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Existence of fractional impulsive functional integro-differential equations in Banach spaces

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Abstract

In this paper, we established the existence of PC-mild solutions for non local fractional impulsive functional integro-differential equations with finite delay. The proofs are obtained by using the techniques of fixed point theorems, semi-group theory and generalized Bellman inequality. In this paper, we have used the distributed characteristic operators to define the mild solution of the system. Results obtained here improve and extend some known results.

Keywords: Fractional differential equations: Impulse: Integro-differential equations: Non local conditions: Fixed point theorem:

2010 Mathematics Subject Classification: 34A08, 34A37, 93B05, 47H10, 45J05.

1 Introduction

Fractional calculus has gained considerable popularity and importance during the past four decades, because fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various processes. Fractional differential equations draw a great applications in many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic models. The most important advantage of using fractional differential equations in these and other applications is their non local property. Also, the study of fractional differential equations have gained considerable importance due to their application in various fields of bio-engineering, mechanics, electrical networks, control theory of dynamical systems, viscoelasticity and so on. In recent years there has been a significant development in fractional differential equations involving fractional derivatives, see the monographs of [1, 20, 22–24, 26, 32, 38] and the papers [2, 5–7, 17, 18, 25, 27, 28, 30, 39–42].

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The study of impulsive differential equation is linked to their utility in simulating processes and phenomena subject to short time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. Integro-differential equations play an important role in many branches of linear and non linear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, biology, economics, and electrostatics. In recent years, impulsive integro-differential equations have become an important object of investigation stimulated by their numerous applications to problems in mechanics, electrical engineering, medicine, biology, ecology etc, we refer [3, 4, 8, 12, 13, 19, 21, 34–36, 45]. Dynamics of many evolutionary processes from various fields such as population dynamics, control theory, physics, biology, and medicine undergo abrupt changes at certain moments of time like earthquack, harvesting, shock, and so forth. These perturbations can be well approximated as instantaneous change of states or impulses. These processes are modeled by impulsive differential equations. The advantage of using non local conditions is they are measurable at more places and those can be incorporated to get better models. The non local Cauchy problem for abstract evolution differential equation was first studied by Byszewski [10]. For the importance of non local conditions in different fields, we refer the reader to [9, 11, 29, 44].
Recently in [33], the authors studied the existence of mild solutions for impulsive fractional semi-linear integro-differential equations using Banach contraction principle and Schaefer's fixed point theorem. They have considered the system without delay and without non local condition. Our work generalizes the work done in [33] with abstract formulation. According to our knowledge this is an untreated article in the literature.
Motivated by the above mentioned paper, we study the existence of mild solutions for non local impulsive fractional semi-linear integro-differential equations of the form

$$ {}^{C}D^{q}_{0,t}x(t) = {}^{C}D^{q}_{t}x(t) = D^{q}_{t}x(t) = Ax(t) + f\Big(t, x_t, \int_0^t h(t,s,x_s)ds, \int_0^b k(t,s,x_s)ds\Big), \quad (1.1)$$

$$x(0) = x_0 + g(x) \in X, \quad (1.2)$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m; \ t \in J = [0, b], \ t \neq t_k, \quad (1.3)$$

where ${}^{C}D^{q}_{t}$ is the Caputo fractional derivative of order q, $0 < q < 1$, with lower limit zero, the histories $x_t : (-r, 0] \to X$ are defined by $x_t(\theta) = x(t+\theta)$ belongs to a Banach space X. $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semi-group $(T(t))_{t\geq 0}$ of a uniformly bounded operator on X, and A is a bounded linear operator. $f : J \times X \times X \times X \to X$ is jointly continuous, $h, k : J \times J \times X \to X$ are continuous, $I_k : X \to X$ are impulsive functions, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively. Our method avoids the compactness conditions on the semi-group $\{T(t)\}_{t\geq 0}$, and some other hypotheses are more general compared with the previous research papers.
System (1.1)-(1.3) described above fits to the mechanical system with impact, the biological

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phenomenon involving thresholds, the bursting rhythm models and industrial robotics, and many more. In particular, above mentioned system (1.1)-(1.3) and its PC-mild solution (in Section 2) is helpful to generate tuning and auto-tuning of fractional order controllers for industry applications. Monje et al.[47] design the fractional order $PI^{\lambda}D^{\mu}$ controllers to ensuring a robust performance of the controlled system with respect to gain variations and noise, without using the delay part, but can be generalized in a more specific way using the delay for a closed loop and the open loop systems. System (1.1)-(1.3) can also be analyzed for a finite time stability test procedure for robotic system where it appears a time delay in fractional control system (refer [48]) but with impulses.
In Section 2, we give some preliminary definitions and lemmas those are to be used later to prove our main results. In Section 3, the existence of PC-mild solutions for equations (1.1)-(1.3) with non-local conditions is discussed. The results are obtained by using Banach contraction principle and Schaefer's fixed point theorem. An example is given in Section 4 to illustrate the application of our main results.

2 Preliminaries

Let us consider the set of functions $PC[J,X] = \{x : J \to X \mid x \in C[(t_k, t_{k+1}), X]$ and there exists $x(t_k^+)$ and $x(t_k^-)$, $k = 0, 1, 2, \cdots, m$ and $x(t_k^-) = x(t_k)\}$, endowed with the norm $||x||_{PC} = \sup_{t\in J} ||x(t)||$. It is easy to know that $(PC[J,X], ||.||_{PC})$ is a Banach space. Throughout this paper, let A be the infinitesimal generator of a C_0 semi-group $(T(t))_{t\geq 0}$ of a uniformly bounded operators on X and let $L_B(X)$ be the Banach space of all linear and bounded operators on X. For a C_0 semi-group $(T(t))_{t\geq 0}$, we set $M_1 = \sup_{t\in J} ||T(t)||_{L_B(X)}$.
For each positive constant r, set $B_r = \{x \in PC[J,X] : ||x|| \leq r\}$.

