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On the characteristic properties of geodesic sub-(α, b, s)-preinvex

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Abstract

In the present work we study the properties of geodesic sub-(α, b, s)-preinvex functions on Hadamard manifolds and establish some basic properties in both general and differential cases. Further, we study sufficient conditions of optimality and proved some new inequalities under geodesic sub-(α, b, s)-preinvexity.

Keywords : E-convex sets; preinvex; E-convex functions; geodesic; sub-preinvex; Riemannian manifolds; Hadamard manifolds.
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1 Introduction

The convexity is an important property in mathematics and economics and in the recent years, and many researchers developed new generalizations for the classical convexity and also many properties were established in new generalized cases. For example, in 1981, one of the important generalization of convexity that is called today as invexity presented by Hanson [7]. The preinvexity was presented by Ben-Israel and Mond in [5] which were special case of invex functions. In this row, some properties of preinvex and α-preinvex functions were considered by Jeyakumar [9] and Noor [14, 15], respectively. In 1991, a class of b-vex functions were introduced by Bector and Singh in [4]. The generalizations of preinvex and b-vex functions were studied by by Suneja et al. [19], today called as b-preinvex functions. In 2006, a generalized of b-invex function which is known as semi-b-preinvex was presented in [13]. Further, Chao et. al. in [6] defined a new class of generalized sub-b-convex functions and discussed sufficient conditions of optimality. Hudzik and Maligrand in [8] studied two kinds of s-convexity

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($s \in (0, 1)$) and sub-b-s-convex functions by using modulation s-convexity and sub-b-convexity, see [12]. Similarly, sub-b-s-pre invexity was presented the generalization for s-convex and b-preinvexity, see [11].

Thus there are many properties of convex functions can be established on Riemannian manifolds. For example, Rapcsak [17] studied smooth nonlinear optimization in R^n and Udeiste [20] studied some generalizations of convexity as well as optimization problems on Riemannian Manifolds which differs from the others in the use of Riemannian manifold. The convexity along curves and generalizations with applications to duality theory and optimality conditions on Riemannian manifold were considered by Pini [16]. The concept of geodesic invexity in Riemannian manifold was introduced and preinvexity on a geodesic invex set were defined, further the relationship between geodesic invexity and preinvexity on manifolds was studied by Barani and Pouryayevali [4], while geodesic α-invexity and α-preinvex functions were defined in [2]. Further in the literature there are many more related generalizations of convexity, and new class of generalized convexity such as strongly α-invex and strongly geodesic α-preinvex functions etc., see [1, 10, 11].

Riemannian geometry is considered as generalization of the Euclidean case and smooth Riemannian manifolds accommodate curvature by using the tangent planes. Thus, the metric is not trivial and distances need to reconsidered for those curvature, see Gabriel et al. [21]. Now, we recall some definitions and some results related to Riemannian and Hadamard manifolds that they can be found in [2, 17, 20].

Consider W m-dimensional Riemannian manifold, and T_pW tangent space to W at point p, if $\mu_p(x_1, x_2)$ then the map $\mu\colon p \longrightarrow \mu_p$ is called a Riemannian metric where μ_p to T_pW. Further, a manifold W is equipped with μ is known as a Riemannian manifold, see the details in [2, 17, 20].

The geodesic property is defined as the shortest possible line between two points on a sphere or other curved surface, or more generally in a Riemannian manifold. Thus next we define the length of curve $\alpha\colon [a_1, a_2] \longrightarrow W$ as:

$$L(\alpha) = \int_{a_1}^{a_2} \|\acute{\alpha}(x)\| dx.$$

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Further if we let

$$d(p_1, p_2) = \inf\left\{L(\alpha)\colon \alpha \text{ is a picewise } \mathrm{C}^1 \text{ curve from } \mathrm{p}_1 \text{ to } \mathrm{p}_2\right\}$$

