

The disprove of the Komlos Conjecture

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ABSTRACT

Komlos conjecture is about the existing of a universal constant K such that for all dimension n and any collection of vectors $\vec{V}_1, \dots, \vec{V}_n \in \mathbb{R}^n$ with $\|\vec{V}_i\|_2 \leq 1$, there are weights $\varepsilon_i \in \{-1, 1\}$ in such that $\|\sum_{i=1}^n \varepsilon_i \vec{V}_i\|_\infty \leq K(n) \leq K$. In this paper, the constant $K(n)$ is evaluated for $n \leq 5$ to be $K(2) = \sqrt{2}$, $K(3) = \frac{\sqrt{2} + \sqrt{11}}{3}$, $K(4) = \sqrt{3}$, and $K(5) = \frac{4 + \sqrt{142}}{9}$. For higher dimension, the function $f(n) = \sqrt{n - \lceil \log_2(2^{n-1}/n) \rceil}$ is found to be the lower bound for the constant $K(n)$, from where it can be concluded that the Komlos conjecture is false i.e., the universal constant K does not exist because of

$$\lim_{n \rightarrow \infty} K(n) \geq \lim_{n \rightarrow \infty} \sqrt{\log(n) - 1} = +\infty.$$

Keywords: Komlos Conjecture; optimization; discrepancy theory.

Introduction

J. Komlos has made the following conjecture: For a given dimension n , let $K(n)$ denote the minimum value such that: for any set of n Vectors $\vec{V}_1, \dots, \vec{V}_n \in \mathbb{R}^n$ with $\|\vec{V}_i\|_2 \leq 1$, there exists weights $\varepsilon_i = +1$ or -1 such that

$$\left\| \sum_{i=1}^n \varepsilon_i \vec{V}_i \right\|_\infty \leq K(n).$$

Kolmos has conjectured the existence of a universal constant K such that $K(n) \leq K$ for all dimension n . The l_2 and l_∞ norms in \mathbb{R}^n are denoted by $\|\cdot\|_2$ and $\|\cdot\|_\infty$ respectively.

This conjecture was referred by Joel Spencer [8] in 1994, where he linked kolmos Conjecture to Spencer's famous Six Standard Deviation in 1985, see [9].

The main nontrivial result known, which is due to Joel Spencer [9], is that if $k \leq n$ then $L = O(\log(L))$. The main result of D. Hajela [4] was very close to disprove the Komlos conjecture, where precisely he have proved the following theorem:

THEOREM 1. Let $f(n)$ be a function that's goes to infinity when n goes to infinity with $f(n) = O(n)$ and let $0 < \lambda < 1/2$. Then form $n \geq n_0$ (where n_0 depends only on n and λ) and any $A \subseteq \{1, -1\}^n$ with $|A| \leq 2^{n/f(n)}$, there are orthogonal vectors x_1, \dots, x_n in \mathbb{R}^n , $\|x_i\|_2 \leq 1$ for all $1 \leq i \leq n$, and such that

$$\|\varepsilon_1 x_1 + \dots + \varepsilon_n x_n\|_\infty \geq \exp\left(\frac{\lambda \log \log f(n)}{\log \log \log f(n)}\right),$$

for all $(\varepsilon_1, \dots, \varepsilon_n) \in A$.

The previous theorem disproves the conjecture of Komlos over the set $A \subseteq \{1, -1\}^n$ where $|A| \leq 2^{n/f(n)}$.

The proof of Theorem 1 is based on certain inequalities which arise in the geometry of convex bodies [1], [10], and [2].

Komlos Conjecture is also related to discrepancy theory, paper of J.Becka and T.Fiala [6], where it states that for a global constant K and for any $m \times n$ matrix A , whose columns are inside a unit ball, there exists a vector $X \in \{-1, +1\}^n$ such that $\|AX\|_\infty \leq K$.

The best progress in proving Komlos conjecture is a result given by Banaszczyk [11] who proved the bound

$$\min_{x \in \{-1, +1\}^n} \|AX\| \leq K\sqrt{\log(n)}$$

for a global constant.

This is the best known bound for Becka-Fiala conjecture as well [5].

Discrepancy is a challenging problem that has application in geometry, data analysis, and complexity theory. The books, J. Matousek [7], B. Chazelle [3], and J. Beck and V.T [5], provide references for a wide array of applications.

For lower dimension, the idea is to find a hypercube of minimum side of $2K$, where all vectors formed by different combinations of the weights, $\sum_{i=1}^n \varepsilon_i \vec{V}_i$, should be all inside the hypercube. Also It is not hard to show that \sqrt{n} is an upper bound for the constant K . To prove that for all vectors $\vec{V}_1, \dots, \vec{V}_n \in \mathbb{R}^n$ with $\|\vec{V}_i\|_2 \leq 1$, we can find ε_i^* such that

$$\left\| \sum_{i=1}^n \varepsilon_i^* \vec{V}_i \right\|_\infty \leq K \leq \sqrt{n},$$

It is enough to highlight the below boulets

- We will prove first that $\|\sum_{i=1}^n \varepsilon_i^* \vec{V}_i\|_2 \leq \sqrt{n}$, which it is a sufficient condition to prove that $\|\sum_{i=1}^n \varepsilon_i^* \vec{V}_i\|_\infty \leq \sqrt{n}$.
- For dimension 2: From cosine rule, we can write the following

$$\begin{aligned}
\|\varepsilon_1^* \vec{V}_1 + \varepsilon_2^* \vec{V}_2\|_2 &= \left\| \vec{V}_1 + \frac{\varepsilon_2^*}{\varepsilon_1^*} \vec{V}_2 \right\|_2 \\
&= \sqrt{\|\vec{V}_1\|_2^2 + \|\vec{V}_2\|_2^2 - 2\|\vec{V}_1\|_2 \|\vec{V}_2\|_2 \cos(\vec{V}_1, \vec{V}_2)} \\
&\leq \sqrt{2}
\end{aligned}$$

- If we suppose that $\|\sum_{i=1}^{n-1} \varepsilon_i^* \vec{V}_i\|_2 \leq \sqrt{n-1}$, we need to prove that $\|\sum_{i=1}^{n-1} \varepsilon_i^* \vec{V}_i + \varepsilon_n^* \vec{V}_n\|_2 \leq \sqrt{n}$. Again by cosine rule, we can write the following:

$$\begin{aligned}
\left\| \sum_{i=1}^{n-1} \varepsilon_i^* \vec{V}_i + \varepsilon_n^* \vec{V}_n \right\|_2 &\leq \sqrt{\|\vec{V}_n\|_2^2 + \left\| \sum_{i=1}^{n-1} \varepsilon_i^* \vec{V}_i \right\|_2^2} \\
&\leq \sqrt{1+n-1} \\
&\leq \sqrt{n}.
\end{aligned}$$

- From the principle of induction proof we can say that all vectors $\vec{V}_1, \dots, \vec{V}_n \in \mathbb{R}^n$ with $\|\vec{V}_i\|_2$ is at most equal to 1, we can find ε_i^* such that

$$\|\sum_{i=1}^n \varepsilon_i^* \vec{V}_i\|_2 \leq \sqrt{n}.$$

Since $\|\sum_{i=1}^n \varepsilon_i^* \vec{V}_i\|_\infty \leq \|\sum_{i=1}^n \varepsilon_i^* \vec{V}_i\|_2$, we can conclude that the constant of Komlos has an upper bound of order \sqrt{n} .

We can extend the Komlos conjecture statement to the below lemma, where it summarizes very interesting properties related to special vectors, \vec{V}_i^* , $i = 1, \dots, n$, that cannot cancel each other further than $K(n)$.

Lemma 1: Let C^n be a set of vectors in \mathbb{R}^n have l_2 norm at most 1, and we denote by V^* as a set of vectors in \mathbb{R}^n that satisfies $V^* = \{\vec{V}_1^*, \dots, \vec{V}_n^*\} = \operatorname{argmax}_{\vec{V}_i \in C^n} \min_{\varepsilon_i} \left\| \sum_{i=1}^n \varepsilon_i \vec{V}_i^* \right\|_\infty$

The set V^* satisfies the below properties:

- For all vector \vec{V}_i^* in V^* has l_2 norm equal to 1, $\|\vec{V}_i^*\|_2 = 1$.
- All the vertices have the same distance l_∞ , i.e.,
$$\operatorname{Min}_{\varepsilon_i \in \{-1, +1\}} \left\| \sum_{i=1}^n \varepsilon_i \vec{V}_i^* \right\|_\infty = \operatorname{Max}_{\varepsilon_i \in \{-1, +1\}} \left\| \sum \varepsilon_i \vec{V}_i^* \right\|_\infty.$$
- $K(n)$ is strictly increasing sequence, i.e., for all integers $m > n$, $K(n) < K(m)$.

A proof of the previous lemma will be publish soon in order to prove that $K(n) \sim \sqrt{\log_2(n)}$.

