

1 Article

# 2 A Study on Fatigue State Evaluation of Rail by the 3 Use of Ultrasonic Nonlinearity

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8 **Abstract:** Nonlinear ultrasonic testing has been accepted as a promising manner for evaluating  
9 material integrity in an early stage. Stress fatigue is the main threats to train safety, railways  
10 examinations for stress fatigue are more significant and necessary. A series of ultrasonic nonlinear  
11 wave experiments are conducted for rail specimens extracted from railhead with different degree  
12 of fatigue produced by three-point bent loading condition. The nonlinear parameter is the indicator  
13 of nonlinear waves for expressing the degree the fatigue. The experimental results show that the  
14 sensitivity of a third harmonic longitudinal wave is higher than second harmonic longitudinal wave  
15 testing. As the same time, collinear wave mixing shows strong relative with fatigue damages than  
16 a second longitudinal wave NDT method and provides more reliable results than third harmonic  
17 longitudinal waves nonlinear testing method.

18 **Keywords:** Railway, Fatigue, Ultrasonic, Nonlinear, Wave mixing

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## 20 1. Introduction

21 Rails are an important element in railway systems. Maintenance of railway systems today is  
22 more important and necessary as traffic volume increases. Various types of defects occur in rails, of  
23 which manufacturing defects, improper operation, and rolling contact fatigue are key components of  
24 the rail defect development process [1]. Railways are mass transit systems that will cause many  
25 casualties and massive economic losses. These accidents can arise from various deficiencies in the  
26 railway system, and in particular rail defects are a significant threat to the safe operation of rail  
27 transport. Most of the rail condition evaluation uses a method of measuring the position change such  
28 as external shape measurement or distortion using a laser. However, this method can measure only  
29 the surface shape and can't evaluate the internal defect and the state. Ultrasonic wave velocity and  
30 wave attenuation are the most basic material state evaluation methods. However, such a linear  
31 ultrasound parameter is insufficient to evaluate microscopic damage. Nonlinear techniques are  
32 useful for diagnosing defects at the initial stage and for micro-defects based on signal changes in the  
33 frequency domain [2]. Nonlinear ultrasonic test methods mainly use harmonic or quasi-harmonic  
34 components [3]. There is an ultrasonic mixing technique in which ultrasonic waves having different  
35 center frequencies are mixed to generate a new center frequency ultrasonic wave. Ultrasonic mixing  
36 techniques include bulk mixing [2-9] and guided wave mixing [10-13]. Ultrasonic nonlinearities  
37 measured through frequency spectrum analysis of received signals include system and material  
38 nonlinearities. Experimental systems such as wave generators, amplifiers, transducers and couplants  
39 in the experiments will result in strong system nonlinearities [2]. Therefore, reducing system non-  
40 linearity is necessary to obtain reliable results. However, Croxford et al. [9] noted that the wave  
41 mixing method is advantageous in reducing the nonlinearity of the system.

42 An initial theory study to utilize this ultrasonic mixing technique for material condition  
43 monitoring is described in Jones et al. [14] and Taylor et al. [15], and numerical simulations have been  
44 performed on experimental studies and ultrasonic mixing in various research areas [2-11]. The main  
45 threat of stress fatigue and rail safety is a kind of microstructural defect caused by internal stresses,

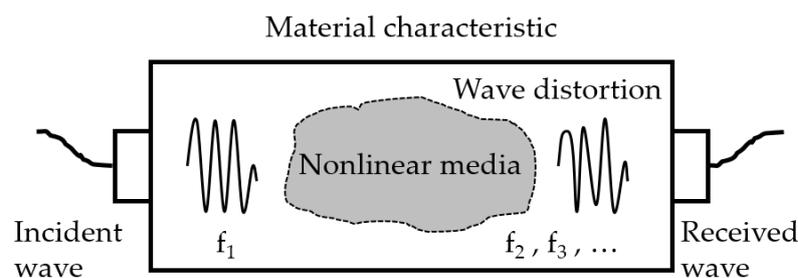
46 and the degree of fatigue defects can be represented by ultrasonic nonlinear parameters. The  
 47 relationship between nonlinearity and fatigue lifetime and fatigue life is expressed in [12, 13, 16].  
 48 There are several studies that have applied various nonlinear techniques to monitor the state of  
 49 mechanical materials. To compare and analyze the characteristics of each technique, many earlier  
 50 researchers conduct a nonlinear study using higher harmonic, sub-harmonic ultrasonic nonlinearity  
 51 and ultrasonic mixing technique. Also, it is necessary to study on early stage damage detection  
 52 technique to diagnose the fatigue condition of railway rail.

