tu(t) = \int_0^1 G(t, s) f(s, s^{2-\alpha} u(s)) ds, \qquad 0 < t \leq 1, \tag{13}$$

where G is given by expression (8).

Now, fix $c_1 \in (0, 1)$, denote $m_0 = \min_{t \in [c_1, 1]} m(t) > 0$ and $c = m_0/M'$, and define the following cone

$$K = \{u \in C_{2-\alpha}(I) : u \geq 0 \, \text{on} \, (0, 1], \min_{t \in [c_1, 1]} \{t^{2-\alpha} u(t)\} \geq c\|u\|_{2-\alpha}\}. \tag{14}$$

In order to use the properties showed in Theorem 7, it is not difficult to verify that T is a completely continuous operator on K such that $T(K) \subset K$ (see [7, Lemma 12] for details).

To prove the existence of solutions of problem (1), we need to prove that $i_K(T, D_K) = 0$ for an open set $D_K \subset K$. Therefore, we construct a relatively open set $D_K = \Omega_\rho$ for which $\Omega_\rho \neq K_s$ for each $s > 0$ and show that $i_K(T, \Omega_\rho) = 0$. This allows f to satisfy weaker conditions than those used in [7].

Definition 8 *Let us define the following sets for every $\rho > 0$:*

$$K_\rho = \{u \in K : \|u\|_{2-\alpha} < \rho\},$$

and

$$\Omega_\rho = \{u \in K : \ t^{2-\alpha} u(t) < \rho \ \textit{for all} \ t \in [c_1, 1]\}.$$

Peer-reviewed version available at Mathematics 2020, 8, 255; doi:10.3390/math8020255

It is clear that

$$K_\rho \subset \Omega_\rho \subset K_{\frac{\rho}{c}},$$

and, in particular, both K_ρ and Ω_ρ are open and bounded sets of $C_{2-\alpha}(I)$ for all $\rho > 0$.

In the two following lemmas we give some sufficient conditions to ensure that, for a suitable $\rho > 0$, the index is either 1 or 0.

Lemma 9 *Let*

$$0 < H = \left(\max_{0 \le t \le 1} \{\int_0^1 t^{2-\alpha} G(t,s)ds\}\right)^{-1} \in \mathbb{R}$$

and

$$f^\rho := \max\{\frac{f(t,u)}{\rho}; 0 \le t \le 1,\ 0 \le u \le \rho\}.$$

If there exists $\rho > 0$ such that $f^\rho < H$, then $i_K(T, K_\rho) = 1$.

Proof. To apply Lemma 7 (2), we will show that $Tu \neq \gamma u$ for all $u \in \partial K_\rho$ and every $\gamma \ge 1$.

Suppose, on the contrary, that there exists $u \in \partial K_\rho$ and $\gamma \ge 1$ such that

$$\gamma t^{2-\alpha} u(t) = t^{2-\alpha} \int_0^1 G(t,s) f(s, s^{2-\alpha} u(s)) ds.$$

Taking the maximum for $t \in I$, we obtain

$$\begin{aligned} \gamma\rho = \gamma \|u\|_{2-\alpha} &= \max_{t \in I} \{t^{2-\alpha} \int_0^1 G(t,s) f(s, s^{2-\alpha} u(s)) ds\} \\ &\le \rho f^\rho \max_{t \in I} \{t^{2-\alpha} \int_0^1 G(t,s) ds\} \\ &= \rho \frac{f^\rho}{H} < \rho, \end{aligned}$$

which contradicts the fact that $\gamma \ge 1$. Therefore, the result is proved. ∎

Lemma 10 *Let*

$$m_1 = \left(\min_{c_1 \le t \le 1} \{\int_{c_1}^1 t^{2-\alpha} G(t,s)ds\}\right)^{-1}$$

and

$$f_\rho^1 := \min\{\frac{f(t,u)}{\rho}; c_1 \le t \le 1,\ 0 \le u \le \frac{\rho}{c}\}.$$

If there exists $\rho > 0$ such that $f_\rho^1 > m_1$, then $i_K(T, \Omega_\rho) = 0$.

