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Deflection angle of photon through dark matter by black holes/wormholes using the Gauss-Bonnet theorem

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Maxwell's fish eye has been known to be a perfect lens in optics. In this letter, using the Gibbons-Werner method, namely Gauss-Bonnet theorem on optical geometry of black hole, we extend the calculation of the weak gravitational lensing within the Maxwell's fisheye as a perfect lensing in medium composed of an isotropic refractive index that near-field information can be obtained from far-field distances. Moreover, these results provide an excellent tool to observe compact massive object by weak gravitational lensing within the dark matter medium and to understand the nature of the dark matter that may effect the gravitational waves. Moreover, we show that Gauss-Bonnet theorem is a global effect and this method can be used as a new tool on any optical geometry of compact objects in dark matter medium.

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Black holes are an essential component of our universe and one of the most important findings in astrophysics is that when stars die they can collapse to extremely small objects. Black holes provide an important tool for probing and testing the fundamental laws of the universe. Recently gravitational waves from black holes and neutron star mergers have been detected [1]. On the other hand, black holes may hint the nature of the quantum gravity at small scales that change the area law of entropy. Observationally, the quantum gravity is far from understood, though theoretically has seen tremendous progress, in few years the event horizon telescope may provide some information about it [2–4].

In 1854 Maxwell presented the solution to a mathematical problem relating to the passage of rays through a sphere of variable refractive index and he noted that the potential existence of a medium of this kind retaining exceptional optical properties [5]. This is similar to reflection of the crystalline lens in fish. The absolute tool of physics especially in optics is Maxwell's fisheye (MFE), namely the condition that all light rays form circular trajectories. Nonetheless it was an remarkable effort to visualize that a lens whose refractive index enlarged radially towards a point could form perfect images, that make possible to observe a gradient refractive index material [7].

Then Luneburg discovered that the ray propagation is equivalent to ray propagation on a homogeneous sphere with unit radius and unit refractive index within geometrical optics [6]. This show that the imaging of variants which has been applied to microwave devices and the fisheye lenses in photography that form an extreme wide-angle image, almost hemispherical in coverage. In 2009 Leonhardt showed that MFE is also good for waves and enabling in producing super-resolution imaging with perfect lensing, which requires negative refractive index materials, that begin a hot debate and rich area of research to explore [8–10]. The MFE happens when all light rays arising from any point within converge at its conjugate which means that power released from a source can only be fully absorbed at its image point, namely perfect imaging. There has been a rapid increase in the importance of the perfect imaging in theoretical and experimental optics [11–13].

Fermat's principle said that light rays always follow extremal optical paths with a path length measured by the refractive index n geodesics. The formula of MFE indicates the interesting possibility that rays generates perfect image in a black hole region. The refractive index depends only on the distance r from the origin. The equation for the MFE is [11, 12]

$$n = \frac{n_0}{1+(r/a)^2}. \tag{1}$$

Another useful tool of the astronomy and astrophysics is a gravitational lensing [14], that the light rays from distant stars and galaxies are deflected by a planet, a black hole, or dark matter [15, 16] . The detection of the dark matter filaments [17] using the weak gravitational lensing is very hot topic because it can help us to understand the large-scale structure in the Universe [18]. Furthermore, to build a sky maps (the index of refraction of the entire visible universe) there are ongoing research on the observation of cosmological weak lensing effect on temperature fluctuations in the Cosmic Microwave Background (CMB) [19]. Theoretical point of view, new methods have been proposed to calculated the deflection angle. One of them is the Gauss-Bonnet

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theorem (GBT) which is first proposed by Gibbons and Werner, using the optical geometry [20–22]. The deflection angle is seen as topologically global effect that can be calculated by integrating the Gaussian curvature of the optical metric outwards from the light ray using the following equation: [20, 21]

$$\hat{\alpha} = -\int_0^\pi \int_r^\infty K \mathrm{d}A. \tag{2}$$

Since a unique perspective of the Gibbons and Werner's paper on weak gravitational lensing by GBT, this method has been applied in various cases [23–43].