53 In this study, resonance conditions of high-order harmonic generation and collinear wave  
 54 mixing are expressed. Experimental studies performed based on a general nonlinear ultrasound  
 55 method, such as second and third-harmonic longitudinal wave tests and collinear wave mixing on  
 56 the fatigue rail specimen treated with a 3-point cyclic bending load. Finally, a comparison of three  
 57 nonlinear methods to test capability and accuracy is provided.

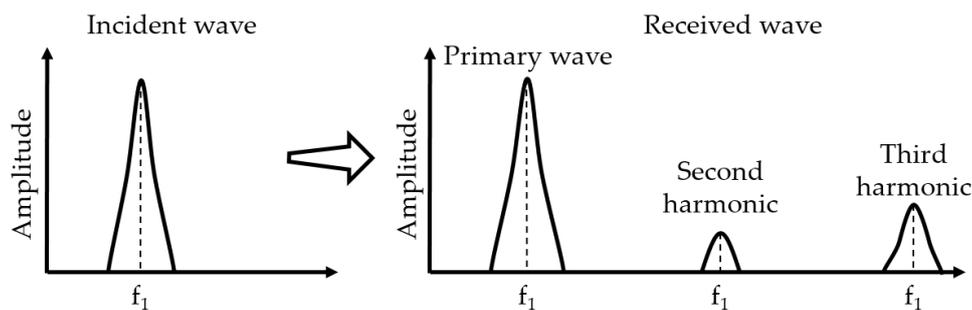
## 58 2. Nonlinear wave theory

### 59 2.1. High-order harmonic nonlinear waves

60 The physical effects of nonlinear ultrasonic technique are the interaction between the incident  
 61 wave and material leads to wave distortion (see Figure 1) and generation of the corresponding  
 62 higher-order harmonic waves (see Figure 2).  
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66 **Figure 1.** Wave propagation in nonlinear media



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68  
69 **Figure 2.** Spectrum changes through nonlinear media

70 Assume that a one-dimensional model, single frequency longitudinal wave propagates without  
 71 attenuation, for a small deformation, wave equation can be express by [17]:

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \quad (1)$$

72 where  $\rho$  is material density,  $\sigma$  is stress term and  $u$  is displacement vector along  $x$  the  
 73 direction. The constitutive equation of nonlinear medium can be described with elastic modulus  $E$ :

$$\sigma = Ef(\varepsilon) \quad (2)$$

74 Power series expansion  $f(\varepsilon)$  is applied to this equation.

$$\sigma = Ef(\varepsilon) = E\left(\varepsilon + \frac{1}{2}\beta_1\varepsilon^2 + \frac{1}{3}\beta_2\varepsilon^3 + \dots + \frac{1}{n}\beta_n\varepsilon^n\right) \approx E\left(\varepsilon + \frac{1}{2}\beta_1\varepsilon^2 + \frac{1}{3}\beta_2\varepsilon^3\right) \quad (3)$$

75 where  $\beta_n$  ( $n = 1, 2, 3, \dots$ ) is a factor indicating the order of nonlinear parameter of the material.  
76 Subsequently, substituting Eq. (3) into Eq. (1), one can be regrouped as:

$$C^2 \frac{\partial^2 u}{\partial t^2} = \frac{\partial f(\varepsilon)}{\partial x} = \frac{\partial\left(\varepsilon + \frac{1}{2}\beta_1\varepsilon^2 + \frac{1}{3}\beta_2\varepsilon^3\right)}{\partial x} = \frac{\partial^2 u}{\partial x^2} \left[1 + \beta_1 \frac{\partial u}{\partial x} + \beta_2 \left(\frac{\partial u}{\partial x}\right)^2\right] \quad (4)$$

77 In order to solve this problem, a perturbation method is applied. The displacement  $u$  is  
78 assumed as:

$$u(x, y) = u_0(x, t) + xu_1(x, t) + x^2u_2(x, t) + \dots + x^nu_n(x, t) \quad (5)$$

79 By equating the terms with the same order, the following equation can be derived:

$$u(x, t) = A_1 \cos(kx - \omega t) - \frac{\beta_1}{8} k^2 A_1^2 x \cos 2(kx - \omega t) + \frac{\beta_2}{24} k^3 A_1^3 x [\cos 3(kx - \omega t) + 3 \cos(kx - \omega t)] \quad (6)$$