The following sections are consecrated to evaluate the constant K for a different dimension, the exact value of K will be calculated for a dimension less or equals to 5 and a lower bound will be evaluated for all dimension.

The constant K for dimension two

It is obvious to say that the constant K for dimension one is equal to one, and it is quite easy to calculate K for dimension 2, denoted by $K(2)$.

To find the value of $K(2)$, it is useful to analyze the parallelogram formed by four vertices centered at the origin, resulting from the four combinations of ε_i , i.e., $\mp \vec{V}_1 \mp \vec{V}_2$ (see Figure1)

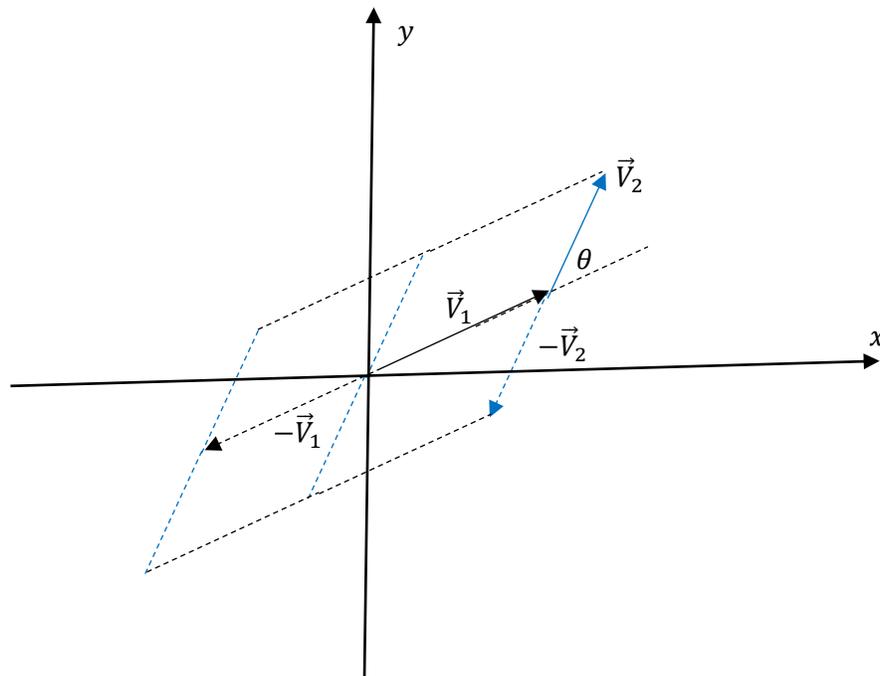


Figure 1. A parallelogram formed by four vertices, $\vec{V}_1 + \vec{V}_2, \vec{V}_1 - \vec{V}_2, -\vec{V}_1 + \vec{V}_2, -\vec{V}_1 - \vec{V}_2$.

By using the cosine rule, we can find the length of the big and the small diagonals respectively as follows:

$$\begin{cases} L^2 = \|\vec{V}_1\|_2 + \|\vec{V}_2\|_2 + 2\|\vec{V}_1\|_2\|\vec{V}_2\|_2\cos(\theta) \\ l^2 = \|\vec{V}_1\|_2 + \|\vec{V}_2\|_2 - 2\|\vec{V}_1\|_2\|\vec{V}_2\|_2\cos(\theta), \end{cases}$$

where θ is the acute angle between the two vectors \vec{V}_1 and \vec{V}_2 .

We can notice that the small diagonal l has $\sqrt{2}$ as an upper bound, i.e., $l \leq \sqrt{2}$.

The two weights, ε_1 and ε_2 , can be chosen in such a way the length of $\varepsilon_1 \vec{V}_1 + \varepsilon_2 \vec{V}_2$ is smaller than the length of diagonal l , in which it implies that for all vectors \vec{V}_i inside the circle of center $(0,0)$ and Radius $=1$, we can find ε_1 and ε_2 such that $\|\varepsilon_1 \vec{V}_1 + \varepsilon_2 \vec{V}_2\|_\infty \leq \sqrt{2}$.

By using the proof by contradiction, we can prove that $K(2) \leq \sqrt{2}$. Let's assume the case where the vertices A, B, C, and D are located outside of the square of side $2\sqrt{2}$ as it is shown in Figure 2.

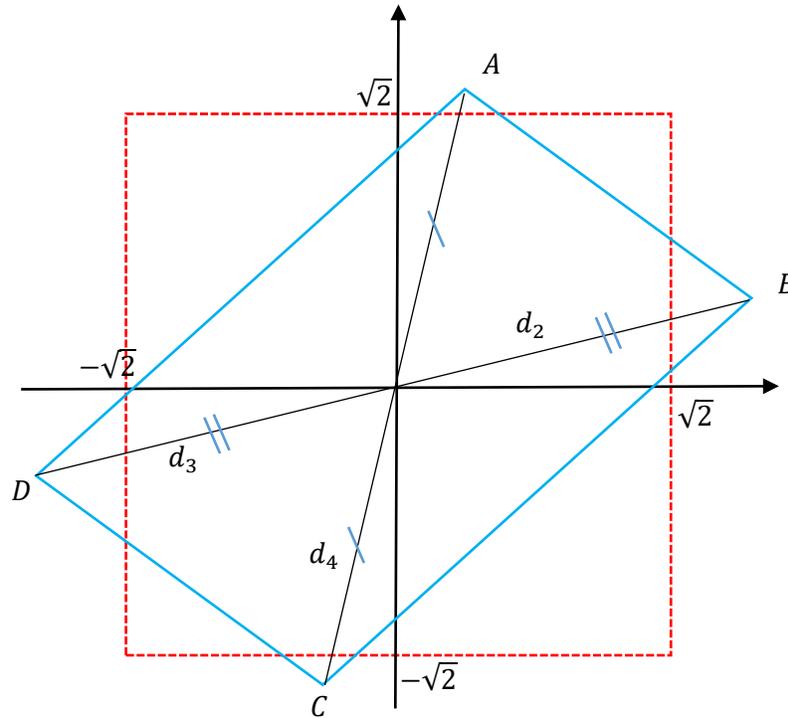


Figure 2. ABCD is a parallelogram with for vertices located outside of square whose side is $\sqrt{2}$ has $\min_i(d_i) \geq \sqrt{2}$.

The possibility to have all the vertices, $\mp \vec{V}_1 \mp \vec{V}_2$, outside the red square, refer to Figure 2, is impossible! Because it contradicts with the fact that small diagonal length is at most $\sqrt{2}$.

From previous proof, we can conclude that $K(2) \leq \sqrt{2}$, and it is enough to find a particular case where $\min_{\varepsilon_i} \|\varepsilon_1 \vec{V}_1 + \varepsilon_2 \vec{V}_2\|_\infty = \sqrt{2}$ in order to prove $K = \sqrt{2}$.

If we consider two vectors as $\vec{V}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ +1 \end{pmatrix}$ and $\vec{V}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ +1 \end{pmatrix}$, and for all possible values of ε_1 and ε_2 , we can calculate $\min_{\varepsilon_i} \|\varepsilon_1 \vec{V}_1 + \varepsilon_2 \vec{V}_2\|_\infty$ as follows:

$$\max\left(\frac{|\varepsilon_1 + \varepsilon_2|}{\sqrt{2}}, \frac{|\varepsilon_1 - \varepsilon_2|}{\sqrt{2}}\right) = \max\left(\frac{|1 + \varepsilon_2/\varepsilon_1|}{\sqrt{2}}, \frac{|1 - \varepsilon_2/\varepsilon_1|}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}}.$$

Finally, we conclude that $K(2) = \sqrt{2}$.

The constant K for dimension three

Given the vector space \mathbb{R}^3 , the span of the set S of finite vectors is defined as the set of all finite linear combinations of elements of S

$$Span(S) = \left\{ \sum_{i=1}^k \alpha_i \vec{v}_i; k \in \mathbb{N}, \vec{v}_i \in S, \alpha_i \in \mathbb{R} \right\}.$$

The calculation of $K(3)$ will be splitted to several cases related to different configuration of the three vectors, \vec{v}_1, \vec{v}_2 , and \vec{v}_3 in \mathbb{R}^3 .

Case 1: $\vec{v}_3 \perp Space(\vec{v}_2, \vec{v}_1) = XY\text{-plane}$.