Dark matter is the mystery of the universe with the 27% of the total massenergy of universe [44]. We only detect them from its gravitational interactions and it is only known that dark matter is non-baryonic, non-relativistic (or cold), and it has weak non-gravitational interactions. There are many dark matter candidates such as weakly interacting massive particles (WIMPs), superWIMPs, axions, and sterile neutrinos [45]. Furthermore, it is proposed the composite model such as dark atom model which we investigate here using the deflection of light through it. Dark matter, although suppressed, generally has electromagnetic interactions [46] so that the medium of dark matter should have some optical properties where the travelling photon feel it because of the a frequency-dependent refractory index. The refraction index regulates the speed at which a wave is propagated via a medium. On the other hand, the particles of dark matter has not got electric charge but they can couple to other particles which has virtual electromagnetic charge that can also couple to photons [47–50]. For finding the amplitude of dark matter annihilation into two photons, first one should calculate the scattering amplitude which provides the crossing symmetry that provide the indirect detection due to the final states of two-photon. Then from the unity, once can also obtain the index of refraction of light where the real part is related with the speed of propagation.

To investigate the gravitational lensing through the dark matter, we consider propagation effects for the case that particle of dark matter (warm thermal relics or axion-like particles) has low mass, whose number density is larger than ordinary matter. Generically, dark matter interact with photons (if only through quantum fluctuations), resulting in a refractive index. The relationship between the refractive index and the forward Compton amplitude at relatively low photon energies [46] ($\mathcal{M}_{\text{fwd}} \sim -\varepsilon^2 e^2$) is

$$n = 1 + \frac{\rho}{4m_{dm}^2 \omega^2} \mathcal{M}_{\text{fwd}} \tag{3}$$

where ω is the measured photon frequency, and $\rho = 1.1 \times 10^6 GeV/cm^3$ the present day dark matter density [46]. Neglecting spin, the amplitude will be a real and even function of ω (for photon energies below the inelastic threshold); additionally, the coefficients of the $O(\omega^{2n})$ terms are positive and the spin dependent interactions can lead to odd powers in the expansion about ω. There presence could inform of the spin of dark matter. Hence, the refractive index becomes [46]

$$n = 1 + \frac{\rho}{4m_{dm}^2} \left[\frac{A}{\omega^2} + B + C\omega^2 + \mathcal{O}\left(\omega^4\right) \right] \tag{4}$$

To do so, we suppose that the photons can be deflected through the dark matter due to the dispersive effects. Hence, we use the index of refraction $n(\omega)$, which is manipulated by the scattering amplitude of the light and dark-matter in the forward [46].

Main motivation of this letter is to shed light on unexpected features of spacetimes according to MFE and deriving the deflection angle using the Gauss-Bonnet theorem in weak limits for dark matter medium. We show that this kind perfect tool can be used in general relativity within the geometry of light by matter of suitable distribution choice, hence creating an interesting light deflector. We also investigate the effect of varying parameters on the refraction index of medium, which was not covered in previous studies.

I. MAXWELL FISHEYE EFFECT OF THE SCHWARZSCHILD BLACK HOLE USING THE GAUSS-BONNET THEOREM

In this section, first we shall first describe the black hole solution in a static and spherically symmetric spacetime. Then we apply the MFE within the GBT to calculate the weak gravitational lensing.

The Schwarzschild black hole spacetime reads

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{5}$$

with the metric functions

$$f(r) = g(r)^{-1} = 1 - \frac{2M}{r}. \tag{6}$$

Analysis of the geodesics equation, the ray equation is calculated by

$$\varphi = \int \frac{b\sqrt{g(r)}\,\mathrm{dr}}{r^2\sqrt{\frac{1}{f(r)} - \frac{b^2}{r^2}}}, \tag{7}$$

where b is the impact parameter of the unperturbed photon.