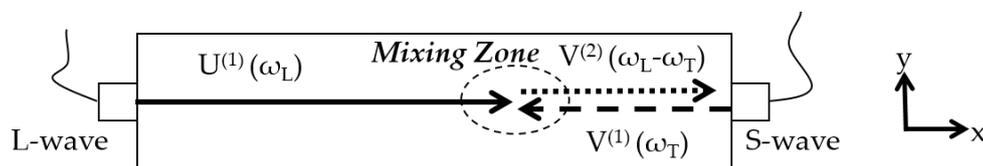
80 where  $\omega$  is angular frequency, and  $k$  is wavenumber. So the amplitude of the second harmonic  
81 propagation in the material,  $A_2$ , relative with  $\cos 2(kx - \omega t)$  the term, is  $\beta_1 k^2 A_1^2 x / 8$ . Similarly, third  
82 harmonic propagation in the material,  $A_3$ , relative with  $\cos 3(kx - \omega t)$  the term, is  $\beta_2 k^3 A_1^3 x / 24$ .  
83 Regrouping those two relations, nonlinear parameters can be expressed by amplitude ratio directly:

$$\beta_1 = \frac{8}{k^2 x} \frac{A_2}{A_1^2}, \quad \beta_2 = \frac{24}{k^3 x} \frac{A_3}{A_1^3} \quad (7)$$

84 Obviously,  $\beta_1$  and  $\beta_2$  are in proportion to  $A_2/A_1^2$  and  $A_3/A_1^3$ , respectively.

## 85 2.2. Collinear wave mixing

86 The method is based on the fact that a resonant wave might be generated by two incident waves  
87 if resonant conditions are satisfied. The acoustic nonlinear parameter  $\beta_T$  is the amplitude ratio of  
88 primary incident waves and receiving the resonant wave introduced from [5]. The dominant wave  
89 mixing technique is that experimental nonlinearity contains less system nonlinearity than  
90 conventional higher-harmonic longitudinal ultrasonic testing. Because the received wave obtained  
91 from the wave mixing zone of the internal material directly and the subtracting frequency term can  
92 eliminate system nonlinearity produced by two incident harmonic waves through measurement  
93 system. Hence, the total system nonlinearity is reduced and reliable experimental results are  
94 provided [8, 9]. The mixed resonant shear wave  $V^{(2)}$  could be generated by a pair of primary  
95 longitudinal wave  $U^{(1)}$  and shear wave  $V^{(1)}$  excited in the opposite direction. Here, the resonant  
96 shear wave propagation direction is contrary to primary incident shear wave, which called as two-  
97 way wave mixing [4, 5] see Figure 3.  
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99  
100 **Figure 3.** Schematic of collinear wave mixing

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The two-way wave mixing with subtracting frequency term is conducted:

$$L(\omega_L) + T(\omega_T) \rightarrow T(\omega_L - \omega_T) \quad (8)$$

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And, the primary excited longitudinal and shear waves are defined by:

$$U^{(1)} = A \sin(k_L x - \omega_L t) \quad (9)$$

$$V^{(1)} = B \sin(k_T x + \omega_T t) \quad (10)$$

104 Nonlinear wave equations are used for the description of waves propagating in nonlinear media,  
105 Goldberg expressed the nonlinear wave equations as below.

$$v_{tt} - C_L^2 u_{xx} = (3C_L^2 + C_{111}/\rho)u_x u_{xx} + (C_L^2 + C_{166}/\rho)(v_x v_{xx} + w_x w_{xx}) \quad (11)$$

$$v_{tt} - C_T^2 v_{xx} = (C_L^2 + C_{166}/\rho)(u_x v_{xx} + v_x u_{xx}) \quad (12)$$

$$w_{tt} - C_T^2 w_{xx} = (C_L^2 + C_{166}/\rho)(u_x w_{xx} + w_x u_{xx}) \quad (13)$$

106 The nonlinearity parameter is defined as  $\beta_T = -(C_L^2 + C_{166}/\rho)$ . Substituting primary  
107 longitudinal and shear waves' equations into the governing equation Eq.(11)~Eq.(13), one can be  
108 derived:

$$v_{tt}^{(2)} - C_T^2 v_{aa}^{(2)} = \frac{1}{2} \beta_T A B k_L k_T (k_L - k_T) \sin[(k_L - k_T)x - (\omega_L + \omega_T)t] \\ + \frac{1}{2} \beta_T A B k_L k_T (k_L + k_T) \sin[(k_L + k_T)x - (\omega_L - \omega_T)t] \quad (14)$$