Proof. We will prove that there exists $e \in K \setminus \{0\}$ such that $u \neq Tu + \gamma e$ for all $x \in \partial\Omega_\rho$ and all $\gamma > 0$.

Let us take $e(t) = t^{\alpha-2+r}$ in I, with $r \in (0,1)$ such that $c_1^r > m_0/M'$. It is clear that $e \in K \backslash \{0\}$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 January 2020 doi:10.20944/preprints202001.0101.v1

Peer-reviewed version available at Mathematics 2020, 8, 255; doi:10.3390/math8020255

Suppose, on the contrary, that there exist $u \in \partial\Omega_\rho$ and $\gamma > 0$ such that $u = Tu + \gamma e$. Then, for all $t \in [c_1, 1]$, the following inequalities hold:

$$\begin{aligned} t^{2-\alpha}u(t) &= t^{2-\alpha}\left(\int_0^1 G(t,s)f(s,s^{2-\alpha}u(s))ds + \gamma t^{\alpha-2+r}\right) \\ &\geq \int_0^1 t^{2-\alpha}G(t,s)f(s,s^{2-\alpha}u(s))ds \\ &\geq \int_{c_1}^1 t^{2-\alpha}G(t,s)f(s,s^{2-\alpha}u(s))ds \end{aligned}$$

Now, since

$$\rho \geq s^{2-\alpha}\,u(s) \geq \frac{m_0}{M'}\|u\|_{2-\alpha} = c\,\|u\|_{2-\alpha}, \quad \text{for all } s \in [c_1, 1],$$

we have that previous expression is bigger than or equals to

$$\rho f_\rho^1 \int_{c_1}^1 t^{2-\alpha}G(t,s)ds > \rho\frac{f_\rho^1}{m_1} > \rho$$

and we arrive at a contradiction.
Thus, from Lemma 7 (1), we deduce that $i_K(T, \Omega_\rho) = 0$. ∎

The above results allows us to give the following new result on existence of solutions for problem (1).

Theorem 11 *Let H and m_1 be as in Lemmas 9 and 10 and $0 < \rho_1 < c\,\rho_2$. Suppose that $f^{\rho_2} < H$ and $f_{\rho_1}^1 > m_1$. Then Problem (1) has at least one solution u such that $\|u\|_{2-\alpha} \leq \rho_2$ and there is $t_0 \in [c_1, 1]$ for which $t_0^{2-\alpha}u(t_0) \geq \rho_1$.*

Proof. As we have proved along this section, the solutions of Problem (1) coincide with the fixed points of operator T.

Of course, if T has a fixed point $u \in K$, such that $\|u\|_{2-\alpha} = \rho_2$, we have that Problem (1) has a solution satisfying such property. Moreover, we have, for any $t \in I$,

$$t^{2-\alpha}u(t) \geq \int_{c_1}^1 t^{2-\alpha}G(t,s)f(s,s^{2-\alpha}u(s))ds.$$

So, if $t^{2-\alpha}u(t) < \rho_1$ for all $t \in [c_1, 1]$, we arrive at a contradiction as in the proof of Lemma 10.

So, suppose that $u \neq Tu$ for all $u \in \partial K_{\rho_2}$. By Lemmas 9 and 10, it is fulfilled that $i_K(T, K_{\rho_2}) = 1$ and $i_K(T, \Omega_{\rho_1}) = 0$. In addition, since $\rho_1 < c\,\rho_2$, we have that $\Omega_{\rho_1} \subset K_{\frac{\rho_1}{c}} \subset K_{\rho_2}$. Therefore, from Lemma 7 (3), we have that

$$i_K(T, K_{\rho_2} \setminus \Omega_{\rho_1}) = i_K(T, K_{\rho_2}) - i_K(T, \Omega_{\rho_1}) = 1,$$

and, from Lemma 7, (4), we have that T has a fixed point u in $K_{\rho_2} \setminus \Omega_{\rho_1}$. As a consequence we know that Problem (1) has at least one solution u such that $\|u\|_{2-\alpha} < \rho_2$ and there is $t_0 \in [c_1, 1]$ for which $t_0^{2-\alpha}u(t_0) \geq \rho_1$. ∎ Analogously, we may prove the following existence result.