Our universe is homogeneous and isotropic on large scales. Now we consider isotropic coordinates which are non-singular at the horizon and the time direction is a Killing vector. Moreover, time slices become Euclidean with a conformal factor and one can calculate the index of refraction, n, of light rays around the black hole. Other important feature of the isotropic coordinates is

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that they satisfy Landau's condition of the coordinate clock synchronization

$$\frac{\partial}{\partial x_j}\left(-\frac{g_{0i}}{g_{00}}\right)=\frac{\partial}{\partial x_i}\left(-\frac{g_{0j}}{g_{00}}\right)(i,j=1,2,3). \quad (8)$$

Using the following transformation

$$r=\rho\left(1+\frac{M}{2\rho}\right)^2, \quad (9)$$

the Schwarzschild black hole rewritten in isotropic coordinates (where ρ is isotropic radial coordinate) [41]

$$ds^2=-F(\rho)dt^2+G(\rho)(d\rho^2+\rho^2d\Omega^2), \quad (10)$$

with

$$F(\rho)=\left(\frac{\rho-\frac{M}{2}}{\rho+\frac{M}{2}}\right)^2, \text{and } G(\rho)=\left(\frac{\rho+\frac{M}{2}}{\rho}\right)^4. \quad (11)$$

The metric becomes nonsingular at the horizon $r=2M$. It can be also written in Fermat form of metric:

$$ds^2=F(\rho)[-dt^2+n(\rho)^2(d\rho^2+\rho^2d\Omega^2], \quad (12)$$

with the index of refraction $n(\rho)=\frac{c}{v(\rho)=\frac{d\rho}{dt}}$. For the Schwarzschild black hole medium, the refractive index reads

$$n=\frac{\left(1+\frac{M}{\rho}\right)^3}{\left(1-\frac{M}{\rho}\right)}, \quad (13)$$

and it can be approximated for large $\rho\gg M$

$$n\approx 1+\frac{4M}{\rho}. \quad (14)$$

Now the ray equation becomes

$$\varphi=\int\frac{bd\rho}{r^2\sqrt{n^2-\frac{b^2}{\rho^2}}}. \quad (15)$$

To discuss the gravitational lensing and extract information of MFE, the GBT will be used instead of the null geodesics method. The GBT is calculated using the negative Gauss curvature of the optical metric.

A. Case 1:

Let us start from the constant case for the medium $n(r)$ as refractive index:

$$n_m=n_0, \quad (16)$$

where n_0 is a constant refractive index of the medium, and here we consider the GBT to obtain the deflection angle in a medium in weak field limits.

First lets write the optical Schwarzschild spacetime in an equatorial plane [41]:

$$d\sigma^2=\frac{n_m^2}{f(r)}[g(r)dr^2+r^2d\varphi^2], \quad (17)$$

and one can calculate

$$\left.\frac{d\sigma}{d\varphi}\right|_{C_R}=n_m\left(\frac{r^3}{r-2M}\right)^{1/2}. \quad (18)$$

Then we calculate the Gaussian optical curvature

$$\mathcal{K}=-2\frac{M}{{n_0}^2r^3}+3\frac{M^2}{{n_0}^2r^4}+O(M^3), \quad (19)$$

where is everywhere negative that gives a universal property of black hole metrics [21].

It reduces to this form at linear order of M:

$$\mathcal{K}\approx -2\frac{M}{{n_0}^2r^3}. \quad (20)$$

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This result will be used to evaluate the deflection angle using a non-singular domain outside the light ray ($\mathcal{D}_r$, with boundary $\partial\mathcal{D}_r=\gamma_{\tilde{g}}\cup C_r$) [20]:

$$\iint_{\mathcal{D}_r}K\,\mathrm{d}S+\oint_{\partial\mathcal{D}_r}\kappa\,\mathrm{d}t+\sum_i\theta_i=2\pi\chi(\mathcal{D}_r), \quad (21)$$