Let us recall the following known definitions. For more details, see [6, 22].

Definition 2.1. *The fractional integral of order α with the lower limit zero for a function f is defined as*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \quad \alpha > 0,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$.

Definition 2.2. *The Riemann-Liouville (R-L) fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in N$, is defined as*

$$^{(R-L)}D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3. *The (strong or classical) Caputo derivative of order α for a function $f \in$*

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$L^1([0,\infty),R)$ *given on the interval* $[0,\infty)$ *is defined by (if it exists)*

$$^{C}D_s^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds, \quad t>0, \quad n-1<\alpha\leq n,\ n\in N.$$

Remark 2.1. *1. R-L and Caputo are just two different operators that are related to each other in a quite simple way. Quite a few details about this are given in the book of Diethelm [46]. There we can also see the exact description of when they are equivalent. The most important difference between them is, of course, the structure of their kernels (i.e the set of functions that is mapped to zero). Depending on what we want from our operator, one of them or the other one may be the right choice for us. We are using here Caputo because derivative of a constant is zero for Caputo but not for R-L.*

2. If derivative of Caputo type are used instead of R-L type then initial conditions for the corresponding Caputo fractional differential equations can be formulated as for classical ordinary equations, namely $x(0)=x_0$.

3. One has to make sure using a constant function and the Heaviside unit step. They must be considered different so they must have different fractional derivatives. The Heaviside unit step is expected to have a non zero fractional derivative. If one given derivative gives zero it is useless due to the importance it enjoys in practice and its relation with the Dirac delta.

Definition 2.4. *The (weak or generalization of classical) Caputo derivative of order* α *for a function* $f\in L^1([0,\infty),R)$ *given on the interval* $[0,\infty)$ *is defined by (if it exists)*

$$^{C}D_w^{\alpha}f(t) = {}^{C}D^{\alpha}f(t) = D^{\alpha}\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t>0, \quad n-1<\alpha\leq n,\ n\in N.$$

Remark 2.2. *1. If* $f(t)\in C^n[0,\infty)$*, then*

$$^{C}D_w^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds = I^{n-\alpha}f^{(n)}(t) = {}^{C}D_s^{\alpha}f(t),\ t>0,\ n-1<\alpha\leq n.$$

Note that for strong Caputo derivative in Definition 2.3, $f\in C^n([0,\infty),R)$ *is not necessarily required. In fact,* $f^{(n)}\in L^1([0,\infty),R)$*, for example,* $f^{(n-1)}$ *be of bounded variation, can guarantee the existence of* $D_s^{\alpha}f(t)$ *on* $[0,\infty)$.

2. The Caputo derivative of a constant is equal to zero.

3. If f *is an abstract function with values in* X*, then integrals which appear in Definition* 2.1 *and* 2.2 *are taken in Bochner's sense.*

Definition 2.5. *[42] By a PC-mild solution of the equations (1.1)-(1.3) we mean that a func-*

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tion $x \in PC[J, X]$ *which satisfies the following integral equation:*

$$x(t) = \begin{cases} \mathcal{T}(t)(x_0 + g(x)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \qquad (\times) f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau)ds, \quad t \in [0, t_1], \\ \mathcal{T}(t)(x_0 + g(x)) + \mathcal{T}(t-t_1)I_1(x(t_1^-)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \qquad (\times) f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau)ds, \quad t \in (t_1, t_2], \\ \vdots \\ \mathcal{T}(t)(x_0 + g(x)) + \sum_{k=1}^{m} \mathcal{T}(t-t_k)I_k(x(t_k^-)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \qquad (\times) f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau)ds, \quad t \in (t_m, b], \end{cases}$$

where $\mathcal{T}(.)$ *and* $\mathcal{S}(.)$ *are called characteristic solution operators and given by*

$$\mathcal{T}(t) = \int_0^\infty \xi_q(\theta)T(t^q\theta)d\theta, \quad \mathcal{S}(t) = q\int_0^\infty \theta\xi_q(\theta)T(t^q\theta)d\theta,$$

and for $\theta \in (0, \infty)$,

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\varpi_q(\theta^{-\frac{1}{q}}) \geq 0, \quad \varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}sin(n\pi q),$$

where ξ_q *is a probability density function defined on* $(0, \infty)$, *that is*

$$\xi_q(\theta) \geq 0, \quad \theta \in (0, \infty) \quad and \quad \int_0^\infty \xi_q(\theta)d\theta = 1.$$

Remark 2.3. *Controversy on the solution operator, Definition 2.5, based on Definition 2.3 and Definition 2.4:*

1. *In this paper we emphasize that we use the generalized Caputo derivative with the lower bound at zero for the equation (1.1). However, we have not chosen the classical Caputo derivative and have not changed it in each sub-intervals for the equation (1.1), where the impulses start at the lower bound* t_k. *Obviously, we mean keeping a different one, in each of the impulses the lower bound is at zero. Moreover, Definition 2.5 is more reasonable since the generalized Caputo derivative in the equation (1.1) should be fixed at the lower bound at zero once we set initial time at zero. So we do not expect to change the lower bound again and again in the definition of Caputo derivative for the same equation.*

2. *We use Definition 2.4 (generalized Caputo derivative), where the integrable function* f *can be discontinuous. Definition 2.4 is more general with respect to Remark 2.2 (1) (relationship between strong and weak Caputo derivatives). So result would be wrong if we have used strong Caputo derivative.*

3. *Finally, we would like to mention the recently published paper written by Liu and Ahmed*

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[43], where the formula of solutions for semi-linear impulsive fractional Cauchy problems (see (20) in [43]) is coincided with ours (see Definition 2.5), if one imposes that the semi-linear term and the impulsive term have the same expression in the given interval.