As the vector \vec{v}_3 is parallel to z-axis, then without losing generality, we can write the following:

$$\min_{\varepsilon_i} \|\varepsilon_3 \vec{v}_3 + \varepsilon_2 \vec{v}_2 + \varepsilon_1 \vec{v}_1\|_{\infty} = \min_{\varepsilon_i} \|\vec{v}_3 + \varepsilon_2 \vec{v}_2 + \varepsilon_1 \vec{v}_1\|_{\infty}$$

by consequence,

$$\min_{\varepsilon_i} \|\vec{v}_3 + \varepsilon_2 \vec{v}_2 + \varepsilon_1 \vec{v}_1\|_{\infty} = \text{Max} \left\{ \|\vec{v}_3\|_2, \min_{\varepsilon_i} \|\varepsilon_2 \vec{v}_2 + \varepsilon_1 \vec{v}_1\|_{\infty} \right\}.$$

From previous section, we know that the constant $K(2) = \sqrt{2}$ and from the fact that $\varepsilon_2 \vec{v}_2 + \varepsilon_1 \vec{v}_1 \in XY\text{-plane}$, we have

$$\begin{aligned} \left\{ \|\vec{v}_3\|_2, \min_{\varepsilon_i} \|\varepsilon_2 \vec{v}_2 + \varepsilon_1 \vec{v}_1\|_{\infty} \right\} &\leq \text{Max} \left\{ \|\vec{v}_3\|_2, \sqrt{2} \right\} \\ &\leq \sqrt{2}. \end{aligned}$$

By considering $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ +1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ +1 \\ 0 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ +1 \end{pmatrix}$, we can calculate

$$\begin{aligned} \left\| \sum_{i=1}^3 \varepsilon_i \vec{v}_i \right\|_{\infty} &= \max \left(\frac{|\varepsilon_1 + \varepsilon_2|}{\sqrt{2}}, \frac{|\varepsilon_1 - \varepsilon_2|}{\sqrt{2}}, |\varepsilon_3| \right) \\ &= \sqrt{2}. \end{aligned}$$

Therefore, under the case 1, the constant $K(3)$ is equal to $\sqrt{2}$.

Case 2: $\text{Span}(\vec{V}_1, \vec{V}_2) = XY\text{-plane}$.

We split the vector \vec{V}_3 as follows:

$\vec{V}_3 = \vec{V}_{31} \oplus \vec{V}_{32}$, where $\vec{V}_{31} \perp XY\text{-plane}$. Without losing generality, we will consider the weight ε_3 as one value in our calculations.

Therefore, for all vectors $\vec{V}_1, \vec{V}_2, \vec{V}_3 \in \mathbb{R}^3$ with $\|\vec{V}_i\|_2 \leq 1$

$$\begin{aligned} \min_{\varepsilon_i} \left\| \sum_{i=1}^3 \varepsilon_i \vec{V}_i \right\|_{\infty} &= \min_{\varepsilon_i} \left\| \vec{V}_3 + \sum_{i=1}^2 \varepsilon_i \vec{V}_i \right\|_{\infty} \\ &= \max \left\{ \|\vec{V}_{31}\|_2, \min_{\varepsilon_i} \left\| \vec{V}_{32} + \sum_{i=1}^2 \varepsilon_i \vec{V}_i \right\|_{\infty} \right\} \end{aligned}$$

From previous equation, we can see that the calculation is moved from dimension 3 to dimension 2 by just calculating the following:

For all vectors $\vec{V}_1, \vec{V}_2, \vec{V}_{32} \in \mathbb{R}^2$ with $\|\vec{V}_i\|_2 \leq 1$, the below minimum is needed to be calculated

$$\min_{\varepsilon_i} \left\| \vec{V}_{32} + \sum_{i=1}^2 \varepsilon_i \vec{V}_i \right\|_{\infty},$$

where $\vec{V}_{32} = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}$, and without losing generality, we can assume that $\alpha^2 + \beta^2 \leq 1$ and $0 \leq \alpha \leq \beta \leq 1$.

To evaluate the constant $K(3)$, the question about the possibility to have all the vertices, $\vec{V}_{32} \pm \vec{V}_2 \pm \vec{V}_1$, outside the square of side $2\sqrt{2}$, as it shown in Figure 3, need to be checked.

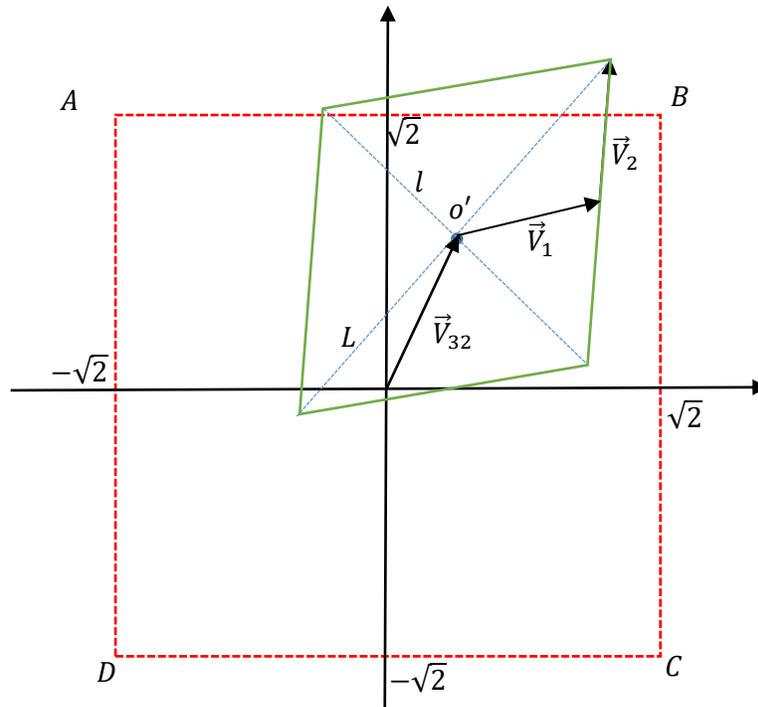


Figure 3 The vector \vec{V}_{32} can be consider with slope bigger than one, $m \geq 1$, without losing generality. The two distances l and L are the length of the small and the big diagonal respectively.

From the Figure 4, the small diagonal, l , of parallelogram centered at the point O' is at most equal to $\sqrt{2}$, consequently, we have to focus only on the green area, highlighted in Figure 5, the possible location of two opposite vertices that form the two small diagonal l .

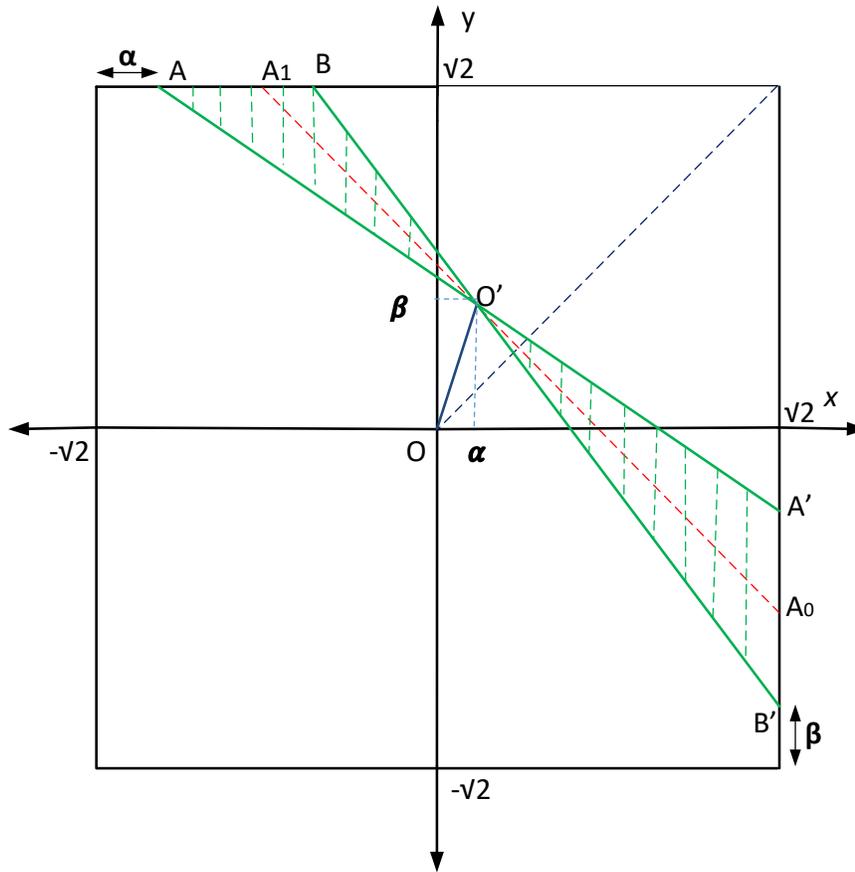


Figure 4 Green Area is the only possible location of the vertices, $\vec{V}_{32} \mp \vec{V}_1 \mp \vec{V}_2$, in order to have maybe $\|\vec{V}_{32} \mp \vec{V}_1 \mp \vec{V}_2\|_{\infty} > \sqrt{2}$

The distance between the point O' and the midpoint of two vertices is equals to either $\|\vec{V}_1\|$ or $\|\vec{V}_2\|$, which implies that the impossibility to have, on one side of square, two vertices outside of red square, refer to Figure 4.

This impossibility can be proved by highlighting the fact that the distance between any point inside the area S_1 and any point inside the area S_2 is bigger than or equal to 1, see Figure 5.