where κ stands for the geodesic curvature $\kappa=\tilde{g}\left(\nabla_{\dot{\gamma}}\dot{\gamma},\ddot{\gamma}\right)$ and K is Gaussian optical curvature, with the exterior angles θ_i=($\theta_\mathcal{O}$, $\theta_\mathcal{S}$) and the Euler characteristic number $\chi(\mathcal{D}_r)=1$. At weak limits, ($\rho\to\infty$), $\theta_O+\theta_S\to\pi$. Then the GBT reduces to

$$\iint_{\mathcal{D}_r}K\,\mathrm{d}S+\oint_{C_r}\kappa\,\mathrm{d}t\overset{\rho\to\infty}{=}\iint_{\mathcal{D}_\infty}K\,\mathrm{d}S+\int_0^{\pi+\hat{\alpha}}\mathrm{d}\varphi=\pi. \quad (22)$$

For the geodesics property of $\gamma_{\tilde{g}}$, geodesic curvature vanishes $\kappa(\gamma_{\tilde{g}})=0$, and we have

$$\kappa(C_r)=|\nabla_{\dot{C}_r}\dot{C}_r|, \quad (23)$$

with $C_r:=\rho(\varphi)=r=$constant. The GBT becomes

$$\lim_{R\to\infty}\int_0^{\pi+\alpha}\left.\left[\kappa_g\frac{d\sigma}{d\varphi}\right]\right|_{C_R}d\varphi=\pi-\lim_{R\to\infty}\int\int_{D_R}\mathcal{K}dS. \quad (24)$$

where for very large radial distance

$$\kappa(C_R)\mathrm{d}t=\mathrm{d}\,\varphi. \quad (25)$$

Therefore, as expected for this number density profile and physical metric (which imply that the optical metric is asymptotically Euclidean) we corroborate that

$$\lim_{R\to\infty}\kappa_g\left.\frac{d\sigma}{d\varphi}\right|_{C_R}=1. \quad (26)$$

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At linear order in M, it follows using (24) in the limit $R \to \infty$, and taking the geodesic curve γ_p approximated by its flat Euclidean version parametrized as $r = b/\sin\varphi$, with b representing the im pact parameter in the physical spacetime that

$$\alpha = -\lim_{R\to\infty}\int_0^{\pi}\int_{\frac{b}{\sin\varphi}}^{R} \mathcal{K} dS. \tag{27}$$

After nontrivial calculation, we calculate the deflection angle of the Schwarzschild black hole in medium for the leading order terms is

$$\alpha = 4\,\frac{M}{n_0\,b}, \tag{28}$$

which agrees with the well-known results in the limit at which its presence is negligible ($n_0 = 1$) this expression reduces to the known vacuum formula $\alpha = 4\,\frac{M}{b}$. So that GBT provides a global and even topological effect. This method as a quantitative tool can be used in any metrics.

B. Case 2:

Now we apply the different model of medium which is MFE that is the archetype of the absolute optical instrument with the profile [9]

$$n = \frac{z_0}{1+z^2}, \tag{29}$$

where z_0 and z are a constant.

The Gaussian curvature of the optical metric approximating in leading orders is everywhere negative and found as :

$$\mathcal{K} = -2\,\frac{\left(z^2+1\right)^2 M}{{z_0}^2 r^3} + O(M^3), \tag{30}$$

Then using the same method we calculate the deflection angle as follows:

$$\hat{\alpha} \simeq 4\,\frac{M z^2}{z_0\,b} + 4\,\frac{M}{z_0\,b}\,. \tag{31}$$

At the $z = 0$ and $z_0 = 1$, it reduces to exact Schwarzschild case.

C. Case 3:

The refractive index for the dark matter medium [46]

$$n(\omega) = 1 + \beta A_0 + A_2\omega^2 \tag{32}$$

where $\beta = \frac{\rho_0}{4m^2\omega^2}$ and ρ_0 the mass density of the scatterers and $A_0 = -2\varepsilon^2 e^2$ and $A_{2j} \geq 0$.