Peer-reviewed version available at *Mathematics* **2020**, *8*, 255; doi:10.3390/math8020255

Theorem 12 *Let H and m_1 be as in Lemmas 9 and 10 and $0 < \rho_2 < c\,\rho_1$. Suppose that $f^{\rho_1} < H$ and $f^1_{\rho_2} > m_1$. Then Problem (1) has at least one solution u such that $\|u\|_{2-\alpha} \le \rho_1$ and there is $t_0 \in [c_1, 1]$ for which $t_0^{2-\alpha} u(t_0) \ge \rho_2$.*

Proof. In this case, it is enough to take into account that $\bar{\Omega}_{\rho_2} \subset K_{\rho_2/c} \subset K_{\rho_1}$

Since, in this case, if $u \neq Tu$ in ∂K_{ρ_1}, we have that $i_K(T, K_{\rho_1}) = 1$ and $i_K(T, \Omega_{\rho_2}) = 0$. The proof follows from Lemma 7 (3) and (4). ■

4.2 Non-existence results

In this section, we give some sufficient conditions of the nonlinear part of the equation of Problem (1) that ensure that such problem has no nontrivial and nonnegative solution in $C_{2-\alpha}(I)$.

Theorem 13 *Suppose that $f : I \times [0, \infty) \to [0, \infty)$ is a continuous function and one of the following conditions holds*

(i) $f(t, u) \le \tilde{m} u$ for $u \ge 0$ and $t \in I$, where $0 < \tilde{m} < \frac{\alpha(\alpha-1)}{M'}$.

(ii) $f(t, u) \ge \tilde{M} u$ for $u \ge 0$ and $t \in [c_1, 1]$, with $\tilde{M} > m_1$ (m_1 given in Lemma 10).

Then Problem (1) has no nontrivial and nonnegative solution in $C_{2-\alpha}(I)$.

Proof.

(i) Suppose, on the contrary, that there exists $u \in C_{2-\alpha}(I)$, $u \ge 0$ on I, u not identically zero on I, that solves (1). As we have seen, this property is equivalent to the fact that $u = Tu$. As a consequence, since $\|u\|_{2-\alpha} > 0$, for $t \in I$, we have

$$\begin{aligned} 0 \le t^{2-\alpha} u(t) &= t^{2-\alpha} \int_0^1 G(t,s) f(s, s^{2-\alpha} u(s)) ds \\ &\le M' \int_0^1 s(1-s)^{\alpha-2} f(s, s^{2-\alpha} u(s)) ds \\ &\le M' \tilde{m} \int_0^1 s(1-s)^{\alpha-2} s^{2-\alpha} u(s) ds \\ &\le \frac{M' \tilde{m}}{\alpha(\alpha-1)} \|u\|_{2-\alpha} \\ &< \|u\|_{2-\alpha}. \end{aligned}$$

Therefore, we get $\|u\|_{2-\alpha} < \|u\|_{2-\alpha}$, which is a contradiction.

(ii) In this case, it the result is false, we have that there exists $u \in C_{2-\alpha}(I)$, $u \ge 0$ on I, with $\|u\|_{2-\alpha} > 0$, such that $u = Tu$.

Then, for $t \in [c_1, 1]$, we have

$$\begin{aligned} t^{2-\alpha} u(t) &\ge t^{2-\alpha} \int_{c_1}^1 G(t,s) f(s, s^{2-\alpha} u(s)) ds \\ &\ge \tilde{M} t^{2-\alpha} \int_{c_1}^1 G(t,s) s^{2-\alpha} u(s) ds. \end{aligned}$$

Peer-reviewed version available at Mathematics 2020, 8, 255; doi:10.3390/math8020255

Using that $t^{2-\alpha}G(t,s) > 0$ for all $t, s \in [c_1, 1]$ and, since $s^{2-\alpha}u(s)$ is a continuous, non negative and non trivial function on $[c_1, 1]$, we have that

$$\min_{t\in[c_1,1]} \left\{ t^{2-\alpha} \int_{c_1}^{1} G(t,s)s^{2-\alpha}u(s)ds \right\} > 0.$$