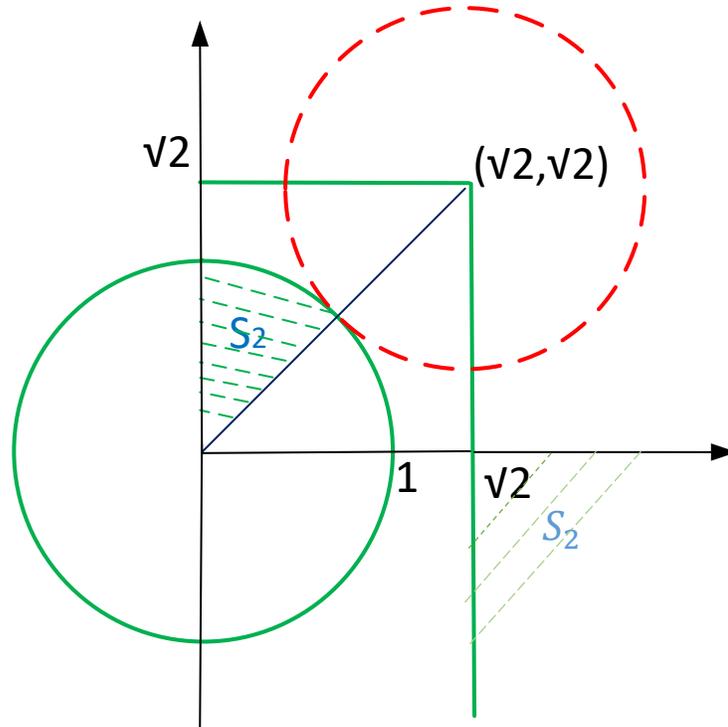


Figure 5. Distance between any point in the area S_1 and any point in the area S_2 has minimum distance of 1, $S_1 = \{x \geq \sqrt{2} \text{ and } y \leq 0\}$ and $S_2 = \{x^2 + y^2 \leq 1 \text{ and } 0 \leq x \leq y \leq 1\}$

Therefore, under the case 2, the constant $K(3)$ is upper bounded by $\sqrt{2}$.

To say that $K(3) = \sqrt{2}$, we need to consider a particular example as follows:

If $\vec{V}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ +1 \\ 0 \end{pmatrix}$, $\vec{V}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ +1 \\ 0 \end{pmatrix}$, and $\vec{V}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ +1 \end{pmatrix}$, then

$$\min_{\varepsilon_i} \left\| \sum_{i=1}^3 \varepsilon_i \vec{V}_i \right\|_{\infty} = \sqrt{2}.$$

Case 3: General case

By symmetry, without losing generality, we can consider the weight $\varepsilon_3 = 1$ in our calculations, as it is proven below

$$\begin{aligned}
\min_{\varepsilon_i} \left\| \sum_{i=1}^3 \varepsilon_i \vec{V}_i \right\|_{\infty} &= \min_{\varepsilon_i} \left\| \varepsilon_3 \sum_{i=1}^3 \frac{\varepsilon_i}{\varepsilon_3} \vec{V}_i \right\|_{\infty} \\
&= \min_{\varepsilon_i} \left\| \vec{V}_3 + \sum_{i=1}^2 \frac{\varepsilon_i}{\varepsilon_3} \vec{V}_i \right\|_{\infty} \\
&= \min_{\varepsilon_j} \left\| \vec{V}_3 + \sum_{j=1}^2 \varepsilon_j \vec{V}_i \right\|_{\infty}.
\end{aligned}$$

The vector $\sum_{i=1}^3 \varepsilon_i \vec{V}_i$ will be evaluated over two perpendicular spaces, XY-plane and Z-axis, and a link between the two spaces will be found in order to maximize the l_{∞} norm of $\left\| \sum_{i=1}^3 \varepsilon_i \vec{V}_i \right\|_{\infty}$.

The projection of the vector \vec{V}_i over the space XY-plane is denoted by vector $\vec{U}_i = Proj_{XY-plane}(\vec{V}_i)$.

From case 2, we have proven that it is not possible to have all vertices, $\vec{U}_3 \pm \vec{U}_2 \pm \vec{U}_1$, outside the square of side $2\sqrt{2}$ centered at the origin. A question arises of the possibility to increase the l_{∞} norm beyond $\sqrt{2}$ for two vertices and compensate the l_{∞} norms of the two other vertices by l_{∞} norm over Z-axis?

To answer the previous question, we need to find Z-coordinates of three vectors \vec{V}_i that satisfy the following statement:

For each possible weight's vector $(1, \varepsilon_1, \varepsilon_2)$ where $\left\| \vec{U}_3 + \sum_{j=1}^2 \varepsilon_j \vec{U}_i \right\|_{\infty} < \sqrt{2}$ then

$$\left\| Z_3 + \sum_{j=1}^2 \varepsilon_j Z_i \right\|_{\infty} = |Z_3 + \sum_{j=1}^2 \varepsilon_j Z_i| > \sqrt{2}.$$

To summarize the above idea, we create an example of vectors \vec{V}_i , where $K = \left\| \sum_{i=1}^3 \varepsilon_i \vec{V}_i \right\|_{\infty} > \sqrt{2}$, as follows:

$$\vec{V}_1 = \begin{pmatrix} -x_1 \\ y_1 \\ z_1 \end{pmatrix}, \vec{V}_2 = \begin{pmatrix} -x_2 \\ y_2 \\ -z_2 \end{pmatrix}, \text{ and } \vec{V}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix},$$

where x_i, y_i, z_i are all non-negative values with $\left\| \vec{V}_i \right\|_2 = \sqrt{x_i^2 + y_i^2 + z_i^2} \leq 1$,

We assume the following equations

$$\begin{aligned}
\text{if } \begin{cases} \left\| \vec{V}_3 - \vec{V}_2 - \vec{V}_1 \right\|_{\infty} = x_1 + x_2 + x_3 = K_1 \\ \left\| \vec{V}_3 + \vec{V}_2 + \vec{V}_1 \right\|_{\infty} = y_1 + y_2 + y_3 = K_2 \end{cases} \\
\text{then } \begin{cases} \left\| \vec{V}_3 - \vec{V}_2 + \vec{V}_1 \right\|_{\infty} = z_1 + z_2 + z_3 = K_3 \\ \left\| \vec{V}_3 + \vec{V}_2 - \vec{V}_1 \right\|_{\infty} = |z_3 - z_1 - z_2| = K_4. \end{cases}
\end{aligned}$$

By symmetry, we can consider $K_1 = K_2$ and $K_3 = K_4$.

Then the system that is needed to be solved is summarized by the following equations:

$$\begin{cases} x_1 + x_2 + x_3 = K_1 \\ y_1 + y_2 + y_3 = K_1 \\ z_1 + z_2 + z_3 = K_3 \\ z_1 + z_2 - z_3 = K_3 \end{cases}$$

From the last two equations, we conclude that

$$z_3 = 0$$

By symmetry, we can conclude that

$$x_3 = y_3$$

Since $\|\vec{V}_3\| \leq 1$, it is convenient to increase x_3 & y_3 as much as we can, then the maximum value of K_1 can be found when the coordinates of \vec{V}_3 are as follows

$$x_3 = y_3 = \frac{1}{\sqrt{2}}$$

Therefore, the system will be simplified as follows

$$\begin{cases} x_1 + x_2 = K_1 - \frac{\sqrt{2}}{2} \\ y_1 + y_2 = K_1 - \frac{\sqrt{2}}{2} \\ z_1 + z_2 = K_3 \end{cases}$$

Again by symmetry, we can consider the following equations:

$$x_1 = y_1 = \alpha$$

$$x_2 = y_2 = \beta$$

$$z_1 = z_2 = \gamma$$

In order to maximize K and by symmetry, we need to impose that $K_1 = K_3$, then the final system that need to be solved is as follows:

$$\begin{cases} \alpha + \beta = K - \frac{\sqrt{2}}{2} \\ \gamma = \frac{K}{2} \\ \alpha^2 + \beta^2 + \gamma^2 \leq 1, \end{cases}$$

the last inequality comes from the constraint that $\|\vec{V}_i\|_2 \leq 1$, for $i=1,2$.

Again, without losing generality, we can assume that $\alpha = \beta$,

The maximum value of K can be calculated by

$$\begin{cases} \alpha = \frac{K}{2} - \frac{\sqrt{2}}{4} \\ \gamma = \frac{K}{2} \\ 2\alpha^2 + \gamma^2 = 1 \end{cases}$$

So, we end up to solve the below quadratic equation

$$2\left(\frac{K}{2} - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{K}{2}\right)^2 = 1$$

After simplification, we find:

$$3K^2 - 2\sqrt{2}K - 3 = 0$$

The value K is equal to $\frac{\sqrt{2}+\sqrt{11}}{3}$, then we can conclude that

$$K(3) \geq \frac{\sqrt{2}+\sqrt{11}}{3}.$$

By using a simulation the question if $K(3) = \frac{\sqrt{2}+\sqrt{11}}{3}$ or not can be checked. A cylindrical Coordinate has been used in our simulation to check most of the cases, the possible coordinate's values of the vector \vec{V}_i are summarized as follows:

$$x = r \cos(\theta) \sin(\alpha)$$

$$y = r \sin(\theta) \sin(\alpha)$$

$$z = r \cos(\alpha)$$

where $\theta = [\text{start value: step: end value}] = [0: 0.001: 2\pi]$, $\alpha = [\text{start value: step: end value}] = [0: 0.001: 2\pi]$, and $r = [\text{start value: step: end value}] = [0: 0.01: 1]$.