The terms in $\mathcal{O}\left(\omega^2\right)$ and higher are related to the polarizability of the dark-matter candidate. Note that, the order of ω^{-2} is due to the charged dark matter candidate and ω^2 for a neutral dark matter candidate. Moreover, there is a linear terms such as only ω when parity and charge-parity asymmetries are present.

The Gaussian curvature is obtained as:

$$\mathcal{K} \approx -2\,\frac{M}{\left(A_2\,\omega^2 + \beta\,A_0 + 1\right)^2 r^3} + O(M^3) \tag{33}$$

The deflection angle is found as follows:

$$\alpha = 4\,\frac{M}{\left(A_2\,\omega^2+1\right)b} - 4\,\frac{M A_0}{\left(A_2\,\omega^2+1\right)^2 b}\beta + O\left(\beta^2\right) \tag{34}$$

The effect of the dark matter can be seen for the above deflection angle by Schwarzcshild black hole.

II. MAXWELL FISHEYE EFFECT OF THE SCHWARZSCHILD-LIKE WORMHOLE USING THE GAUSS-BONNET THEOREM

In this section, we consider the static Schwarzschild-like wormhole solution [51] with metric:

$$ds^2 = -(1-2M/r+\lambda^2)dt^2 + \frac{dr^2}{1-2M/r} + r^2 d\Omega^2_{(2)}\,, \tag{35}$$

which reduces to the black hole metric in Eq. 5 at $\lambda = 0$. Using the transformation of $t \to t/\sqrt{1+\lambda^2}$ and $M \to M(1+\lambda^2)$, the metric functions of the Schwarzschild-like wormhole spacetime becomes:

$$f(r) = 1 - \frac{2M}{r}\,, \quad g(r)^{-1} = 1 - \frac{2M(1+\lambda^2)}{r}\,. \tag{36}$$

A. Case 1

We first use the constant profile as refractive $n_m = n_0$ to calculate the deflection angle in the medium in weak field limits. Using the same procedure, we obtain the optical metric and calculate the Gaussian optical curvature for the Schwarzschild-like wormhole at linear order of M as follows:

$$\mathcal{K} \approx -\frac{\left(\lambda^2+2\right)M}{r^3 {n_0}^2} + O(M^3), \tag{37}$$

and after similar calculations, the corresponding deflection angle in the leading order terms is

$$\alpha = 2\,\frac{M\lambda^2}{n_0\,b} + 4\,\frac{M}{n_0\,b} \tag{38}$$

which agrees with the well-known results in the limit at which its presence is negligible ($n_0 = 1$) [38].

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B. Case 2:

To see the effect of the medium of MFE, we use the archetype of the absolute optical instrument with the profile [9]

$$n = \frac{z_0}{1+z^2}, \tag{39}$$

where n_0 and z are a constant. The Gaussian curvature of the optical metric approximating in leading orders is everywhere negative and found as :

$$\mathcal{K} \approx -\frac{\left(z^2+1\right)^2\left(\lambda^2+2\right)M}{r^3{z_0}^2} \tag{40}$$

Using the GBT, the deflection angle is calculated as follows:

$$\hat{\alpha} \simeq 2\,\frac{M\lambda^2 z^2}{z_0\,b} + 2\,\frac{M\lambda^2}{z_0\,b} + 4\,\frac{Mz^2}{z_0\,b} + 4\,\frac{M}{z_0\,b}. \tag{41}$$

At the $z=0$ and $z_0=1$, it reduces to previous result [38].

C. Case 3:

Last we use the refractive index for the dark matter given in Eq. 32 to calculate the deflection angle of wormhole in medium. The Gaussian curvature is obtained as:

$$\mathcal{K} \approx -\frac{\left(\lambda^2+2\right)M}{r^3\left(A_2\omega^2+A_0\,\beta+1\right)^2} \tag{42}$$

The deflection angle is found as follows:

$$\alpha = 2\,\frac{M\lambda^2}{\left(A_2\,\omega^2+A_0\,\beta+1\right)b} + 4\,\frac{M}{\left(A_2\,\omega^2+A_0\,\beta+1\right)b} \tag{43}$$

We find that the deflecting photon through the dark matter around the Schwarzcshild-like wormhole has small deflection angle compared to the normal case.