for any points $p_1, p_2 \in W$, then, d is a metric induced by topology on W. Note that for every Riemannian manifold W there exists only one covariant derivation and it is also known as Levi-Civita connection $\bigtriangledown_X Y$, for $X, Y \in W$. Further, a geodesic smooth path α having tangent and satisfies $\bigtriangledown_{\acute{\alpha}(t)}\acute{\alpha}(t) = 0$. Any path α joining p_1 and p_2 in W that $L(\alpha) = d(p_1, p_2)$ is called a minimal geodesic. Similarly, Hadamard manifold is complete, simply connected manifolds and has non-positive sectional curvature on W, that is having an exponential map $\exp_p : T_pW \longrightarrow W$ such that $\exp_p(v) = \alpha_v(1)$, on the whole tangent space of a point then α_v is also geodesic and applied as velocity of α.
Now, we recall the following definition and the details can be found in [2].

Definition 1.1. *Assume that W is a Hadamard manifold, and $\eta : W \times W \longrightarrow TW$ is a function and while $\alpha : W \times W \longrightarrow \mathbb{R}\setminus\{0\}$ defined such that $\alpha(j_1, j_2)\eta(j_1, j_2) \in T_lW$, for all $j_1, j_2 \in W$. A non-empty subset $Y \subset W$ is called a geodesic α-invex ($G\alpha$ invex) set with respect to(w.r.t) η if there is a unique geodesic $\alpha_{j_1,j_2} : [0, 1] \longrightarrow W$ such that*

$$\alpha_{j,l}(0) = v, \acute{\alpha}_{j_1,j_2}(0) = \alpha(j_1, j_2)\eta(j_1, j_2)$$

for $\alpha_{j_1,j_2}(t) \in Y$, and $0 \le t \le 1$.

The set Y is called G.α- invex set on a Hadamard manifold if

$$\exp_l\left(t\alpha(j_1, j_2)\eta(j_1, j_2)\right) \in Y,$$

for $j_1, j_2 \in Y$ and $0 \le t \le 1$.

Remark 1.2. *If $\alpha(j_1, j_2) = 1$, the Definition 1.1 reduces to geodesic invex set [3].*

2 Main Results

Definition 2.1. *Assume that Y is G.α- invex set. The function $h : Y \longrightarrow TW$ is called a geodesic sub-(α, b, s)-preinvex, if $b(j_1, j_2, t) : Y \times Y \times [0, 1] \longrightarrow \mathbb{R}$ such that*

$$h(\exp_{j_2} t\alpha(j_1, j_2)\eta(j_1, j_2)) \le t^s h(j_1) + (1-t)^s h(j_2) + b(j_1, j_2, t),$$

for $j_1, j_2 \in Y$, $t \in [0, 1]$ and $s \in (0, 1]$.

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Remark 2.2. 1. *If $s = 1$ and $b(j_1, j_2, t) \leq 0$, the Definition 2.1 reduces to geodesic α-preinvex.*

2. *If $s = 1$, $\alpha(j_1, j_2) = 1$ and $b(j_1, j_2, t) \leq 0$, the Definition 2.1 reduces to geodesic convex.*

Example 2.3. *Consider $h : [0, +\infty) \longrightarrow \mathbb{R}$ is defined by*

$$h(x) = (x^2 + 4x)^s, \text{ and } b(x, y, t) = tx^2 + 4ty^2 \text{ for } s \in (0, 1).$$

Now assume that

$$\alpha(t) = \exp_{j_2}(t\alpha(j_1, j_2)\eta(j_1, j_2))$$

where $\alpha(j_1, j_2) = 1$ and $\eta(j_1, j_2) = exp_{j_2}^{-1} j_1$. Then h is a sub $-$ (α, b, s)-preinvex.