109 Assume that the receiving resonant shear is,

$$V^{(2)} = C \sin((k_L + k_T)x - (\omega_L - \omega_T)t) \quad (15)$$

110 Where  $C$  is the amplitude of the mixed wave demonstrated by:

$$C = \frac{1}{2} \beta_T A B \frac{\omega_T (C_T \omega_L + C_L \omega_T)}{(C_L + C_T)(C_T \omega_L + 2C_L \omega_T - C_L \omega_L)} \quad (16)$$

111 Obviously, amplitude  $C$  in Eq. (16) is in proportion to the primary waves' frequencies. For the  
112 resonant condition, meaning that the denominator equals zero and the mixed wave amplitude  
113 approaches to infinity. Therefore, the frequency relation of the excited waves can be described as  
114 follows:

$$\frac{\omega_T}{\omega_L} = \frac{C_L - C_T}{2C_L} \quad (17)$$

### 115 3. Experimental setup

116 KR 60 rails are widely used in Korean transport systems. The material properties are listed in  
117 Table 1 [18]. The specimen used in this study is a rectangular bar of 350 x 30 x 60 mm in rail head.  
118 Three-point bending fatigue tests were performed on the specimens. The load was 12 ton and the  
119 fatigue cycle was 2.5 Hz. In order to measure the fatigue life of the specimen, the fatigue cycle  
120 measured using two specimens up to failure was about 51,700 cycles on average. Based on the fatigue  
121 life curve as depicted in Figure 4, fatigue test specimens of 60% and 80% were prepared based on the  
122 life span of undamaged test specimens and fatigued specimen.

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124

**Table 1.** Chemical composition and material properties of KR 60

Chemical composition (%)					Mechanical properties		
Carbon	Silicon	Manganese	Phosphorus	Sulfur	Tensile strength (MPa)	Elongation (%)	Hardness (HBW)
0.63 ~ 0.75	0.15 ~ 0.3	0.7 ~ 1.10	<0.03	<0.025	>880	>10	260 ~ 300

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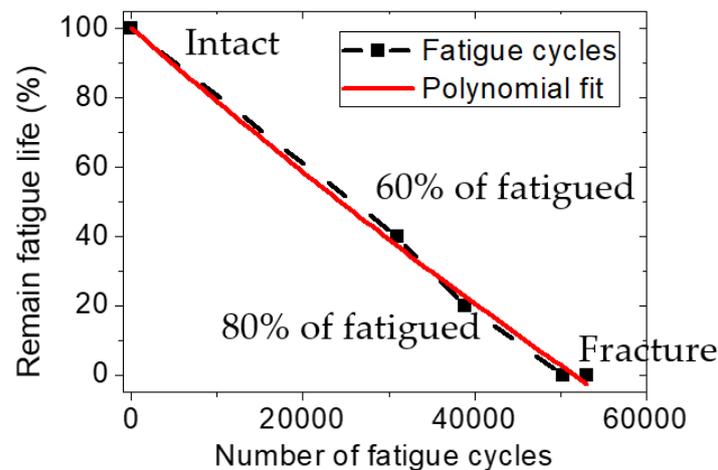
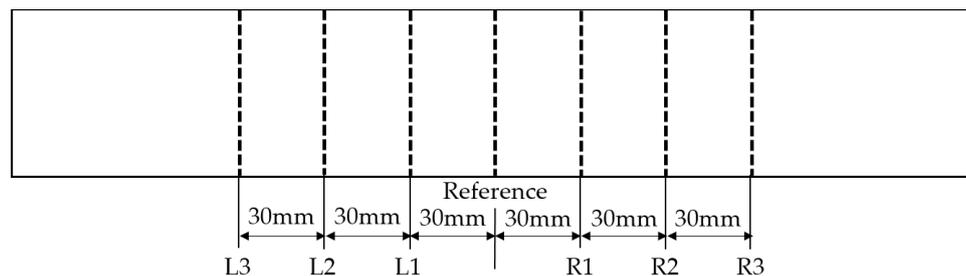


Figure 4. Fatigue life of KR 60 rail specimens

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High-order harmonic longitudinal wave and wave mixing tests are applied to seven detection points defined at the same distance in the central region (see Figure 5). Use the high-power tone burst system to generate narrowband signals at seven detection points. The transducer is mounted on the specimen with couplants and consistent pressure to ensure constant contact conditions. Signal processing uses the Fast Fourier Transform (FFT) to obtain the spectrogram.