Remark 2.4. *Problems associated with impulsive effects and hereditary property are modeled by impulsive delay differential equations. So we use impulsive finite delay system (1.1)-(1.3) and the solution is a piecewise continuous with discontinuities at impulses time. So, here the mild solution is called PC-mild solution. (As we know a function x is continuous is said to be a mild solution. A function x which is differential almost everywhere on $[0, T]$ is called a strong solution. Clearly, every strong solution is a mild solution, since differentiability implies continuity).*

Definition 2.6. *[31] Let X be a Banach space, a one parameter family $T(t)$, $0 \leq t < +\infty$, of bounded linear operators from X to X is a semi-group of bounded linear operators on X if*
(1) $T(0) = I$, (here I is the identity operator on X)
(2) $T(t + s) = T(t)T(s)$ for every $t, s \geq 0$, (the semi-group property)
A semi-group of bounded linear operator, $T(t)$, is uniformly continuous if $\lim_{t \downarrow 0} ||T(t) - I|| = 0$.

Lemma 2.1. *[31] Linear operator A is the infinitesimal generator of a uniformly continuous semi-group if and only if A is the bounded linear operator.*

Lemma 2.2. *[37] (Schaefer's fixed point theorem) Let X be a Banach space and $F : X \to X$ be a completely continuous operator. If the set*

$$E(F) = \{x \in X \ : \ x = \lambda F x \ for \ some \ 0 \leq \lambda \leq 1\}.$$

is bounded, then F has at least a fixed point.

Lemma 2.3. *[42] The operator $\mathcal{T}(t)$ and $\mathcal{S}(t)$ have the following properties:*

1. *For any fixed $t \geq 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ linear and bounded operator, i.e. for any $x \in X$,*

$$||\mathcal{T}(t)x|| \leq M_1||x||, \quad ||\mathcal{S}(t)x|| \leq \frac{qM_1}{\Gamma(1+q)}||x||.$$

2. *$\{\mathcal{T}(t), \ t \geq 0\}$ and $\{\mathcal{S}(t), \ t \geq 0\}$ are strongly continuous.*

3. *$\{\mathcal{T}(t), \ t \geq 0\}$ and $\{\mathcal{S}(t), \ t \geq 0\}$ are uniformly continuous, that is, for each fixed $t > 0$, and $\epsilon > 0$, there exists $h > 0$ such that*

$$||\mathcal{T}(t+\epsilon) - \mathcal{T}(t)|| \leq \epsilon, \ for \ t+\epsilon \geq 0 \ and \ |\epsilon| < h,$$
$$||\mathcal{S}(t+\epsilon) - \mathcal{S}(t)|| \leq \epsilon, \ for \ t+\epsilon \geq 0 \ and \ |\epsilon| < h.$$

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3 Existence results

In this section, we give the existence of mild solutions of the system (1.1) – (1.3). To establish our results, we introduce the following hypotheses:

(**H1**) $f : J \times X \times X \times X \to X$ is continuous, and there exists functions $\mu_1, \mu_2, \mu_3 \in L[J, R^+]$ such that

$$||f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)|| \leq \mu_1(t)||x_1 - y_1|| + \mu_2(t)||x_2 - y_2|| + \mu_3(t)||x_3 - y_3||,$$
$$x_i, y_i \in X, \;\; i = 1, 2, 3.$$

(**H2**) $h, k : J \times J \times X \to X$ is continuous and there exist $M_h,\; M_k > 0$ such that

$$\begin{aligned} ||h(t, s, x_1) - h(t, s, y_1)|| &\leq M_h||x_1 - y_1||, \\ ||k(t, s, x_1) - k(t, s, y_1)|| &\leq M_k||x_1 - y_1||, \quad x_1,\; y_1 \in X. \end{aligned}$$

(**H3**) $g : PC([0, b], X)$ is continuous and there exists a constant $G > 0$ such that

$$\begin{aligned} ||g(x) - g(y)|| &\leq G||x - y||, \quad \forall x, y \in PC([0, b], X) \\ ||g(0)|| &\leq k_1. \end{aligned}$$

(**H4**) The function $I_k : X \to X$ are continuous and there exist $\rho_k > 0$ such that

$$||I_k(x) - I_k(y)|| \leq \rho_k||x - y||, \quad x, y \in X, \quad k = 1, 2, \cdots, m.$$

(**H5**) The function $\Omega_m(t) \; : J \to \mathbb{R}^+$ is defined by

$$\Omega_m(t) = M_1(G + m\rho_m) + \frac{M_1 b^q}{\Gamma(1 + q)}\Big(\mu_1(t) + \mu_2(t)M_h b + \mu_3(t)M_k b\Big),$$

where $0 < \Omega_m(t) < 1,\; t \in J$.