The simulation proves that $K(3) = \frac{\sqrt{2}+\sqrt{11}}{3}$.

The constant K for dimension four

Before giving the approach for dimension 4, we will review the calculation for dimension 2 and 3 in a different ways.

For dimension 2, we denote by $\vec{V}_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ and $\vec{V}_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ the two particular vectors that verified

$$\min_{\varepsilon_1} \|\vec{V}_1 + \varepsilon_2 \vec{V}_2\|_{\infty} = K(2).$$

By symmetry, we can assume that

$$\|\vec{V}_1 + \vec{V}_2\|_{\infty} = \alpha_1 + \alpha_2 = K(2)$$

$$\|\vec{V}_1 - \vec{V}_2\|_{\infty} = \beta_1 - \beta_2 = K(2)$$

From the definition of $K(2)$, to get the maximum value of it, the coordinates of the two vectors should be non-negatives values except the coordinate β_2 should be a negative value.

By symmetry, we denote $\alpha_1 = \alpha_2 = \alpha$ & $\beta_1 = -\beta_2 = \beta$.

to find $K(2)$, it is a enough to solve the following system:

$$\begin{cases} 2\alpha = K(2) \\ 2\beta = K(2) \end{cases}$$

under the constraint $\alpha^2 + \beta^2 \leq 1$.

The maximum $K(2)$ can be found by considering $\alpha^2 + \beta^2 = 1$, then the previous system is equivalent to the following equation

$$\left(\frac{K(2)}{2}\right)^2 + \left(\frac{K(2)}{2}\right)^2 \leq 1$$

Therefore

$$K(2) \leq \sqrt{2}.$$

For dimension 3, we would like to find $\vec{V}_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}$, $\vec{V}_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix}$ and $\vec{V}_3 = \begin{pmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{pmatrix}$ that verify

$$\min_{\varepsilon_1, \varepsilon_2} \|\vec{V}_3 + \varepsilon_2 \vec{V}_2 + \varepsilon_1 \vec{V}_1\|_{\infty} = K(3)$$

We need to investigate all possible cases of the vector $\vec{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 1 \end{pmatrix}$ under a matrix A, where its rows r_i

form all cases $\vec{\varepsilon}$. The matrix A_3 is defined as follows:

$$A_3 = \begin{bmatrix} +1 & +1 & +1 \\ +1 & -1 & +1 \\ +1 & +1 & -1 \\ +1 & -1 & -1 \end{bmatrix}.$$

The four rows are not independent vectors because it is noted that $r_4 = r_2 + r_3 - r_1$.

By symmetry, we can assume the following equations

$$\|\vec{V}_3 + \vec{V}_2 + \vec{V}_1\|_{\infty} = |\alpha_3 + \alpha_2 + \alpha_1| = K(3)$$

$$\|\vec{V}_3 + \vec{V}_2 - \vec{V}_1\|_{\infty} = |\beta_3 + \beta_2 - \beta_1| = K(3)$$

$$\|\vec{V}_3 - \vec{V}_2 + \vec{V}_1\|_{\infty} = |\gamma_3 - \gamma_2 + \gamma_1| = K(3)$$

$$\|\vec{V}_3 - \vec{V}_2 - \vec{V}_1\|_{\infty} = |\gamma_3 - \gamma_2 - \gamma_1| = K(3)$$

In order to maximize the value of $K(3)$, it is suitable to consider β_1 and γ_2 as a negative values, then the coordinate of the three vectors \vec{V}_i will be summarized as following

$$\vec{V}_1 = \begin{pmatrix} \alpha_1 \\ -\beta_1 \\ \gamma_1 \end{pmatrix}, \vec{V}_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ -\gamma_2 \end{pmatrix} \text{ and } \vec{V}_3 = \begin{pmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{pmatrix},$$

where all parameters, $(\alpha_i, \beta_i, \gamma_i)$ are non-negative values.

To calculate $K(3)$, it is enough to solve the below system:

$$\begin{cases} \alpha_3 + \alpha_2 + \alpha_1 = K(3) \\ \beta_3 + \beta_2 + \beta_1 = K(3) \\ \gamma_3 + \gamma_2 + \gamma_1 = K(3) \\ -\gamma_3 + \gamma_2 + \gamma_1 = K(3) \end{cases}$$

Under the constraints

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 \leq 1$$

$$\alpha_2^2 + \beta_2^2 + \gamma_2^2 \leq 1$$

$$\alpha_3^2 + \beta_3^2 + \gamma_3^2 \leq 1$$

From the last two equations of the system we can conclude that $\gamma_3 = 0$.

By symmetry also, we can assume that

$$\alpha_1 = \alpha_2 = \beta_2 = \beta_1 = \beta$$

$$\gamma_2 = \gamma_1 = \gamma$$

$$\alpha_3 = \beta_3 = \alpha$$

Therefore,

$$\begin{cases} 2\beta + \alpha = K(3) \\ 2\gamma = K(3) \end{cases}$$

Under the constraints

$$2\beta^2 + \alpha^2 \leq 1$$

$$2\alpha^2 \leq 1$$

In order to maximize the value of $K(3)$, the two constraints can be considered as

$$2\beta^2 + \alpha^2 = 1$$

$$2\alpha = 1$$

Then the system will be simplified as follows

$$\begin{cases} 2\beta = K(3) - \frac{\sqrt{2}}{2} \\ 2\gamma = K(3) \end{cases}$$

The below quadratic equation is needed to be solved to calculate the value $K(3)$,

$$2\left(\frac{K(3)}{2} - \frac{\sqrt{2}}{4}\right)^2 + \left(\frac{K(3)}{2}\right)^2 = 1.$$

After simplification, we got

$$3K(3)^2 - 2\sqrt{2}K(3) - 3 = 0$$

Finally, we can conclude with simulation that

$$K(3) = \frac{\sqrt{2} + \sqrt{11}}{3}.$$

The particular vectors that cannot canceled each other further than $K(3)$ are define as follows:

$$\vec{V}_1 = \begin{pmatrix} \frac{K(3)}{4} - \frac{\sqrt{2}}{4} \\ \frac{-K(3)}{4} + \frac{\sqrt{2}}{4} \\ \frac{K(3)}{2} \end{pmatrix}, \vec{V}_2 = \begin{pmatrix} \frac{K(3)}{4} - \frac{\sqrt{2}}{4} \\ \frac{K(3)}{4} - \frac{\sqrt{2}}{4} \\ -\frac{K(3)}{2} \end{pmatrix} \text{ and } \vec{V}_3 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}.$$

Note that these particular three vectors are not unique that verify $\min_{\varepsilon_i} \|\sum \varepsilon_i \vec{V}_i\|_{\infty} = K(3)$.

Our idea is to generalize the previous approach in evaluating the constant K , for that let denote by $V = \{\vec{V}_4, \vec{V}_3, \vec{V}_2, \vec{V}_1\}$ as a set of particular vectors that satisfy the below equation

$$K(4) = \min_{\varepsilon_1, \varepsilon_2, \varepsilon_3} \left\| \vec{V}_4 + \sum_{i=1}^3 \varepsilon_i \vec{V}_i \right\|_{\infty}.$$

The matrix A_4 can be extended to fit the dimension 4, where its rows, r_i , are all possible values of $(1, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ as follows:

$$A_4 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 \\ +1 & +1 & +1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & -1 \end{bmatrix},$$

where it is noted that $r_{i+2} = r_{i+1} + r_i - r_{i-1}$, for $i > 1$, and $\dim(\text{span}(r_1, r_2, r_3)) = 3$.

The idea is to well assign each row r_i to one of fourth dimension in order to avoid zero coordinate in \vec{V}_i , which it is a consequence of maximizing the value of $K(4)$, i.e., the axes where L-infinity norm of $\vec{V}_4 + \sum_{i=1}^3 \varepsilon_i \vec{V}_i$ is located will be distributed over possible combination of $(1, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ in a way to maximize the value $K(4)$.

The below diagram, in Figure 6, identifies which coordinate will be eliminated, being zero, when we associate two rows to same axes.