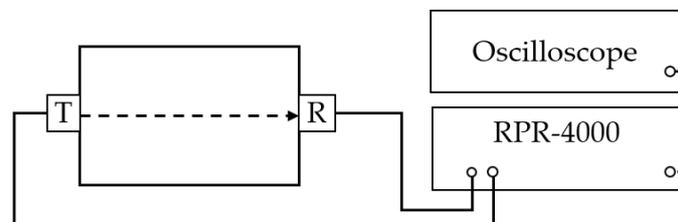


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Figure 5. Measurement points of the specimen

### 137 3.1. High-order harmonic longitudinal wave nonlinear testing

138 The experimental setup for measuring higher order harmonic nonlinearities is shown in Figure  
139 6. High-voltage tone burst system calls such as the RPR-4000 (Ritec Inc.) have been working to  
140 generate 20 cycles with the PZT sensor signal at 5 MHz. The frequency bandwidth can be easily  
141 limited by the window tone burst signal generated by the measurement system.  
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Figure 6. Schematic of higher harmonic longitudinal wave nonlinear testing

146 To detect the corresponding higher-order harmonic frequency components primarily, the  
147 bandwidth of the receiving transducer should cover the frequency range. In the spectrum of the  
148 signal, it can be found that second harmonic wave at the quadratic frequency at 10 MHz and third  
149 harmonic wave at the triple frequency at 15 MHz. The sensor conditions are listed in Table 2.

150

**Table 2.** Experimental conditions for higher harmonic measurement technique

Transducer	Wave type	Frequency
Sender	Primary wave	5 MHz
Receiver	Second harmonic wave	10 MHz
	Third harmonic wave	15 MHz

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152 *3.2. Wave mixing nonlinear testing*

153 Once satisfying the resonant condition, the excited frequency relation can be derived by  
 154 substituting wave speeds terms into Eq. 17. Where, theoretical longitudinal and transverse waves'  
 155 speeds are  $C_L = 5856.4m/s$  and  $C_T = 3130.4m/s$ , respectively, calculated from [19, 20]. One can be  
 156 expressed by:

$$\frac{\omega_T}{\omega_L} = \frac{C_L - C_T}{2C_L} = \frac{5856.4 - 3130.4}{2 \times 5856.4} \approx 0.233 \quad (18)$$

157 One can be simplified as,

$$U^{(1)}(\omega_L) - V^{(1)}(0.233\omega_T) = V^{(1)}(0.767\omega_L) \quad (19)$$

158 The schematic of wave mixing experimental setup is shown in Figure 7. Two tone burst devices  
 159 RPR-4000 are connected to each other and synchronized for transmitting and receiving waves. The  
 160 oscilloscope is used for showing and validating the received signals. In order to fully collect data, a  
 161 sampling rate of 125 MHz was chosen.

162 A 2.5 MHz excited transverse wave and a 10 MHz excited longitudinal wave were meet in the  
 163 internal specimen to produce a mixed ultrasonic wave at 7.5 MHz. Sensors information are listed in  
 164 Table 3.

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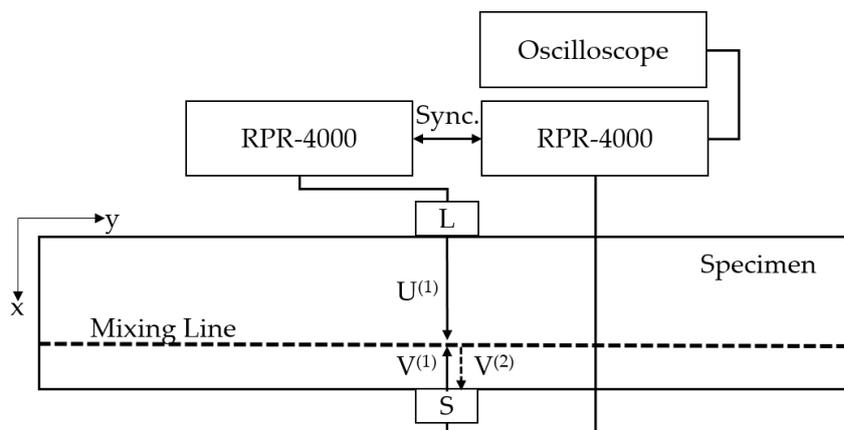
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**Table 3.** Experimental conditions for wave mixing technique