Remark 2.4. *When $\alpha(t) = tj_1 + (1 - t)j_2$ in Example 2.3, then h will be come sub-b-s-convex function [12].*

Theorem 2.5. *Assume that $f_1, f_2 : Y \longrightarrow TW$ are geodesic sub-(α, b, s)-preinvex then $f_1 + f_2$ and $\beta f_1, \beta \geqslant 0$ are also geodesic sub-(α, b, s)-preinvex.*

The above theorem determine that the geodesic sub-preinvex property is a linear property. Then in the same way we can extend to above theorem straightforward and we have the following corollary.

Corollary 2.6. *If $h_\iota : Y \longrightarrow TW, (\iota = 1, 2, \cdots, n)$ are geodesic sub-(α, b, s)-preinvex $b_\iota : Y \times Y \times [0, 1] \longrightarrow \mathbb{R}$, $(i = 1, 2, \cdots, n)$, respectively, then*

$$h = \sum_{\iota=1}^{n} \lambda_\iota h_\iota, \lambda_\iota \geqslant 0$$

is also geodesic sub-(α, b, s)-preinvex where $b = \sum_{\iota=1}^{n} \lambda_\iota b_\iota$.

Theorem 2.7. *Consider $h_1 : Y \longrightarrow TW \subseteq \mathbb{R}$ is a geodesic sub-(α, b, s)-preinvex function and $h_2 : K \longrightarrow \mathbb{R}$ is a non-decreasing convex function where $rang(h_1) \subseteq K$, then $h_1 o h_2$ is a geodesic sub-(α, b, s)-preinvex function b where $b = h_2 o b_1$.*

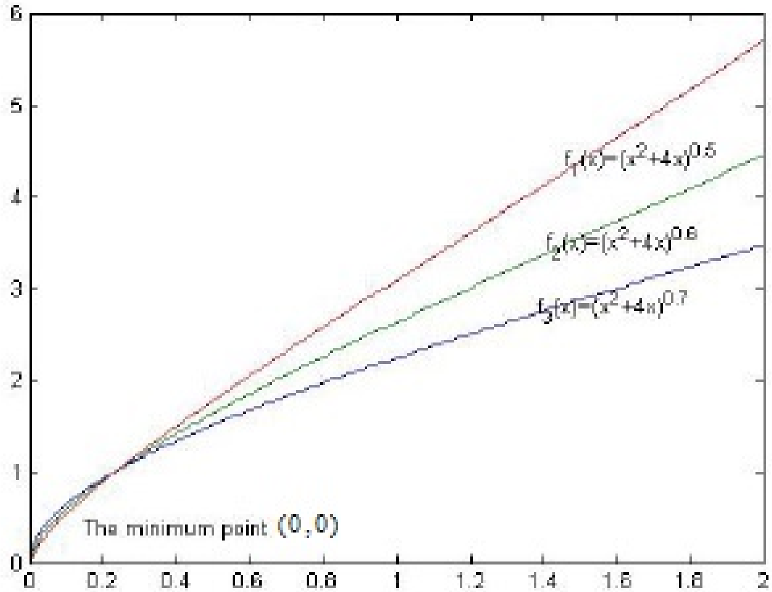

Figure 1: h is sub-b-s-convex function

Proof.

$$
\begin{aligned}
(h_2 o h_1)\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right) &= h_2\left(h_1(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2))\right) \\
&\leq h_2\left(t^s h_1(j_1) + (1-t)^s h_1(j_2) + b_1(j_1,j_2,t)\right) \\
&= t^s h_2\left(h_1(j_1)\right) + (1-t)^s h_2\left(h_1(j_2)\right) + h_2\left(b_1(j_1,j_2,t)\right) \\
&= t^s\left(h_2 o h_1\right)(j_1) + (1-t)^s\left(h_2 o h_1\right)(j_2) + b(j_1,j_2,t)
\end{aligned}
$$

which means that $h_2 o h_1$ is a geodesic sub-(α, b, s)-preinvex function. □

The above theorem indicates that under certain conditions the composition is invariant. Next, the definition of a geodesic sub-(α, b, s)-preinvex set.