(**H6**) The constants Ω_u and $\Omega'_m(t) : J \to \mathbb{R}^+$ are defined by

$$\begin{aligned} \Omega_u &= M_1K(G + m\rho_m) + \frac{M_1 b^q K}{\Gamma(1 + q)}\Big(\mu_1(t) + \mu_2(t)M_h b + \mu_3(t)M_k b\Big) \\ \Omega'_m(t) &= M_1(G + m\rho_m) + \frac{M_1 b^q}{\Gamma(1 + q)}\Big(\mu_1(t) + \mu_2(t)M_h b + \mu_3(t)M_k b\Big) + \frac{M_1 b^q \Omega_u}{\Gamma(1 + q)} \end{aligned}$$

and $0 < \Omega'_m(t) < 1,\; t \in J$.

Theorem 3.1. *If the hypotheses $(H1) - (H5)$ are satisfied, then the nonlocal fractional impulsive integro-differential equations (1.1)-(1.3) has a unique mild solution $x \in PC[J, X]$.*

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Proof: Define an operator N on $PC[J, X]$ by

$$(Nx)(t) = \begin{cases} \mathcal{T}(t)(x_0 + g(x)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \qquad f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)ds, \quad t \in [0, t_1], \\ \mathcal{T}(t)(x_0 + g(x)) + \mathcal{T}(t-t_1)I_1(x(t_1^-)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \qquad f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)ds,, \quad t \in (t_1, t_2], \\ \vdots \\ \mathcal{T}(t)(x_0 + g(x)) + \sum_{k=1}^{m} \mathcal{T}(t-t_k)I_k(x(t_k^-)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \qquad f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)ds, \quad t \in (t_m, b]. \end{cases} \tag{3.1}$$

We shall show that N is well defined on $PC[J, X]$. For $0 \le \tau < t \le t_1$, applying (3.1), we obtain

$$\begin{aligned} ||(Nx)(t) - (Nx)(\tau)|| &\le ||T(t) - T(\tau)||\ ||x_0 + g(x)|| \\ &\quad || \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)ds|| \\ &\le ||T(t) - T(\tau)||\ [||x_0|| + G||x|| + k_1] \\ &\quad + \Big\| \int_\tau^t (t-s)^{q-1}\mathcal{S}(t-s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)ds \Big\| \\ &\quad + \Big\| \int_0^\tau (t-s)^{q-1} [\mathcal{S}(t-s) - \mathcal{S}(\tau-s)] \\ &\qquad\qquad (\times) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)ds \Big\| \\ &\quad + \Big\| \int_0^\tau \left[(t-s)^{q-1} - (\tau-s)^{q-1}\right] \mathcal{S}(\tau-s) \\ &\qquad\qquad (\times) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)ds \Big\|. \end{aligned}$$

We know that the inequality $|t^\sigma - \tau^\sigma| \le (t-\tau)^\sigma$ for $\sigma \in (0, 1]$ and $0 < \tau \le t$ and Lemma 2.3, it is obviously that $||(Nx)(t) - (Nx)(\tau)|| \to 0$ as $t \to \tau$. Thus, $Nx \in [(0, t_1], X]$.

For $t_1 < \tau < t \le t_2$, we have

$$\begin{aligned} ||(Nx)(t) - (Nx)(\tau)|| &\le ||\mathcal{T}(t) - \mathcal{T}(\tau)||\ [||x_0|| + G||x|| + k] \\ &\quad + ||\mathcal{T}(t-t_1) - \mathcal{T}(\tau-t_1)||\ ||I_1(x(t_1^-))|| \\ &\quad + \Big\| \int_\tau^t (t-s)^{q-1}\mathcal{S}(t-s) \\ &\qquad\qquad (\times) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)ds \Big\| \end{aligned}$$

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$$+\left\|\int_0^\tau (t-s)^{q-1}\left[\mathcal{S}(t-s)-\mathcal{S}(\tau-s)\right]\right.$$
$$\left.(\times) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)ds\right\|$$
$$+\left\|\int_0^\tau \left[(t-s)^{q-1}-(\tau-s)^{q-1}\right]\mathcal{S}(\tau-s)\right.$$
$$\left.(\times) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)ds\right\|$$

It is easy to get, as $t \to \tau$, the right hand side of the above inequality tends to zero. Thus, we can deduce that $Nx \in C[(t_1, t_2], X]$. By repeating the same procedure, we can also obtain that $Nx \in C[(t_2, t_3], X], \cdots, Nx \in C[(t_m, b], X]$. That is, $Nx \in PC[J, X]$. Take $t \in (0, t_1]$, then

$$\begin{aligned} ||(Nx)(t)-(Ny)(t)|| &\le M_1 G||x-y||_{PC} + \frac{qM_1}{\Gamma(1+q)}\int_0^t (t-s)^{q-1} \\ &\Big[\mu_1(s)||x_s-y_s|| + \mu_2(s)M_h b||x_s-y_s|| + \mu_3(s)M_k b||x_s-y_s||\Big]\, ds. \end{aligned}$$

So we deduce that

$$||(Nx)(t)-(Ny)(t)||_{PC} \le \left[M_1 G + \frac{M_1 b^q}{\Gamma(1+q)}\Big(\mu_1(t)+\mu_2(t)M_h b+\mu_3(t)M_k b\Big)\right]||x-y||_{PC} \tag{3.2}$$

For each $t \in (t_1, t_2]$, using hypotheses, and (3.2), we have

$$\begin{aligned} ||(Nx)(t)-(Ny)(t)||_{PC} &\le [M_1(G+\rho_1) \\ &+\frac{M_1 b^q}{\Gamma(1+q)}\Big(\mu_1(t)+\mu_2(t)M_h b+\mu_3(t)M_k b\Big)\Big]\, ||x-y||_{PC} \end{aligned}$$