Transducer	Sensor Type	Frequency
Sender	L-wave	10 MHz
	T-wave	2.5 MHz
Receiver	T-wave	7.5 MHz

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**Figure 7.** Schematic of wave mixing experimental setup

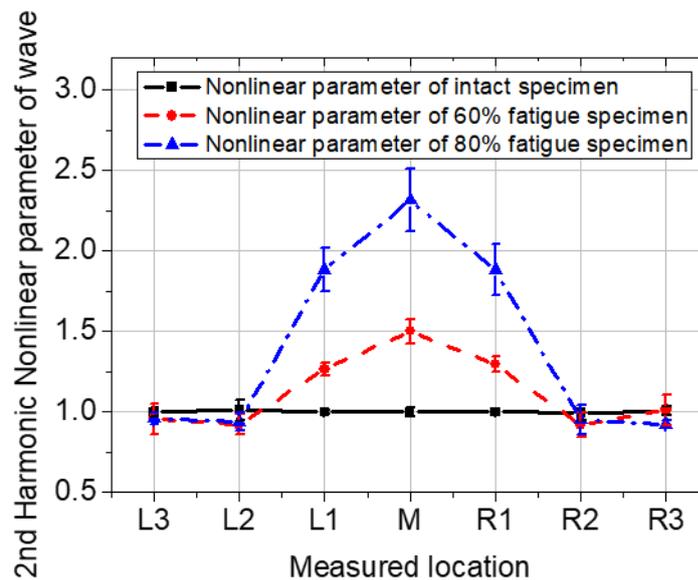
172 It should be noticed that due to the difference in wave speed, the mixing line is close to the  
 173 bottom shear transducer. The mixing time  $T$  is  $6.676\mu s$ , which can be calculated directly by the  
 174 equation  $(C_T + C_L)T = L$ , where,  $L$  is the height of the specimen equal to 60mm. Obviously, the

175 distance between mixing lines and the shear transducer is about 20mm as depicted in in Figure 7.  
 176 Certainly, it is feasible for defining a mixing line in the random position with time delay.

#### 177 4. Experimental results

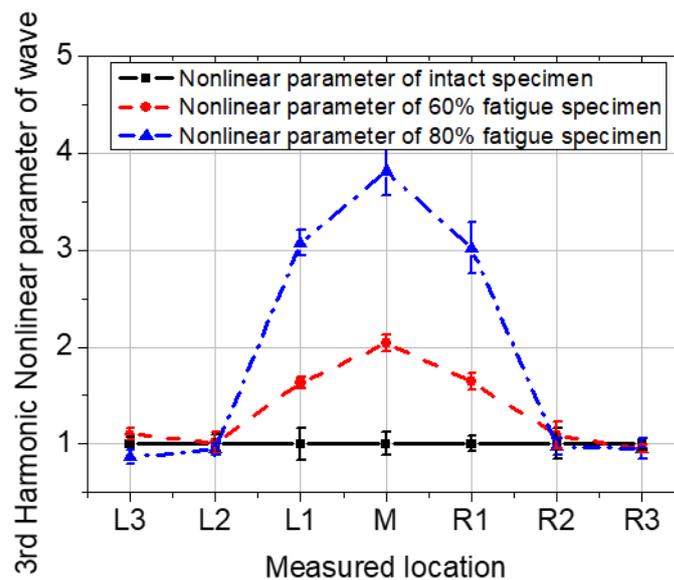
##### 178 4.1. High-order harmonic longitudinal wave nonlinear testing

179 The signal of higher-order harmonic longitudinal waves can be received by receiving transducers and stored  
 180 by RPR-4000 equipment. After signal processing, the corresponding spectrum can be easy obtained. The second  
 181 and third nonlinear value can be described by amplitude ratio of second harmonic and third harmonic wave  
 182 amplitude to primary wave amplitude, respectively. Relative second and third harmonic nonlinear parameter  
 183 for application is defined by  $\beta_1 \propto A_2/A_1^2$  and  $\beta_2 \propto A_3/A_1^3$ , respectively. The normalized nonlinear parameter  
 184 value at each detecting point is shown in Figure 8 and Figure 9 for second and third order harmonic longitudinal  
 185 wave, respectively.