Definition 2.8. *A set $Y \subseteq W$ is said to be a geodesic sub-(α, b, s)-preinvex set, if*

$$\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2), t^s\beta_1 + (1-t)^s\beta_2 + b(u_1,u_2,t)\right) \in Y,$$

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$\forall (j_1, \beta_1), (j_2, \beta_2) \in Y$, $j_1, j_2 \in W$, $t \in [0,1]$, $s \in (0,1]$ *and* b *defined as* $b : Y \times Y \times [0,1] \longrightarrow \mathbb{R}$.

The epigraph of a geodesic sub-(α, b, s)-preinvex function $h : Y \longrightarrow TW$ can be explained as

$$\omega(h) = \{(j, r) : j \in Y, \beta \in \mathbb{R}, h(j) \leq \beta\}.$$

Now, in order to prove the sufficient and necessary rule for h to be a geodesic sub-(α, b, s)-preinvexn we need to study properties of geodesic sub-(α, b, s)-preinvex in term of their epigraph $\omega(h)$.

Proposition 2.9. *Assume that* $h_i : Y \longrightarrow TW$ *be geodesic sub-*(α, b, s)*-preinvex functions with respect to maps* $b_i : Y \times Y \times [0,1] \longrightarrow \mathbb{R}$*, for* $(i = 1, 2, \cdots, n)$*, then* $H = \max h_i$ *is also geodesic sub-*(α, b, s)*-preinvex where* $b = \max b_i$*.*

Theorem 2.10. *A function* $h : Y \longrightarrow TW$ *is geodesic sub-*(α, b, s)*-preinvex if and only if its epigraph is also a geodesic sub-*(α, b, s)*-preinvex.*

Proof. Assume that h be a geodesic sub-(α, b, s)-preinvex function and $(j_1, \beta_1), (j_2, \beta_2) \in \omega(h)$, then by hypothesis, $h(j_1) \leq \beta_1$ and $h(j_2) \leq \beta_2$. Further,

$$\begin{aligned} h\left(\exp_{j_2} t\alpha(j_1, j_2)\eta(j_1, j_2)\right) &\leq t^s h(j_1) + (1-t)^s h(j_2) + b(j_1, j_2, t) \\ &\leq t^s \beta_1 + (1-t)^s \beta_2 + b(j_1, j_2, t). \end{aligned} \tag{2.1}$$

Then,

$$\left(\exp_{j_2} t\alpha(j_1, j_2)\eta(j_1, j_2), t^s\beta_1 + (1-t)^s\beta_2 + b(j_1, j_2, t)\right) \in \omega(h).$$

Thus, $\omega(h)$ is geodesic sub-(α, b, s)-preinvex set.
Next, let that $\omega(h)$ be geodesic sub-(α, b, s)-preinvex set, then

$$(j_1, h(j_1)), (j_2, h(j_2)) \in \omega(h),$$

where $j_1, j_2 \in Y$.
$\left(\exp_{j_2} t\alpha(j_1, j_2)\eta(j_1, j_2), t^s h(j_1) + (1-t)^s h(j_2) + b(j_1, j_2, \delta)\right) \in \omega(h)$ which shows that

$$h\left(\exp_{j_2} t\alpha(j_1, j_2)\eta(j_1, j_2)\right) \leq t^s h(j_1) + (1-t)^s h(j_2) + b(j_1, j_2, t).$$

Then h is geodesic sub-(α, b, s)-preinvex function. $\square$

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Proposition 2.11. *If Y_i is a family of geodesic sub-(α, b, s)-preinvex sets then the intersection $\cap_{i\in K} Y_i$ is also a geodesic sub-(α, b, s)-preinvex.*

Proof. Suppose that $(j_1, \beta_1), (j_2, \beta_2) \in \cap_{i\in K} Y_i$. Then $(j_1, \beta_1), (j_2, \beta_2) \in Y_i, \forall i \in K$

$$\left(\exp_{j_2} t\alpha(j_1, j_2)\eta(j_1, j_2), t^s\beta_1 + (1-t)^s\beta_2 + b(j_1, j_2, t)\right) \in Y_i, \forall i \in K.$$