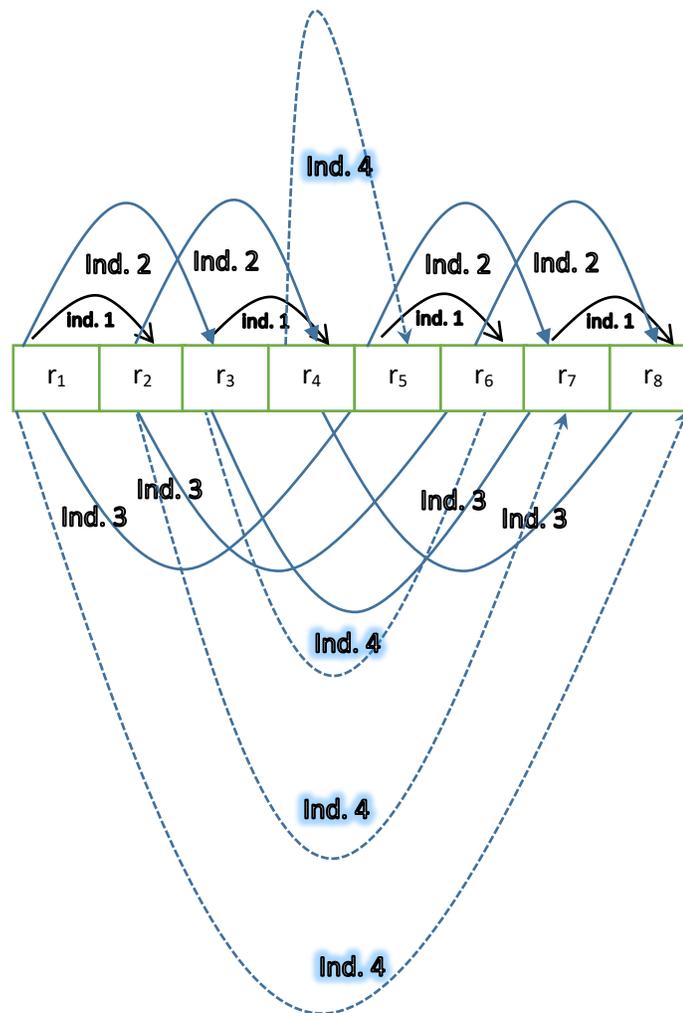


Figure 6 A link between gathering two rows as system of equation and the index variable that will be eliminated.

From previous diagram, the rows are gathered as follows

- $(r_1, r_2): \|\vec{V}_4 \pm \vec{V}_3 + \vec{V}_2 + \vec{V}_1\|_\infty = |V_4^1 \pm V_3^1 + V_2^1 + V_1^1|$
- $(r_5, r_7): \|\vec{V}_4 + \vec{V}_3 \pm \vec{V}_2 - \vec{V}_1\|_\infty = |V_4^2 + V_3^2 \pm V_2^2 - V_1^2|$

- $(r_4, r_8): \|\vec{V}_4 - \vec{V}_3 - \vec{V}_2 \pm \vec{V}_1\|_\infty = |V_4^3 - V_3^3 - V_2^3 \pm V_1^3|$
- $(r_3, r_6): \|\pm \vec{V}_4 + \vec{V}_3 - \vec{V}_2 + \vec{V}_1\|_\infty = |\pm V_4^4 + V_3^4 - V_2^4 + V_1^4|,$

where V_i^j is the j -th coordinate of vector \vec{V}_i .

From previous system of equations and to maximize the value $K(4)$ the coordinate of \vec{V}_i will be as follows:

$$\vec{V}_1 = \begin{pmatrix} \alpha_1 \\ -\beta_2 \\ \gamma_2 \\ w_2 \end{pmatrix}, \vec{V}_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ -\gamma_2 \\ -w_2 \end{pmatrix}, \vec{V}_3 = \begin{pmatrix} \alpha_3 \\ \beta_3 \\ -\gamma_3 \\ w_4 \end{pmatrix} \text{ and } \vec{V}_4 = \begin{pmatrix} \alpha_4 \\ \beta_4 \\ \gamma_4 \\ w_4 \end{pmatrix},$$

where $\alpha_i, \beta_i, \gamma_i$ and w_i are non-negative values.

The negative sign highlighted at the coordinate of \vec{V}_i comes from rows $r_1, r_5, r_4,$ and r_3 . For instance, if we assume that $\|\vec{V}_4 + \sum_{i=1}^3 \varepsilon_i \vec{V}_i\|_\infty = \sum_{i=1}^3 |\varepsilon_i \alpha_i| + |\alpha_4|$, where $r_4 = (1, -1, -1, 1)$, in order to get maximum of $K(4)$, it is preferable to consider the two coordinates α_2 , and α_3 , as negative values such that the equation $1(\alpha_1) - 1(\alpha_2) - 1(\alpha_3) + 1(\alpha_4) = K(4)$ will be equivalent to the equation

$$|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = K(4).$$

The row distribution can be formulated by the following systems of equations

$$\begin{aligned} (r_1, r_2): & \begin{cases} \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 = K \\ \alpha_4 - \alpha_3 + \alpha_2 + \alpha_1 = K \end{cases} \Rightarrow \alpha_3 = 0 \\ (r_5, r_7): & \begin{cases} \beta_4 + \beta_3 + \beta_2 + \beta_1 = K \\ \beta_4 + \beta_3 - \beta_2 + \beta_1 = K \end{cases} \Rightarrow \beta_2 = 0 \\ (r_4, r_8): & \begin{cases} \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 = K \\ \gamma_4 + \gamma_3 + \gamma_2 - \gamma_1 = K \end{cases} \Rightarrow \gamma_1 = 0 \\ (r_3, r_6): & \begin{cases} w_4 + w_3 + w_2 + w_1 = K \\ -w_4 + w_3 + w_2 + w_1 = K \end{cases} \Rightarrow w_4 = 0 \end{aligned}$$

The system can be simplified further by

$$\begin{cases} \alpha_4 + \alpha_2 + \alpha_1 = K \\ \beta_4 + \beta_3 + \beta_1 = K \\ \gamma_4 + \gamma_3 + \gamma_2 = K \\ w_3 + w_2 + w_1 = K \end{cases}$$

Under the below constraints

$$\begin{cases} \alpha_4^2 + \beta_4^2 + \gamma_4^2 = 1 \\ w_4^2 + \beta_3^2 + \gamma_3^2 = 1 \\ \alpha_2^2 + w_2^2 + \gamma_2^2 = 1 \\ \alpha_1^2 + \beta_1^2 + w_1^2 = 1 \end{cases}$$

As before, the previous system of equations needs to be matched with coordinates of the four vectors in order to maximize the value of $K(4)$, then

$$[\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4] = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 & \alpha_4 \\ -\beta_1 & 0 & \beta_3 & \beta_4 \\ 0 & -\gamma_2 & -\gamma_3 & \gamma_4 \\ w_1 & -w_2 & w_3 & 0 \end{bmatrix}$$

By symmetry, we can assume that

$$\alpha_4 = \alpha_2 = \alpha_1 = \alpha$$

$$\beta_4 = \beta_3 = \beta_1 = \beta$$

$$\gamma_4 = \gamma_3 = \gamma_2 = \gamma$$

$$w_3 = w_2 = w_1 = w$$

Therefore

$$\begin{cases} \alpha = K/3 \\ \beta = K/3 \\ \gamma = K/3 \\ w = K/3 \end{cases}$$

To maximize K , the constraints can be assumed to be as follows:

$$\begin{aligned} 1 &= \alpha^2 + \beta^2 + \gamma^2 \\ &= \alpha^2 + \beta^2 + w^2 \\ &= \alpha^2 + w^2 + \gamma^2 \\ &= w^2 + \beta^2 + \gamma^2 \end{aligned}$$

To find the value of $K(4)$, it is enough to solve the below quadrature equation

$$\alpha^2 + \beta^2 + \gamma^2 = 3 \left(\frac{K}{3} \right)^2 = 1.$$

It implies that

$$K(4) \geq \sqrt{3},$$

and the coordinates of the particular set of vectors \vec{V}_i are summarized under the below matrix

$$[\vec{V}_1 \ \vec{V}_2 \ \vec{V}_3 \ \vec{V}_4] = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

Note: Other distribution can formulated by the following the below configuration:

The matrix can be formulated differently as follows:

$$A_4 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 \\ -1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 \\ -1 & -1 & -1 & +1 \end{bmatrix}$$

From the below Table 1, the row distribution can configured by the following systems of equations

$$(r_1, r_2): \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = K \\ -\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = K \end{cases} \Rightarrow \alpha_1 = 0$$

$$(r_6, r_8): \begin{cases} \beta_1 + \beta_2 + \beta_3 + \beta_4 = K \\ \beta_1 - \beta_2 + \beta_3 + \beta_4 = K \end{cases} \Rightarrow \beta_2 = 0$$

$$(r_3, r_7): \begin{cases} \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = K \\ \gamma_1 + \gamma_2 - \gamma_3 + \gamma_4 = K \end{cases} \Rightarrow \gamma_3 = 0$$

$$(r_4, r_5): \begin{cases} w_1 + w_2 + w_3 + w_4 = K \\ w_1 + w_2 + w_3 - w_4 = K \end{cases} \Rightarrow w_4 = 0$$

Rows	alpha	beta	Gamma	Lambda	Index vector and coordinate= 0								Dimension					
1	1	1	1	1	a											dim=2	dim=3	dim=4
2	-1	1	1	1	a													
3	1	-1	1	1						g								
4	-1	-1	1	1										w	dim=2			
5	1	1	-1	1										w	dim=2			
6	-1	1	-1	1		b										dim=2		
7	1	-1	-1	1						g						dim=3		
8	-1	-1	-1	1		b										dim=2		

Table 1 How to gather two rows of the matrix in order to eliminate a given index coordinate.