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Figure 8. Second harmonic wave for measuring nonlinear parameter at various positions

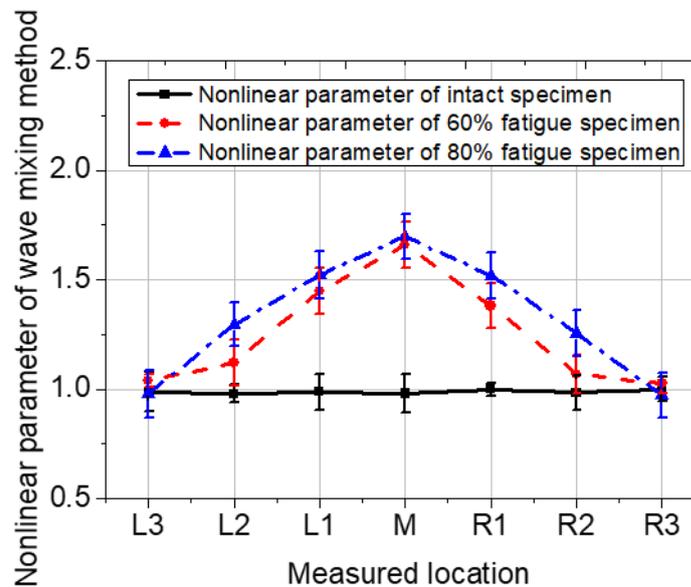


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Figure 9. Third harmonic wave for measuring nonlinear parameter at various positions

#### 191 4.2. Collinear wave mixing nonlinear testing

192 Butterworth filter is applied for signal processing. Finally, the spectrum of collinear wave mixing can be  
 193 obtained. In order to measure the material nonlinearity of the rail specimens, the amplitude of the new generated  
 194 shear wave  $V^{(2)}$  was normalized to the product of the two incident amplitudes  $U^{(1)}$  and  $V^{(1)}$  measured in volts  
 195 [9]. Figure 10 shows the normalized nonlinear parameter values of the detecting points [21].



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Figure 10. Wave mixing for measuring nonlinear parameter at various positions

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#### 199 5. Conclusions

200 A feasibility study is carried out for high-dimensional longitudinal tests and simulant-line mixed nonlinear  
 201 tests to demonstrate fatigue rail specimens. Three different stress fatigue test specimens are extracted from the  
 202 rail head and subjected to three point cyclic loading. The theoretical analysis focuses on the phenomenon of  
 203 wave distortion and the generation of new harmonics when waves propagate in a nonlinear medium.  
 204 Experiments are performed on fatigue test specimens based on high-order longitudinal test and wave mixing  
 205 test methods. The value of the non-linear parameter at the detection point is used to indicate the degree of  
 206 fatigue. Nonlinear trends are evident by forcing the value of a nonlinear parameter normalized to an undamaged  
 207 specimen. In general, the nonlinearity is proportional to the stress fatigue.

208 The nonlinear parameter values of the third harmonic and second harmonic in the central region are about 3  
 209 ~ 4 times and 1.5 ~ 2.5 times higher than the uninjured sample, respectively. In other words, the third harmonic  
 210 longitudinal nonlinear test is more sensitive than the second harmonic longitudinal nonlinear test. It should be  
 211 noted that higher order harmonics represent low energy transfer reflected by the experimental operation and the  
 212 value of the wave amplitude which is heavily influenced by the experimental system. Therefore, the accuracy  
 213 of the third harmonic nonlinearity can't be guaranteed.

214 The nonlinearity of the wave mixing method for these fatigue test specimens is much clearer and more  
 215 remarkable than the second harmonic longitudinal test method, and the value of the nonlinear parameter at the  
 216 center portion is lower than that described later. Obviously, wave mixing is more accurate by eliminating  
 217 nonlinearities in the system.

218 This study validates the probabilities of nonlinear techniques for fatigue failures and provides quantitative  
219 comparisons between three common nonlinear techniques. High-order harmonics and wave mixing Wave is  
220 sensitive to microstructural defects, so you can draw conclusions, but the results of wave mixing are more  
221 reliable.

222

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224 this research. Jaesun Lee designed the methodology, implemented the theoretical derivation and gave  
225 guidance and helped to improve the quality of the manuscript.

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