$\Rightarrow$

$$\left(\exp_{j_2} t\alpha(j_1, j_2)\eta(j_1, j_2), t^s\beta_1 + (1-t)^s\beta_2 + b(j_1, j_2, t)\right) \in \cap_{i\in K} Y_i.$$

Thus, the intersection $\cap_{i\in K} Y_i$ is a geodesic sub-(α, b, s)-preinvex set. □

The above proposition indicates that the arbitrary intersection of geodesic sub-preinvex sets again is geodesic sub-preinvex. As per Theorem 2.10 and Proposition 2.11, the following proposition is holds:

Proposition 2.12. *If h_i are geodesic sub-(α, b, s)-preinvex functions then a function $H = \sup_{i\in K} h_i$ is also geodesic sub-(α, b, s)-preinvex function.*

Definition 2.13. *For a mapping $h : Y \longrightarrow \mathbb{R}$, if the next limit*

$$\lim_{t\longrightarrow 0} \frac{h\left(\exp_{j_2} t\alpha(j_1, j_2)\eta(j_1, j_2)\right) - h(j_2)}{t\|\alpha(j_1, j_2)\eta(j_1, j_2)\|},$$

exists, then h is called a (α, η)-differentiable mapping at $j_2 \in W$. Also, the (α, η)-differentiable mapping of h at j_2 is given by

$$d_{\alpha(j_1,j_2)\eta(j_1,j_2)} h(j_2) = \lim_{t\longrightarrow 0} \frac{h\left(\exp_{j_2} t\alpha(j_1, j_2)\eta(j_1, j_2)\right) - h(j_2)}{t\|\alpha(j_1, j_2)\eta(j_1, j_2)\|}.$$

Theorem 2.14. *Assume that Y is a G.α-invex set. If $h : Y \longrightarrow \mathbb{R}$ is (α, η)-differentiable geodesic sub-(α, b, s)-preinvex then following holds*

$$\begin{aligned} d_{\alpha(j_1,j_2)\eta(j_1,j_2)} h(j_2)\|\alpha(j_1, j_2)\eta(j_1, j_2)\| \leq\ & t^{s-1}h(j_1) \\ & + \frac{h(j_2)}{2t} + \lim_{t\longrightarrow 0_+} \frac{b(j_1, j_2, t)}{t}. \end{aligned}$$

Proof. Since h is a geodesic sub-(α, b, s)-preinvex, then it follows that

$$h\left(\exp_{j_2} t\alpha(j_1, j_2)\eta(j_1, j_2)\right) \leq t^s h(j_1) + (1-t)^s h(j_2) + b(j_1, j_2, t),$$

$\forall j_1, j_2 \in Y, t \in [0,1]$ and for some $s \in (0,1]$. Also, since h is (α,η)-differentiable, then

$$d_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2) = \lim_{t\longrightarrow 0} \frac{h\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right) - h(j_2)}{t\|\alpha(j_1,j_2)\eta(j_1,j_2)\|},$$

hence

$$\begin{aligned} h(j_2) + td_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| + O^2(t) &= h\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right) \\ &\leq t^s h(j_1) + (1-t)^s h(j_2) + b(j_1,j_2,t) \\ &\leq t^s h(j_1) + (1+t^s) h(j_2) + b(j_1,j_2,t). \end{aligned}$$

Then

$$td_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| + O^2(t) \leq t^s\left[h(j_1) + h(j_2)\right] + b(j_1,j_2,t).$$

Since $\lim\limits_{t\longrightarrow 0_+} \dfrac{b(j_1,j_2,t)}{t}$ is the maximum of $\frac{b(j_1,j_2,t)}{t} - \frac{O^2(t)}{t}$, then we obtain that

$$d_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| \leq t^{s-1}\left[h(j_1) + h(j_2)\right] + \lim_{t\longrightarrow 0_+} \frac{b(j_1,j_2,t)}{t}. \tag{2.2}$$