By finishing the calculation, we find that

$$K(4) \geq \sqrt{3}.$$

The Cauchy-Schwarz inequality, also known as the Cauchy–Bunyakovsky–Schwarz inequality, can be used to optimized the following system:

Maximizing the variable K , under the fourth objective functions:

$$\begin{cases} \alpha_2 + \alpha_3 + \alpha_4 = K \\ \beta_1 + \beta_3 + \beta_4 = K \\ \gamma_1 + \gamma_2 + \gamma_4 = K \\ w_1 + w_2 + w_3 = K \end{cases}$$

Under the below constraints:

$$\begin{cases} \alpha_4^2 + \beta_4^2 + \gamma_4^2 = 1 \\ w_3^2 + \beta_3^2 + \gamma_3^2 = 1 \\ \alpha_2^2 + w_2^2 + \gamma_2^2 = 1 \\ \gamma_1^2 + \beta_1^2 + w_1^2 = 1 \end{cases}$$

The system can be modified by Cauchy as follows:

$$\begin{cases} K^2 = (\alpha_2 + \alpha_3 + \alpha_4)^2 \leq (\alpha_2^2 + \alpha_3^2 + \alpha_4^2)(1^2 + 1^2 + 1^2) \\ K^2 = (\beta_1 + \beta_3 + \beta_4)^2 \leq (\beta_1^2 + \beta_3^2 + \beta_4^2)(1^2 + 1^2 + 1^2) \\ K^2 = (\gamma_1 + \gamma_2 + \gamma_4)^2 \leq (\gamma_1^2 + \gamma_2^2 + \gamma_4^2)(1^2 + 1^2 + 1^2) \\ K^2 = (w_1 + w_2 + w_3)^2 \leq (w_1^2 + w_2^2 + w_3^2)(1^2 + 1^2 + 1^2) \end{cases}$$

By adding all the four equation will get

$$4K^2 \leq 3(\alpha_4^2 + \alpha_2^2 + \alpha_3^2 + \beta_4^2 + \beta_3^2 + \beta_1^2 + \gamma_4^2 + \gamma_2^2 + \gamma_1^2 + w_3^2 + w_2^2 + w_1^2),$$

from constraints, the maximum of the value K can be calculated as follows:

$$4K^2 = 12,$$

then the constant of the optimization is found to be as follows

$$K = \sqrt{3},$$

Then the komlos constant has a lower bound as follows

$$K(4) \geq \sqrt{3}.$$

The coordinate of the four vectors can be calculated from the equality of Cauchy-Schwarz inequality property that states that

$$\begin{cases} (\alpha_4 + \alpha_2 + \alpha_3)^2 = (\alpha_4^2 + \alpha_2^2 + \alpha_3^2)(1^2 + 1^2 + 1^2) \Rightarrow \frac{\alpha_3}{1} = \frac{\alpha_2}{1} = \frac{\alpha_4}{1} \\ (\beta_4 + \beta_3 + \beta_1)^2 = (\beta_4^2 + \beta_3^2 + \beta_1^2)(1^2 + 1^2 + 1^2) \Rightarrow \frac{\beta_1}{1} = \frac{\beta_3}{1} = \frac{\beta_4}{1} \\ (\gamma_4 + \gamma_2 + \gamma_1)^2 = (\gamma_4^2 + \gamma_1^2 + \gamma_2^2)(1^2 + 1^2 + 1^2) \Rightarrow \frac{\gamma_1}{1} = \frac{\gamma_2}{1} = \frac{\gamma_4}{1} \\ (w_3 + w_2 + w_1)^2 = (w_3^2 + w_2^2 + w_1^2)(1^2 + 1^2 + 1^2) \Rightarrow \frac{w_1}{1} = \frac{w_2}{1} = \frac{w_3}{1} \end{cases}$$

Therefore

$$\alpha_i = \beta_i = \gamma_i = w_i = \frac{\sqrt{3}}{3}.$$

The coordinate of the particular vectors \vec{V}_i are summarized under the below matrix

$$[\vec{V}_1 \ \vec{V}_2 \ \vec{V}_3 \ \vec{V}_4] = \frac{\sqrt{3}}{3} \begin{bmatrix} 0 & +1 & +1 & +1 \\ +1 & 0 & -1 & +1 \\ +1 & -1 & 0 & +1 \\ -1 & -1 & +1 & 0 \end{bmatrix}.$$

In the case where the dimension is under the form of 2^m , for any integer m , the optimization is perfect but for other cases of dimension we can find upper bound of the constant K if Cauchy-Schwarz inequality is applied as above.

The constant K for dimension five

By using the same idea of the previous section, dimension 4, we denote by $\vec{V}_1, \dots, \vec{V}_5$ as a special vector satisfying

$$K(5) = \min_{\varepsilon_i} \left\| \vec{V}_5 + \sum_{i=1}^4 \varepsilon_i \vec{V}_i \right\|_{\infty}$$

All the different combination of $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 1)$ are summarized at the rows of the matrix A_5 defined as follows

$$A_5 = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & +1 \\ +1 & +1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & +1 \\ -1 & -1 & -1 & +1 & +1 \\ +1 & +1 & +1 & -1 & +1 \\ -1 & +1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & -1 & -1 & +1 \\ -1 & +1 & -1 & -1 & +1 \\ +1 & -1 & -1 & -1 & +1 \\ -1 & -1 & -1 & -1 & +1 \end{bmatrix}$$

where it is noted that $r_{i+2} = r_{i+1} + r_i - r_{i-1}$, for $i > 1$ and $\dim\{r_i, i = 1, \dots, 16\} = 5$.

The target is to distribute the 16 rows among to five dimensions, named $\{\alpha, \beta, \gamma, \lambda, w\}$ in such way to

minimize number of zeros in the 5 vector coordinate, $\vec{V}_i = \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \\ \lambda_i \\ w_i \end{pmatrix}, i = 1, \dots, 5.$

The rows distribution is summarized as following:

- Four rows will be assigned to each axe except axe α , where one row is a linear combination of the others $r_{i+2} = r_{i+1} + r_i - r_{i-1}$, it looks like each three independent rows will be assigned to one axe,
- Two rows will be assigned to axe α

Formulating the previous distribution of the 16 rows to the below 16 equations as follows:

For α -Axe: $\|\vec{V}_5 + \sum_{i=1}^4 \varepsilon_i \vec{V}_i\|_{\infty} = \sum_{i=1}^4 |\varepsilon_i \alpha_i| + |\alpha_5|$

$$\begin{cases} r_{15}: \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1 = K(5) \\ r_{16}: \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 - \alpha_1 = K(5) \end{cases} \Rightarrow \alpha_1 = 0$$

For β -Axe: $\|\vec{V}_5 + \sum_{i=1}^4 \varepsilon_i \vec{V}_i\|_{\infty} = \sum_{i=1}^4 |\varepsilon_i \beta_i| + |\beta_5|$

$$\begin{cases} r_1: \beta_5 + \beta_4 + \beta_3 + \beta_2 + \beta_1 = K(5) \\ r_3: \beta_5 + \beta_4 + \beta_3 - \beta_2 + \beta_1 = K(5) \\ r_5: \beta_5 + \beta_4 - \beta_3 + \beta_2 + \beta_1 = K(5) \\ r_7: \beta_5 + \beta_4 - \beta_3 - \beta_2 + \beta_1 = K(5) \end{cases} \Rightarrow \beta_2 = \beta_3 = 0$$

Note that the last equations depend on the 3 first equations.

For γ -Axe: $\|\vec{V}_5 + \sum_{i=1}^4 \varepsilon_i \vec{V}_i\|_{\infty} = \sum_{i=1}^4 |\varepsilon_i \gamma_i| + |\gamma_5|$

$$\begin{cases} r_2: \gamma_5 + \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 = K(5) \\ r_6: \gamma_5 + \gamma_4 - \gamma_3 + \gamma_2 + \gamma_1 = K(5) \\ r_{10}: \gamma_5 - \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 = K(5) \\ r_{14}: \gamma_5 - \gamma_4 - \gamma_3 + \gamma_2 + \gamma_1 = K(5) \end{cases} \Rightarrow \gamma_4 = \gamma_3 = 0$$

Note that the last equations depend on the 3 first equations.