In particular, previous inequalities show us that

$$\overline{u} = \min_{t\in[c_1,1]} \{t^{2-\alpha}u(t)\} > 0.$$

Moreover

$$\overline{u} \geq \tilde{M} \min_{t\in[c_1,1]} \{t^{2-\alpha} \int_{c_1}^{1} G(t,s)s^{2-\alpha}u(s)ds\} \geq \tilde{M}\overline{u} \min_{t\in[c_1,1]} \{\int_{c_1}^{1} t^{2-\alpha}G(t,s)ds\} > \overline{u},$$

which is a contradiction.

■

5 Examples

In this section, we present an example where the applicability of the existence results given in Section 4 is pointed out.

Example 14 *Let us consider Problem* (1) *with* $\lambda_1^* < \lambda \leq 0$, $\alpha = \frac{3}{2}$, $\mu = \frac{1}{12}$, $\eta = \frac{1}{16}$, $\lambda = -\frac{1}{2}$, $c_1 = \frac{1}{3}$.
A simple calculation yields to $(1-\mu\theta-\eta\sigma) \approx 0.0998338 > 0$ *and* $c \approx 0.0160729$. *Let*

$$f(t,u) = \delta\sqrt{1+t}(1+u^{\frac{3}{2}}), \textit{for } \delta > 0.$$

Let $\rho_1,\ \rho_2 >$. *Then*

$$f_{\rho_1}^1 = \min\left\{ \frac{f(t,u)}{\rho_1} :\ \ t \in [\frac{1}{3}, 1],\ \ u \in [0, \frac{\rho_1}{c}] \right\} = \frac{2}{\sqrt{3}\rho_1}\delta,$$

and

$$f^{\rho_2} = \max\left\{ \frac{f(t,u)}{\rho_2} :\ \ t \in [0,1],\ \ u \in [0,\rho_2] \right\} = \frac{\sqrt{2}(\rho_2^{\frac{3}{2}}+1)}{\rho_2}\delta.$$

Moreover, it is not difficult to verify that $H \geq 0.0213044$ *and* $m_1 \leq 0.0504397$.

Hence, from Theorem 11, for any ρ_1, ρ_2 *such that* $\rho_1 < c\rho_2$ *and*

$$\frac{\sqrt{3}\rho_1}{2} m_1 < \delta < \frac{\rho_2}{\sqrt{2}(1+\rho_2^{\frac{3}{2}})} H$$

Problem (1) has at least one solution $u \in C_{1/2}(I)$, *such that* $\|u\|_{2-\alpha} \leq \rho_2$ *and there is* $t_0 \in [c_1, 1]$ *for which* $t_0^{2-\alpha}u(t_0) \geq \rho_1$.

In particular, since the minimum on the left hand side of previous inequality is 0 *and, by defining*

$$f(x) = \frac{x}{1+x^{3/2}}$$

we have that

$$\max_{x \geq 0} \{f(x)\} = f(\sqrt[3]{4}) \approx 0.529134,$$

we have that Problem (1) has a non negative and nontrivial solution for all

$$0 < \delta < 0.00797113,$$

such that $\|u\|_{1/2} \leq \sqrt[3]{4}$.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976) 620-709.

[2] B. Ahmad, J. J. Nieto, Existence results for nonlinear boundary value problems of fractional integro differential equations with integral boundary conditions, Boundary Value Problems, vol. 2009, Article ID 708576, 11 pages, 2009.

[3] B. Ahmad, A. Alsaedi, and B. S. Alghamdi, Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions, Nonlinear Analysis: Real World Applications, vol. 9, no. 4,(2008) 1727-1740.

[4] P. L. Butzer, U. Westphal, *An introduction to fractional calculus*, in: R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*, World Scientific, New Jersey, 2000.

[5] M. Benchohra, A. Cabada, D. Seba, An existence result for nonlinear fractional differential equations on Banach spaces, Boundary Value Problems 2009 (2009) 11 p.