On the other hand, because of

$$\begin{aligned} h(j_2) &+ td_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| + O^2(t) \\ &\leq t^s h(j_1) + (1-t)^s h(j_2) + b(j_1,j_2,t) \\ &= t^s h(j_1) + (1-t)^s h(j_2) - t^s h(j_2) + t^s h(j_2) + b(j_1,j_2,t) \\ &= t^s\left(h(j_1) - h(j_2)\right) + b(j_1,j_2,t) + \left((1-t)^s + t^s\right) h(j_2). \end{aligned}$$

Hence, $\left((1-t)^s + t^s\right) \leq 2$ for $t \in [0,1]$ and for $s \in (0,1]$. Here h is a non-negative function, then we have

$$h(j_2) + td_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| + O^2(t) \leq t^s\left(h(j_1) - h(j_2)\right) + b(j_1,j_2,t) + 2h(j_2),$$

which implies that

$$td_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| + O^2(t) \leq t^s\left(h(j_1) - h(j_2)\right) + h(j_2) + b(j_1,j_2,t),$$

it follows that

$$d_{\alpha(j_1,j_2)\eta(j_1,j_2)}h(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| \leq t^{s-1}\left(h(j_1) - h(j_2)\right) + \frac{h(j_2)}{t} + \lim_{t\longrightarrow 0_+} \frac{b(j_1,j_2,t)}{t}. \tag{2.3}$$

Hence, by adding equations (2.2) and (2.3), the result is obtained. $\square$

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Theorem 2.15. *Assume that $g : Y \longrightarrow \mathbb{R}$ is (α,η)-differentiable geodesic sub-(α,b,s)-preinvex then*

$$d_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| \leq t^{s-1}\left(g(j_1)-g(j_2)\right)+\lim_{t\longrightarrow 0_+}\frac{b(j_1,j_2,t)}{t}.$$

Proof. If g is a geodesic sub-(α,b,s)-preinvex and also (α,η)-differentiable, then

$$d_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2) = \lim_{t\longrightarrow 0}\frac{g\left(\exp_{j_2} t\alpha(j_1,j_2)\eta(j_1,j_2)\right)-g(j_2)}{t\|\alpha(j_1,j_2)\eta(j_1,j_2)\|},$$

hence

$$g(j_2)+td_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\|+O^2(t) \leq t^s g(j_1)+(1-t)^s g(j_2)+b(j_1,j_2,t).$$

Since $t \in [0,1]$ and $s \in (0,1]$, then $(t^s+(1-t)^s \geqslant 1)$, which implies that

$$g(j_2)+td_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\|+O^2(t) \leq t^s g(j_1)+(1-t^s)g(j_2)+b(j_1,j_2,t),$$

$$td_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\|+O^2(t) \leq t^s\left(g(j_1)-g(j_2)\right)+b(j_1,j_2,t),$$

$$d_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| \leq t^{s-1}\left(g(j_1)-g(j_2)\right)+\lim_{t\longrightarrow 0_+}\frac{b(j_1,j_2,t)}{t}.$$

□

Next, we apply the above associated results to the non-linear programming. First, the following unconstraint problem (P) is considering

$$(P) : \min f(x), x \in Y.$$

Theorem 2.16. *Assume that $g : Y \longrightarrow \mathbb{R}$ is a non-negative (α,η)-differentiable and sub-(α,b,s). If $\tilde{j} \in Y$ and the inequality*

$$d_{\alpha(j,\tilde{j})\eta(j,\tilde{j})}f(\tilde{j})\|\alpha(j,\tilde{j})\eta(j,\tilde{j})\| \geqslant \frac{g(\tilde{j})}{t}+\lim_{t\longrightarrow 0_+}\frac{b(j,\tilde{j},t)}{t} \tag{2.4}$$

holds for $j \in Y$, $t \in (0,1]$ and $s \in (0,1]$, then $\tilde{j}$ is the optimal solution for problem (P) w.r.t. g on Y.