III. CONCLUSIONS

We have constructed a MFE on the black hole using the GBT in medium. This has been achieved by constructing an optical metrics. In summary, we have investigated that GBT is global effect. We have showed it by using three different cases. In the first case, we have used the constant profile as a refractive index. Then constructing the optical geometry, and used the GBT, we have obtain the deflection angle in weak limits. The deflection angle of the Schwarzschild black hole has been calculated correctly in the medium which has constant n_0 refractive index. In the second case, we have used the different model of MFE and repeat the calculation for this model and showed that it gives similar effect. Here all the emitted light rays travel along the great circles, then meet at a antipodal point. Clearly, the agreement has been shown to arise from optics to general relativity where we exploit the MFE into a finite circle. Thus, this method as a quantitative tool can be used in any metrics.

In the section 2, we have repeated our method to Schwarzcshild-like wormhole to see the effect of the dark matter medium when light propagating through it. In the first case, again we have used the constant refractive index, then we have considered the MFE profile, and last the medium for the dark matter is taken to find the deflection angle in weak fields limit.

We have concluded that the deflection angle is decreasing if there is a medium of dark matter as seen in Eq.(34) and Eq. (43).

The main message of this article is that, these results provide an excellent tool to observe compact massive object by weak gravitational lensing within the dark matter medium or MFE, namely perfect imaging and to understand the nature of the dark matter that may effect the gravitational waves.

Needless to say, since the discovery of gravitational waves by LIGO, new discoveries are waiting to happen in future, we wish to be part of this cutting-edge research or become pioneer of a new field of physics.

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[1] B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. **116**, no. 6, 061102 (2016); Phys. Rev. Lett. **119**, no. 16, 161101 (2017).
[2] Y. Nutku and M. Halil, Phys.Rev.Lett. **39** 1379-1382 (1977).
[3] S. B. Giddings and D. Psaltis, Phys. Rev. D **97**, no. 8, 084035 (2018).
[4] A. Barrau, Scholarpedia **12**, no. 10, 33321 (2017).
[5] Maxwell's original piece in the Cambridge and Dublin Mathematical Journal for Feb. 1854, Ed. W. D. Niven, The Scientific Papers of James Clerk Maxwell, vol. 1, pp 76 -79,

Peer-reviewed version available at *Universe* **2019**, *5*, 115; doi:10.3390/universe5050115

CUP 1890.

[6] R. K. Luneburg, (1944). Mathematical Theory of Optics. Providence, Rhode Island: Brown University. pp. 189-213.

[7] J. B. Pendry, Phys. Rev. Lett. **85**, 3966-9 (2000).

[8] Ulf Leonhardt, New J. Phys. **11**, 093040 (2009).

[9] U. Leonhardt and S. Sahebdivan Phys. Rev. A **92**, 053848 (2015).

[10] U. Leonhardt, Science **23** Jun 2006:312, 5781, 1777.

[11] T. Tyc and A. Danner, New J. Phys. **16** 063001 (2014).

[12] Y. Liu and H. Chen, J. Opt. **17** 125102 (2015).

[13] S. Guenneau, A. Diatta and R.C . McPhedran, Journal of Modern Optics, **57**:7, 511 (2010).

[14] V. Perlick, Living Rev. Relativ. (2004) 7: 9.

[15] V. Bozza, Gen. Rel. Grav. **42**, 2269 (2010).

[16] Ivan Zh. Stefanov, Stoytcho S. Yazadjiev, and Galin G. Gyulchev, Phys. Rev. Lett. **104**:251103, (2010).