For λ -Axe: $\|\vec{V}_5 + \sum_{i=1}^4 \varepsilon_i \vec{V}_i\|_{\infty} = \sum_{i=1}^4 |\varepsilon_i \lambda_i| + |\lambda_5|$

$$\begin{cases} r_4: \lambda_5 + \lambda_4 + \lambda_3 + \lambda_2 + \lambda_1 = K(5) \\ r_5: \lambda_5 + \lambda_4 - \lambda_3 - \lambda_2 - \lambda_1 = -K(5) \\ r_{12}: \lambda_5 - \lambda_4 + \lambda_3 + \lambda_2 + \lambda_1 = K(5) \\ r_{13}: \lambda_5 - \lambda_4 - \lambda_3 - \lambda_2 - \lambda_1 = K(5) \end{cases} \Rightarrow \lambda_4 = \lambda_5 = 0$$

Note that the last equations depend on the 3 first equations.

$$\text{For } w\text{-Axe: } \|\vec{V}_5 + \sum_{i=1}^4 \varepsilon_i \vec{V}_i\|_{\infty} = \sum_{i=1}^4 |\varepsilon_i w_i| + |w_5|$$

$$\begin{cases} r_6: w_5 + w_4 + w_3 + w_2 + w_1 = K(5) \\ r_8: w_5 + w_4 + w_3 - w_2 + w_1 = -K(5) \\ r_9: w_5 - w_4 - w_3 + w_2 + w_1 = K(5) \\ r_{11}: w_5 - w_4 - w_3 - w_2 - w_1 = K(5) \end{cases} \Rightarrow w_5 = w_2 = 0.$$

Note that the last equations depend on the 3 first equations.

From the previous systems of equations, we can shape our five vectors \vec{V}_i in order to maximize $K(5)$ as follows

$$[\vec{V}_1 \quad \vec{V}_2 \quad \vec{V}_3 \quad \vec{V}_4 \quad \vec{V}_5] = \begin{bmatrix} 0 & -\alpha_2 & -\alpha_3 & -\alpha_4 & \alpha_5 \\ \beta_1 & 0 & 0 & \beta_4 & \beta_5 \\ -\gamma_1 & \gamma_2 & 0 & 0 & \gamma_5 \\ -\lambda_1 & -\lambda_2 & \lambda_3 & 0 & 0 \\ -w_1 & 0 & -w_3 & w_4 & 0 \end{bmatrix},$$

where $\alpha_i, \beta_i, \gamma_i, \lambda_i$, and w_i are non-negative values.

The negative sign highlighted at the coordinate of \vec{V}_i comes from rows r_{15}, r_1, r_2, r_4 and r_6 , for example we have assume that $\|\vec{V}_5 + \sum_{i=1}^4 \varepsilon_i \vec{V}_i\|_{\infty} = \sum_{i=1}^4 |\varepsilon_i \alpha_i| + |\alpha_5|$ where $r_5 = (1, -1, -1, -1, 1)$. Our target is to maximize the value of $K(5)$, then it is preferable to consider α_2, α_3 , and α_4 are negative values such that the equation $1(\alpha_1) - 1(\alpha_2) - 1(\alpha_3) - 1(\alpha_4) + 1(\alpha_5) = K(5)$ will be equivalent to the below equation,

$$|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| + |\alpha_5| = K(5),$$

for notation simplification notation, we write the negative parameter α_i as $-\alpha_i$.

To calculate the constant $K(5)$, we need to solve the below system

$$\begin{cases} \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = K(5) \\ \beta_1 + \beta_4 + \beta_5 = K(5) \\ \gamma_1 + \gamma_2 + \gamma_5 = K(5) \\ \lambda_1 + \lambda_2 + \lambda_3 = K(5) \\ w_1 + w_3 + w_4 = K(5) \end{cases}$$

Under the constraint $\|\vec{V}_i\| \leq 1, i = 1, \dots, 5$.

By symmetry, we can assume the following

$$\alpha_5 = \alpha_4 = \alpha_3 = \alpha_2 = \alpha$$

$$\gamma_5 = \gamma_2 = \gamma$$

$$\lambda_3 = \lambda_2 = \lambda$$

$$w_4 = w_3 = w$$

$$\beta_1 = \gamma_1 = \lambda_1 = w_1$$

As $\|\vec{v}_1\|_2 \leq 1$, we can put

$$\beta_1 = \gamma_1 = \lambda_1 = w_1 = \frac{1}{2}$$

The system will be summarized as follow:

$$\begin{cases} 4\alpha = K(5) \\ 2\beta = K(5) - \frac{1}{2} \\ 2\gamma = K(5) - \frac{1}{2} \\ 2\lambda = K(5) - \frac{1}{2} \\ 2w = K(5) - \frac{1}{2} \end{cases}$$

Under the constraints

$$\begin{aligned} 1 &= \alpha^2 + \beta^2 + \gamma^2 \\ &= \alpha^2 + \beta^2 + w^2 \\ &= \alpha^2 + \lambda^2 + w^2 \\ &= \alpha^2 + \lambda^2 + \gamma^2 \end{aligned}$$

The system be equivalent to quadratic equation

$$\left(\frac{K(5)}{4}\right)^2 + 2\left(\frac{K(5)}{2} - \frac{1}{4}\right)^2 = 1.$$

So we can conclude that the value of $K(5)$ is lower bounded by $\frac{4+\sqrt{142}}{9}$ i.e.

$$K(5) \geq \frac{4 + \sqrt{142}}{9}.$$

To see the importance of the way of distributing the rows among the axes is very important, we try to make, as an example, another configuration as follows:

$$\begin{cases} (r_7, r_8, r_9, r_{10}) & \text{for } \alpha - \text{Axe} \\ (r_2, r_3, r_4) & \text{for } \beta - \text{Axe} \\ (r_{12}, r_{14}, r_{16}) & \text{for } \gamma - \text{Axe} \\ (r_5, r_{13}, r_1) & \text{for } \lambda - \text{Axe} \\ (r_6, r_{11}, r_{15}) & \text{for } w - \text{Axe} \end{cases}$$

The five vectors coordinate will be summarized under the below matrix

Conclusion

In dimension n , it is very crucial to find a best way to distribute all possible combinations of the vectors $\vec{\varepsilon} = (1, \varepsilon_1, \dots, \varepsilon_{n-1})$ among the n axes. We assume that we have $\left\lfloor \frac{2^{n-1}}{n} \right\rfloor$ different combinations of the vector of $\vec{\varepsilon}$ for which

$$\left\| \vec{V}_n + \sum_{i=1}^{n-1} \varepsilon_i \vec{V}_i \right\|_{\infty} = \left| \sum \varepsilon_i x_i + x_n \right| = K(n),$$

where x_i is the coordinate of vector \vec{V}_i corresponding to X-Axe.

The $\left\lfloor \frac{2^{n-1}}{n} \right\rfloor$ vectors that have been assign to one axes has a dimension of order $\left\lfloor \log_2 \left(\frac{2^{n-1}}{n} \right) \right\rfloor$, and as consequence, it implies that each vector \vec{V}_i has $\left\lfloor \log_2 \left(\frac{2^{n-1}}{n} \right) \right\rfloor$ null coordinate.

To evaluate the constant $K(n)$, it is enough to solve the below optimization equation

$$K(n) = \max_{x_i} \left(\begin{array}{c} n - \left\lfloor \log_2 \left(\frac{2^{n-1}}{n} \right) \right\rfloor \\ \sum_{i=1} \quad x_i \end{array} \right).$$

By imposing the symmetry conditions by choosing a good way of distribution, the non-null coordinate in each axes as constant values, i.e., $x_i = x$.

Let B a subset $\{1, \dots, n\}$ of cardinality around $n - \left\lfloor \log_2 \left(\frac{2^{n-1}}{n} \right) \right\rfloor$ and from the condition that $\|\vec{V}_n\|_2 \leq 1$, the upper bound of x can be found as following

$$\sum_{j \in B} (x_j^j)^2 = \sum_{j \in B} (x)^2 \leq 1 \Rightarrow x \leq \frac{\sqrt{n - \left\lfloor \log_2 \left(\frac{2^{n-1}}{n} \right) \right\rfloor}}{n - \left\lfloor \log_2 \left(\frac{2^{n-1}}{n} \right) \right\rfloor}.$$

The lower bound of Kmolos conjecture can be calculated as follows:

$$\begin{aligned} K(n) &\geq \sqrt{n - \left\lfloor \log_2 \left(\frac{2^{n-1}}{n} \right) \right\rfloor} \\ &\geq \sqrt{\log(n) - 1}. \end{aligned}$$

Under our lemma, if it exists an natural n such that $n = 2^k$, then the symmetry conditions can be used always in order to conclude that

$$K(n) = \sqrt{n - \left\lfloor \log_2 \left(\frac{2^{n-1}}{n} \right) \right\rfloor} = \sqrt{\log_2(n)}.$$

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