In general, for each $t \in (t_i, t_{i+1}]$, using (H5)

$$\begin{aligned} ||(Nx)(t)-(Ny)(t)||_{PC} &\le [M_1(G+m\rho_m) \\ &+\frac{M_1 b^q}{\Gamma(1+q)}\Big(\mu_1(t)+\mu_2(t)M_h b+\mu_3(t)M_k b\Big)\Big]\, ||x-y||_{PC} \\ &\le \Omega_m(t)||x-y||_{PC}. \end{aligned}$$

From the assumption (H5) and in the view of the contraction mapping principle, N has a unique fixed point $x \in PC[J, X]$, that is

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$$x(t)=\begin{cases}\mathcal{T}(t)(x_0+g(x))+\int_0^t(t-s)^{q-1}\mathcal{S}(t-s)\\ \qquad(\times)f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau,\int_0^b k(s,\tau,x_\tau)d\tau)ds, \quad t\in[0,t_1],\\ \mathcal{T}(t)(x_0+g(x))+\mathcal{T}(t-t_1)I_1(x(t_1^-))+\int_0^t(t-s)^{q-1}\mathcal{S}(t-s)\\ \qquad(\times)f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau,\int_0^b k(s,\tau,x_\tau)d\tau)ds,, \quad t\in(t_1,t_2],\\ \vdots\\ \mathcal{T}(t)(x_0+g(x))+\sum_{k=1}^{m}\mathcal{T}(t-t_k)I_k(x(t_k^-))+\int_0^t(t-s)^{q-1}\mathcal{S}(t-s)\\ \qquad(\times)f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau,\int_0^b k(s,\tau,x_\tau)d\tau)ds, \quad t\in(t_m,b],\end{cases}$$

is a PC-mild solution of equations (1.1)-(1.3).

Next theorem is based on Schaefer's fixed point theorem, let us list the following hypotheses:

(**H7**) $f: J\times X\times X\times X\to X$ is continuous and there exist functions $c_1,c_2,c_3,c_4\in L(J,R^+)$, such that

$$||f(t,x,y,z)||\le c_1(t)+c_2(t)||x||+c_3(t)||y||+c_4(t)||z||,\ t\in J,\ \ x,y,z\in X.$$

(**H8**) $h,k: J\times J\times X\to X$ is continuous and there exist functions $d_1,d_2,d_3,d_4\in C(I,R^+)$, such that

$$\begin{aligned}||h(t,s,x)|| &\le d_1(s)+d_2(s)||x||\\ ||k(t,s,y)|| &\le d_3(s)+d_4(s)||y||, \quad x,y\in X.\end{aligned}$$

(**H9**) There exist $\Phi_k\in C[J,R^+]$, such that

$$||I_k(x)||\le\Phi_k(t)||x||, \quad x\in X.$$

(**H10**) For all bounded subsets B_r, the set

$$\Pi_{h,\delta}(t)=\Big\{\mathcal{T}_\delta(t)(x_0+g(x))+\int_0^{t-h}(t-s)^{q-1}\mathcal{S}_\delta(t-s)F(s)ds \\ +\sum_{k=1}^{m}\mathcal{T}_\delta(t-t_k)I_k(x(t_k^-)):\ x\in B_r\Big\}$$

is relatively compact in X for arbitrary $h\in(0,t)$ and $\delta>0$, where

$$\mathcal{T}_\delta(t)=\int_\delta^\infty\xi_q(\theta)\mathcal{T}(t^q\theta)d\theta, \quad \mathcal{S}_\delta(t)=q\int_\delta^\infty\theta\xi_q(\theta)\mathcal{T}(t^q\theta)d\theta.$$

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(**H11**) For all bounded subsets B_r, the set

$$\Pi'_{h,\delta}(t) = \Big\{\mathcal{T}_\delta(t)(x_0 + g(x)) + \int_0^{t-h} (t-s)^{q-1}\mathcal{S}_\delta(t-s)F(s)ds + \sum_{k=1}^{m} \mathcal{T}_\delta(t-t_k)I_k(x(t_k^-)) : x \in B_r\Big\}$$

is relatively compact in X for arbitrary $h \in (0,t)$ and $\delta > 0$.

Theorem 3.2. *If the hypotheses $(H6)-(H10)$ are satisfied, then the non local fractional impulsive integro-differential equations (1.1)-(1.3) has at least one mild solution $x \in PC[J,X]$.*

Proof : From Theorem 3.1, the operator N is defined as follows:

$$(Nx)(t) = \begin{cases} \mathcal{T}(t)(x_0 + g(x)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \quad (\times) f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau)ds, \quad t \in [0,t_1], \\ \mathcal{T}(t)(x_0 + g(x)) + \mathcal{T}(t-t_1)I_1(x(t_1^-)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \quad (\times) f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau)ds, \quad t \in (t_1,t_2], \\ \vdots \\ \mathcal{T}(t)(x_0 + g(x)) + \sum_{k=1}^{m} \mathcal{T}(t-t_k)I_k(x(t_k^-)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \quad f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau)ds, \quad t \in (t_m, b]. \end{cases} \tag{3.3}$$