Proof. By using (2.3), then we have

$$d_{\alpha(j,\tilde{j})\eta(j,\tilde{j})}g(\tilde{j})\|\alpha(j,\tilde{j})\eta(j,\tilde{j})\| \leq t^{s-1}[g(j)-g(\tilde{j})]+\frac{g(\tilde{j})}{t}+\lim_{t\longrightarrow 0_+}\frac{b(j,\tilde{j},t)}{t},$$

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$$d_{\alpha(j,\tilde{j})\eta(j,\tilde{j})}g(\tilde{j})\|\alpha(j,\tilde{j})\eta(j,\tilde{j})\| - \frac{g(\tilde{j})}{t} - \lim_{t\longrightarrow 0_+}\frac{b(j,\tilde{j},t)}{t} \leq t^{s-1}[g(j)-g(\tilde{j})],$$

holds for $t \in (0,1]$ and $s \in (0,1]$ On the other hand,

$$d_{\alpha(j,\tilde{j})\eta(j,\tilde{j})}g(\tilde{j})\|\alpha(j,\tilde{j})\eta(j,\tilde{j})\| \geqslant \frac{g(\tilde{j})}{t} + \lim_{t\longrightarrow 0_+}\frac{b(j,\tilde{j},t)}{t},$$

then get $g(j)-g(\tilde{j}) \geqslant 0$. Hence, $\tilde{j}$ is the optimal solution of g on Y. □

Corollary 2.17. *Considering that $g : Y \longrightarrow \mathbb{R}$ is a strictly non-negative sub-(α, b, s)-preinvex. If $\tilde{j} \in Y$ satisfies (2.4), then $\tilde{j}$ is a unique optimal solution.*

Proof. From (2.3) if g is a strictly non-negative and sub-(α, b, s)-preinvex then

$$d_{\alpha(j_1,j_2)\eta(j_1,j_2)}g(j_2)\|\alpha(j_1,j_2)\eta(j_1,j_2)\| \leq t^{s-1}[g(j_1)-g(j_2)]+\frac{g(j_2)}{t}+\lim_{t\longrightarrow 0_+}\frac{b(j_1,j_2,t)}{t}.$$

Assume that $u_1, v_1 \in Y$ are two different optimal solutions for (P). Then $g(u_1) = g(v_1)$, hence

$$d_{\alpha(u_1,v_1)\eta(u_1,v_1)}g(v_1)\|\alpha(u_1,v_1)\eta(u_1,v_1)\|-\frac{g(v_1)}{t}-\lim_{t\longrightarrow 0_+}\frac{b(u_1,v_1,t)}{t} \leq t^{s-1}[g(u_1)-g(v_1)].$$

By applying 2.4, we get

$$t^{s-1}[g(u_1)-g(v_1)] > 0,$$

and since $g(u_1) = g(v_1)$, then it follows that $u_1 = v_1 = \tilde{j}$. Therefore, $\tilde{j}$ is the unique optimal solution of g on Y, then the corollary is proven. □

Next, the following non-linear programming problem will be given

$$(P_Y) : \min\{f(u) : u \in W, g_i(u) \leq 0, i \in I\}, I = \{1, 2, \cdots, m\}.$$

Now assume feasible set of (P_Y) is given by $M = \{u \in W : g_i(u) \leq 0, i \in I\}$, and f and g_i are all differentiable and W_1 is a non-empty set in W, then we have next theorem.