[6] A. Cabada, S. Aleksić, Suzana, T. Tomović, S. Dimitrijević, Existence of Solutions of Nonlinear and Non-local Fractional Boundary Value Problems. Mediterr. J. Math. 16 (2019), 5, Paper No. 119.

[7] A. Cabada, O.M. Wanassi, Existence and uniqueness of positive solutions for nonlinear fractional mixed problems, Submitted. `arXiv:1903.09042 [math.CA]`

[8] A. Cabada, Z. Hamdi, Existence results for nonlinear fractional Dirichlet problems on the right side of the first eigenvalue, Georgian Mathematical Journal, 1-13, 2017.

[9] A. Cabada, G. Infante and F. A. F. Tojo, Nonzero solutions of perturbed Hammerstein integral equations with deviated arguments and applications, Topol. Methods Nonlinear Anal., 47 (2016), 265-287.

[10] M. Feng, X. Liu, H. Feng, The existence of positive solution to a nonlinear fractional differential equation with integral boundary conditions. Adv. Differ. Equ. (2011) 1-15.

[11] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, 1988.

[12] J. R. Graef, L. Kong, Q. Kong, M. Wang, Uniqueness of positive solutions of fractional boundary value problems with non-homogeneous integral boundary conditions, Fract. Calc. Appl. Anal., 15 (2012), 509-528.

[13] R. Gorenflo, F. Mainardi, *Fractional calculus: Integral and differential equations of fractional order*, in: A. Carpinteri, F. Mainardi (Eds.), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, New York, 1997, 223–276.

[14] G. Infante, A short course on positive solutions of systems of ODEs via fixed point index, in Lecture Notes in Nonlinear Analysis (LNNA), 16 (2017), 93-140.

[15] G. Infante, J. R. L. Webb, Nonlinear nonlocal boundary value problems and perturbed Hammerstein integral equations, Proc. Edinb. Math. Soc., 49 (2006), 637-656.

[16] J. Jiang, W. Liu, and H. Wang. Positive solutions to singular Dirichlet-type boundary value problems of nonlinear fractional differential equations. Adv. Difference Equ. 2018; 2018: 169.

[17] A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics studies, Vol. 204, Elsevier, Amsterdam, 2006.

[18] Y. Kong, P. Chen, Positive solutions for periodic boundary value problem of fractional differential equation in Banach spaces, Adv. Difference Equ., 1-8, 2018.

[19] S. Liang and J. Zhang, Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval, Mathematical and Computer Modelling, vol. 54, no. 5-6, pp. 1334-1346, 2011.

[20] Lloyd, N. G. Degree theory. Cambridge Tracts in Mathematics, 73. Cambridge University Press, Cambridge-New York-Melbourne, 1978.

[21] K. Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, J. London Math. Soc, 63 (2001), 690-704.

[22] K. Q. Lan, J. R. L. Webb, Positive solutions of semilinear differential equations with singularities, J. Differential Equations 148 (1998) 407-421.

[23] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, 1993.

[24] C. Pengyu, G, Yabing, Positive solutions for a class of nonlinear fractional differential equations with nonlocal boundary value conditions, Positivity 22(3), (2018) 761-772.

[25] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, Switzerland, 1993.

Peer-reviewed version available at *Mathematics* **2020**, *8*, 255; doi:10.3390/math8020255

[26] J. R. L. Webb, Remarks on positive solutions of three point boundary value problems, Dynamical systems and differential equations (Wilmington, NC, 2002), Discrete Contin. Dyn. Syst. 2003, suppl., 905-915.

[27] B. Tellab, K. Haouam, Solvability of semilinear fractional differential equations with nonlocal and integral boundary conditions, Mathematics in Engineering Science and Aerospace, Vol. 10, No 2, (2019) 341-356.

[28] X. Zhang, L. Wang, Q. Sun, Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter, Applied Mathematics and Computation, (2014) 708-718.

[29] E. Zeidler, Nonlinear Functional Analysis and Its Applications. I. Fixed-Point Theorems, Translated from the German by Peter R. Wadsack, Springer-Verlag, New York, 1986.