We shall prove the result in following steps:
Step 1 : Continuity of N on $(t_i, t_{i+1}]$ $(i = 0,1,2,\cdots,m)$ Let $x_n, x \in PC[J,X]$ such that $||x_n - x^*||_{PC} \to 0$ $(n \to +\infty)$, *then* $r = sup_n||x_n||_{PC} < \infty$ and $||x^*||_{PC} < r$, for every $t \in (t_i, t_{i+1}]$ $(i = 0,1,2,\cdots,m)$, we have from (3.3)

$$\begin{aligned} ||(Nx_n)(t) - (Nx)(t)|| \leq\ & M_1 G||x_n - x|| \\ & + \left\|\sum_{k=1}^{m} T(t-t_k)I_k(x_n(t_k^-)) - \sum_{k=1}^{m} T(t-t_k)I_k(x(t_k^-))\right\| \\ & + \frac{qM_1}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\left\| f\Big(s, x_{n_s}, \int_0^s h(s,\tau,x_{n_\tau})d\tau, \int_0^b k(s,\tau,x_{n_\tau})d\tau\Big)\right. \\ & \left. - f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)\right\| ds. \end{aligned}$$

Since the functions f, I_k and g are continuous,

$$f\Big(s, x_{n_s}, \int_0^s h(s,\tau,x_{n_\tau})d\tau, \int_0^b k(s,\tau,x_{n_\tau})d\tau\Big) \to f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big), \quad n \to \infty.$$

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By conditions (H7)-(H8) we know that

$$\begin{aligned}
&\left\| f\Big(s, x_{n_s}, \int_0^s h(s,\tau,x_{n_\tau})d\tau, \int_0^b k(s,\tau,x_{n_\tau})d\tau\Big) - f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\Big)\right\| \\
&\le c_1(s) + c_2(s)||x_{n_s}|| + c_3(s)\left\|\int_0^s h(s,\tau,x_{n_\tau})d\tau\right\| + c_4(s)\left\|\int_0^b k(s,\tau,x_{n_\tau})ds\right\| \\
&\quad + c_1(s) + c_2(s)||x_s|| + c_3(s)\left\|\int_0^s h(s,\tau,x_\tau)d\tau\right\| + c_4(s)\left\|\int_0^b k(s,\tau,x_\tau)ds\right\| \\
&\le 2c_1(s) + c_2(s)\left(||x_n|| + ||x||\right) + 2c_3(s)\int_0^s d_1(s) + c_3(s)\int_0^s d_2(s)\left(||x_n|| + ||x||\right)ds \\
&\quad + 2c_4(s)\int_0^b d_3(s)ds + c_4(s)\int_0^b d_4(s)\left(||x_n|| + ||x||\right)ds \\
&\le 2c_1(s) + 2c_3(s)\int_0^s d_1(s)ds + 2c_4(s)\int_0^b d_3(s)ds \\
&\quad + \left(c_2(s) + c_3(s)\int_0^s d_2(s)ds + c_4(s)\int_0^b d_4(s)ds\right)\left(||x_n|| + ||x||\right)
\end{aligned}$$

$$\begin{aligned}
&\le 2c_1(s) + 2c_3(s)\int_0^s d_1(s)ds + 2c_4(s)\int_0^b d_3(s)ds \\
&\quad + \left(2c_2(s) + 2c_3(s)\int_0^s d_2(s)ds + 2c_4(s)\int_0^b d_4(s)ds\right) r.
\end{aligned}$$

Hence,

$$\begin{aligned}
(t-s)^{q-1}&\left\| f\left(s, x_{n_s}, \int_0^s h(s,\tau,x_{n_\tau})d\tau, \int_0^b k(s,\tau,x_{n_\tau})d\tau\right)\right. \\
&\left. - f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\right)\right\| \in L^1[J, R^+].
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
\int_0^t (t-s)^{q-1}&\left\| f\left(s, x_{n_s}, \int_0^s h(s,\tau,x_{n_\tau})d\tau, \int_0^b k(s,\tau,x_{n_\tau})d\tau\right)\right. \\
&\left. - f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau\right)\right\| ds \to 0.
\end{aligned}$$

It is easy to get

$$\lim_{n\to\infty} ||(Nx_n)(t) - (Nx)(t)||_{PC} = 0. \tag{3.4}$$

Thus N is continuous on $(t_i, t_{i+1}]$, $(i = 0, 1, 2, \cdots, m)$.

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Step 2: N maps bounded sets into bounded sets in $PC[J,X]$. From (3.4) we get

$$\begin{aligned}\|(Nx)(t)\| &\leq ||\mathcal{T}(t)||\;||x_0+g(x)||+\frac{qM_1}{\Gamma(1+q)}\\ &\qquad\int_0^t (t-s)^{q-1}\left\|f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau,\int_0^b k(s,\tau,x_\tau)d\tau\right)\right\|ds\\ &\quad +m||\mathcal{T}(t-t_k)\;I_k(x(t_k^-))||.\end{aligned}$$

and we know that

$$\begin{aligned}&\left\|f\left(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau,\int_0^b k(s,\tau,x_\tau)d\tau\right)\right\|\\ &\qquad\leq c_1(s)+c_3(s)\int_0^s d_1(\tau)d\tau+c_4(s)\int_0^b d_3(\tau)d\tau\\ &\qquad\quad+\left(c_2(s)+c_3(s)\int_0^s d_2(\tau)d\tau+c_4(s)\int_0^b d_4(\tau)d\tau\right)||x||\\ &\qquad\leq \Psi_1(s)+\Psi_2(s)||x||.\end{aligned}$$

From the above we get,

$$\begin{aligned}||(Nx)(t)|| &\leq M_1\left(||x_0||+G||x||+k_1\right)+mM_1\Phi_k||x||\\ &\quad+\frac{b^qM_1}{\Gamma(1+q)}\int_0^t(\Psi_1(s)+\Psi_2(s)||x||)ds.\end{aligned}$$