Theorem 2.18 (Karush-Kuhn-Tucher sufficient condition)**.** *Assume that $f : W \longrightarrow \mathbb{R}$ is a non-negative (α, η)-differentiable sub-(α, b, s)-preinvex, and $g_i : W \longrightarrow \mathbb{R}$ $(i \in I)$ are (α, η)-differentiable sub-(α, b, s)-preinvex and*

$$d_{\alpha(j_1,j^*)\eta(j_1,j^*)}f(j^*) + \sum_{i\in I} z_i d_{\alpha(j_1,j^*)\eta(j_1,j^*)}g_i(j^*) = 0, z_i g_i(j^*) = 0, \quad (2.5)$$

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where $j^* \in M$ and $z_i \geqslant 0 (i \in I)$.
If

$$\frac{f(j^*)}{t} + \lim_{t \longrightarrow 0_+} \frac{b(j_1, j^*, t)}{t} \leq -\sum_{i \in I} \lim_{t \longrightarrow 0_+} \frac{b(j_1, j^*, t)}{t}, \tag{2.6}$$

then j^* is an optimal solution of (P_Y).

Proof. Assume that $j_1 \in P_Y$, then

$$g_i(j_1) \leq 0 = g_i(j^*), i \in I(j^*) = \{i \in I : g_i(j^*) = 0\}.$$

Since g_i are $sub-(\alpha, b, s)$-preinvex and by Theorem 2.15, we have

$$d_{\alpha(j_1,j^*)\eta(j_1,j^*)} g_i(j^*) \|\alpha(j_1, j^*)\eta(j_1, j^*)\| \leq t^{s-1}[g_i(j_1) - g_i(j^*)] + \lim_{t \longrightarrow 0_+} \frac{b(j_1, j^*, t)}{t},$$

which means that

$$d_{\alpha(j_1,j^*)\eta(u,v^*)} g_i(j^*) \|\alpha(j_1, j^*)\eta(j_1, j^*)\| - \lim_{t \longrightarrow 0_+} \frac{b(j_1, j^*, t)}{t}, \leq t^{s-1}[g_i(j_1) - g_i(j^*)] \leq 0.$$

From 2.5, we get

$$\begin{aligned} &d_{\alpha(j_1,j^*)\eta(j_1,j^*)} f(j^*) \|\alpha(j_1, j^*)\eta(j_1, j^*)\| \\ &= -\sum_{i \in I} z_i d_{\alpha(j_1,j^*)\eta(j_1,j^*)} g_i(j^*) \|\alpha(j_1, j^*)\eta(j_1, j^*)\| \\ &= -\sum_{i \in I(j^*)} z_i d_{\alpha(j_1,j^*)\eta(j_1,j^*)} g_i(j^*) \|\alpha(j_1, j^*)\eta(j_1, j^*)\|. \end{aligned} \tag{2.7}$$

Using 2.6, then

$$\begin{aligned} &d_{\alpha(j_1,j^*)\eta(j_1,j^*)} f(j^*) \|\alpha(j_1, j^*)\eta(j_1, j^*)\| - \frac{f(j^*)}{t} - \lim_{t \longrightarrow 0_+} \frac{b(j_1, j^*, t)}{t} \\ &\geqslant -\sum_{i \in I(j^*)} z_i \left[d_{\alpha(j_1,j^*)\eta(j_1,j^*)} g_i(j^*) \|\alpha(j_1, j^*)\eta(j_1, j^*)\| - \lim_{t \longrightarrow 0_+} \frac{b(j_1, j^*, t)}{t} \right]. \end{aligned} \tag{2.8}$$

From 2.7 and 2.8, we have

$$d_{\alpha(j_1,j^*)\eta(j_1,j^*)} f(j^*) \|\alpha(j_1, j^*)\eta(j_1, j^*)\| - \frac{f(j^*)}{t} - \lim_{t \longrightarrow 0_+} \frac{b(j_1, v^*, t)}{t} \geqslant 0.$$

From Theorem 2.16, we get $f(j_1) - f(j^*) \geqslant 0, \forall j_1 \in M$. Hence, j^* is an optimal solution of the problem (P_Y). $\square$

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