[17] Epps, Seth D. et al. Mon.Not.Roy.Astron.Soc. **468** (2017) 3, 2605-2613.

[18] M. Bartelmann and P. Schneider, Phys. Rept. **340**, 291 (2001).

[19] C. M. Hirata, et al. Phys.Rev. D **78**, 043520 (2008).

[20] G. W. Gibbons and M. C. Werner, Class. Quant. Grav. **25**, 235009 (2008).

[21] G. W. Gibbons and C. M. Warnick, Phys. Rev. D **79**, 064031 (2009).

[22] M. C. Werner, Gen. Rel. Grav. **44**, 3047 (2012).

[23] A. Ishihara, Y. Suzuki, T. Ono, T. Kitamura and H. Asada, Phys. Rev. D **94**, no. 8, 084015 (2016).

[24] I. Sakalli and A. Ovgun, EPL **118**, no. 6, 60006 (2017).

[25] K. Jusufi, M. C. Werner, A. Banerjee and A. Övgün, Phys. Rev. D **95**, no. 10, 104012 (2017).

[26] T. Ono, A. Ishihara and H. Asada, Phys. Rev. D **96**, no. 10, 104037 (2017).

[27] K. Jusufi, I. Sakalli and A. Övgün, Phys. Rev. D **96**, no. 2, 024040 (2017).

[28] A. Ishihara, Y. Suzuki, T. Ono and H. Asada, Phys. Rev. D **95**, no. 4, 044017 (2017).

[29] K. Jusufi, A. Övgün and A. Banerjee, Phys. Rev. D **96**, no. 8, 084036 (2017).

[30] H. Arakida, Gen. Rel. Grav. **50**, no. 5, 48 (2018).

[31] K. Jusufi and A. Övgün, Phys. Rev. D **97**, no. 2, 024042 (2018).

[32] A. Övgün, I. Sakalli and J. Saavedra, JCAP **1810**, no. 10, 041 (2018)

[33] A. Övgün, I. Sakalli and J. Saavedra, arXiv:1806.06453 [gr-qc].

[34] A. Övgün, G. Gyulchev and K. Jusufi, arXiv:1806.03719.

[35] K. Jusufi, A. Övgün, J. Saavedra, P. A. Gonzalez and Y. Vasquez, Phys. Rev. D **97**, 124024 (2018).

[36] A. Övgün, K. Jusufi and I. Sakalli, Annals Phys. **399**, 193 (2018).

[37] K. Jusufi and A. Övgün, Phys. Rev. D **97**, no. 6, 064030 (2018).

[38] A. Övgün, Phys. Rev. D **98**, no. 4, 044033 (2018)

[39] T. Ono, A. Ishihara and H. Asada, Phys. Rev. D **98**, no. 4, 044047 (2018)

[40] T. Ono, A. Ishihara and H. Asada, arXiv:1811.01739 [gr-qc].

[41] G. Crisnejo and E. Gallo, Phys. Rev. D **97**, 124016 (2018).

[42] G. Crisnejo, E. Gallo and A. Rogers, arXiv:1807.00724 [gr-qc].

[43] H. Asada, Mod. Phys. Lett. A **32**, no. 34, 1730031 (2017)

[44] G. Hinshaw *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **208**, 19 (2013)

[45] J. L. Feng, Ann. Rev. Astron. Astrophys. **48**, 495 (2010)

[46] D. C. Latimer, Phys. Rev. D **88**, 063517 (2013)

[47] D. C. Latimer, Phys. Rev. D **95**, no. 9, 095023 (2017).

[48] D. C. Latimer, Phys. Rev. D **94**, no. 9, 093010 (2016)

[49] A. K. Kvam and D. C. Latimer, J. Phys. G **43**, no. 8, 085002 (2016)

[50] K. M. Whitcomb and D. C. Latimer, Am. J. Phys. **85**, 932 (2017)

[51] T. Damour and S. N. Solodukhin, Phys. Rev. D **76**, 024016 (2007)