Thus for any $x\in B_r=\{x\in PC[J,X]:||x||_{PC}\leq r\}$,

$$\begin{aligned}||(Nx)(t)|| &\leq M_1\left(||x_0||+k_1\right)+\frac{b^qM_1}{\Gamma(1+q)}\int_0^b\Psi_1(s)ds\\ &\quad+\left(G+mM_1\Phi_k+\frac{b^qM_1}{\Gamma(1+q)}\int_0^t\Psi_2(s)ds\right)r\\ &= \gamma_1.\end{aligned}$$

Hence $||(Nx)(t)||\leq\gamma_1$, (i.e.,) N maps bounded sets to bounded sets in $PC[J,X]$.
Step 3: $N(B_r)$ is equicontinuous with B_r on $(t_i,t_{i+1}]$ $(i=0,1,2,\cdots,m)$.

For any $x\in B_r$, $t',t''\in(t_i,t_{i+1}]$ $(i=0,1,2,\cdots,m)$, we obtain

$$\begin{aligned}||(Nx)(t'')-(Nx)(t')|| &\leq ||\mathcal{T}(t'')-\mathcal{T}(t')||\;||x_0+g(x)||\\ &\quad+\left\|\int_0^{t''}(t''-s)^{q-1}\mathcal{S}(t''-s)F(s)ds-\int_0^{t'}(t'-s)^{q-1}\mathcal{S}(t'-s)F(s)ds\right\|\\ &\quad+\left\|\sum_{k=1}^m\mathcal{T}(t''-t_k)I_k(x(t_k^-))-\sum_{k=1}^m T(t'-t_k)I_k(x(t_k^-))\right\|,\end{aligned}$$

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after some calculation, we have

$$\begin{aligned} \leq\ & ||\mathcal{T}(t'') - \mathcal{T}(t')||\ ||x_0 + g(x)|| + m||\mathcal{T}(t''-t')||\ ||I_k(x(t_k^-))|| \\ & + \left\| \int_{t'}^{t''} (t''-s)^{q-1} S(t''-s)F(s)ds \right\| \\ & + \left\| \int_0^{t'} \left[(t''-s)^{q-1} - (t'-s)^{q-1}\right] \mathcal{S}(t''-s)F(s)ds \right\| \\ & + \left\| \int_0^{t'} (t'-s)^{q-1} \left[\mathcal{S}(t''-s) - \mathcal{S}(t'-s)\right] F(s)ds \right\|. \end{aligned}$$

Using $\mathcal{T}(t)$ and $\mathcal{S}(t)$ is uniformly continuous and the well known inequality $|t'^{\sigma} - t''^{\sigma}| \leq (t''-t')^{\sigma}$ for $\sigma \in (0,1]$ and $0 < t' \leq t''$

$$\lim_{t'' \to t'} \left\|(Nx)(t'') - (Nx)(t')\right\| = 0.$$

Thus $N(B_r)$ is equi-continuous with B_r on $(t_i, t_{i+1}]$ $(i = 0, 1, 2, \cdots, m)$

Step 4: N maps B_r into a precompact set in X.

Define $\Pi = NB_r$ and $\Pi(t) = \{(Nx)(t)\ :\ x \in B_r\}$ for $t \in J$. Set $\Pi_{h,\delta}(t) = \{(N_{h,\delta}x)(t)\ : x \in B_r\}$ where

$$\Pi_{h,\delta}(t) = \left\{ \mathcal{T}_\delta(t)(x_0 + g(x)) + \int_0^{t-h} (t-s)^{q-1} S_\delta(t-s)F(s)ds\ + \sum_{k=1}^{m} \mathcal{T}_\delta(t-t_k) I_k(x(t_k^-))\ :\ x \in B_r \right\}.$$

From Lemma 2.3(ii),(iii) and (H10), we can verify that the set $\Pi(t)$ can be arbitrary approximated by the relatively compact set $\Pi_{h,\delta}(t)$. Thus, $N(B_r)(t)$ is relatively compact in X.

Step 5: The set $E = \{x \in PC[J, X] : x = \lambda Nx\ for\ 0 < \lambda < 1\}$ is bounded.

Let $x \in E$, then

$$x(t) = \begin{cases} \lambda\mathcal{T}(t)(x_0 + g(x)) + \lambda \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) \\ \qquad (\times) f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau)ds, \quad t \in [0, t_1], \\ \lambda\mathcal{T}(t)(x_0 + g(x)) + \lambda\mathcal{T}(t-t_1)I_1(x(t_1^-)) + \lambda \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \qquad (\times) f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau)ds,, \quad t \in (t_1, t_2], \\ \vdots \\ \lambda\mathcal{T}(t)(x_0 + g(x)) + \lambda \sum_{k=1}^{m} \mathcal{T}(t-t_k)I_k(x(t_k^-)) + \lambda \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \\ \qquad (\times) f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau, \int_0^b k(s,\tau,x_\tau)d\tau)ds, \quad t \in (t_m, b]. \end{cases} \tag{3.5}$$

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From (3.5) we know

$$\begin{aligned}||x(t)|| &\leq \lambda M_1 (||x_0|| + k_1) + \frac{b^q M_1}{\Gamma(1+q)} \int_0^b \Psi_1(s)ds \\ &+\lambda \left(G + mM_1\Phi_k + \frac{b^q M_1}{\Gamma(1+q)} \int_0^t \Psi_2(s)ds \right) ||x(t)||.\end{aligned}$$

Obviously there exists λ sufficiently small such that $\rho = 1 - M_1 k_1 \lambda - \lambda G - \lambda m M_1 \Phi_k > 0$ and then we get

$$\begin{aligned}||x(t)|| &\leq \frac{\lambda M_1}{\rho}||x_0|| + \frac{\lambda b^q M_1}{\rho\Gamma(1+q)} \int_0^b \Psi_1(s)ds \\ &+\frac{\lambda b^q M_1}{\rho\Gamma(1+q)} \int_0^t \Psi_2(s)||x(s)||ds.\end{aligned}$$

Let

$$Q = \lambda M_1 \rho ||x_0|| + \frac{\lambda b^q M_1}{\rho\Gamma(1+q)} \int_0^b \Psi_1(s)ds, \;\; f(t) = \frac{\lambda b^q M_1}{\rho\Gamma(1+q)} \int_0^t \Psi_2(s)ds.$$

It is clear that $f(t)$ is non negative continuous function on $[0, +\infty)$, generalized Bellman inequality implies that

$$||x(t)|| \leq Qe^{\int_0^t f(s)ds} \leq Qe^{\int_0^b f(s)ds} = C_0;$$

where C_0 is a constant. Obviously, the set E is bounded on $(t_i, t_{i+1}]$, $(i = 0, 1, 2, \cdots, m)$. Since N is continuous and compact. From the Schaefer's fixed point theorem, N has a fixed point which is a PC-mild solution of (1.1)-(1.3). This completes the proof.

4 Example

Consider the following fractional partial functional mixed differential equations with impulsive conditions of the form

$$D_t^q(z(t,\eta)) = \frac{\partial}{\partial\eta}z(t,\eta) + \Phi\left(t, z(t,\eta-r), \int_0^t h_1(t, v(x,\eta-r))ds, \int_0^b k_1(t, v(x,\eta-r))ds\right),$$

$$\text{for } (t,\eta-r) \in [0,T_0]\times(0,\pi),\; t \neq \frac{T_0}{2} \tag{4.1}$$

$$z(t,0) = z(t,\pi) = 0,\; 0 \leq t \leq T_0 \tag{4.2}$$

$$z(0,\eta) = z_0(\eta) + g(z(t,\eta)),\; 0 < \eta < \pi \tag{4.3}$$

$$\Delta z|_{t=\frac{T_0}{2}} = I_1\left(\frac{T_0^-}{2}\right) \tag{4.4}$$

where $T_0 > 0$, $0 < q < 1$, D_t^q is a Caputo fractional partial derivative of order $q \in (0,1)$. To write the system (4.1)-(4.4) to the form (1.1)-(1.3), we take

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(i) Let $X = L^2([0,\pi])$ as the state space and $z(t,\cdot) = \{z(t,\eta)\ :\ 0 \le \eta \le \pi\}$ as the state.

(ii) $A : D(A) \subset X \to X$ is defined as $Af = f''$ with domain

$$D(A) = \left\{f \in X\ :\ f',\ f'' \in X \text{ are absolutely continuous, } f(0) = f(\pi) = 0\right\}.$$

Then A is the infinitesimal generator of a strongly continuous semi-group $\{T(t)\ :\ t \ge 0\}$ in $L^2[0,\pi]$. Moreover $T(\cdot)$ is also strongly continuous such that $||T(t)|| \le M_1$ for each $t \ge 0$. A can be written as $Ax = -\sum_{n=1}^{\infty} n^2(x,e_n)e_n,\ x \in D(A)$, where $e_n(x) = \sqrt{\frac{2}{\pi}}sin(nx),\ n \in N$ is an orthonormal set of eigen functions of A.

Furthermore, for $x \in X$, we get $T(t)x = \sum_{n=1}^{\infty} exp\left(-n^2 t\right)(x,e_n)e_n$.

We define the operators $f : J \times X \times X \times X \to X,\ g : J^2 \times X \to X,$ and $k : J^2 \times X \to X$ by

$$x_t = z(t,n-r),\ h(t,s,x_s) = h_1(t,v(x,n-r)),\ k(t,s,x_s) = k_1(t,v(x,n-r)).$$

Obviously, h_1, k_1 satisfy Lipschitz condition (H2), g satisfies Lipschitz condition (H3), and I_1 satisfies Lipschitz condition (H4). All together satisfies (H1). Also it is easy to verify conditions (H5) and (H6).

Thus functions Φ, h_1, k_1, g and I_1 of the system (4.1)-(4.4) satisfies the hypotheses of the Theorem 3.1 and Theorem 3.2. Thus all the conditions of Theorem 3.1 and Theorem 3.2 are satisfied. Therefore the system (4.1)-(4.4) can be written to the abstract form (1.1)-(1.3). That phenomenon model equations (4.1)-(4.4), (refer [36]). Hence we conclude that the system (4.1)-(4.4) has a mild solution.

5 Conclusion

Here we have established the existence of PC-mild solutions for non local fractional impulsive functional integro-differential equations with finite delay. The proofs are obtained by using Banach contraction principle and a fixed point theorem due to Schaefer with generalized Bellman inequality. We have used the distributed characteristic operators to define the mild solution of the system taking care of all the controversy related to the solution operator (refer Remark 2.3). Also we have considered a bounded linear operator which gave the standard semi-group in the exponential form, but one can consider unbounded operator and prove the results. For this we refer the techniques given in [14, 15]. The same problem can also be extended for Trajectory controllability problem, refer [16].

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