

*Article*

# The rôle of $\text{Spin}(9)$ in Octonionic Geometry

*Dedicated to the memory of Thomas Friedrich***Maurizio Parton**<sup>1</sup> and **Paolo Piccinni**<sup>2</sup><sup>1</sup> Università di Chieti-Pescara, Dipartimento di Economia, Viale della Pineta 4, I-65129 Pescara, Italy; [parton@unich.it](mailto:parton@unich.it)<sup>2</sup> Dipartimento di Matematica, Sapienza - Università di Roma, Piazzale Aldo Moro 2, I-00185, Roma, Italy; [piccinni@mat.uniroma1.it](mailto:piccinni@mat.uniroma1.it)

**Abstract:** Starting from Thomas Friedrich's work [22], we review several interactions between  $\text{Spin}(9)$  and geometries related to octonions. Several topics are discussed in this respect: explicit descriptions of the  $\text{Spin}(9)$  canonical 8-form and its analogies with quaternionic geometry, the role of  $\text{Spin}(9)$  both in the classical problems of vector fields on spheres and in the geometry of the octonionic Hopf fibration. Next, we deal with locally conformally parallel  $\text{Spin}(9)$  manifolds in the framework of intrinsic torsion. Finally, we discuss applications of Clifford systems and Clifford structures to Cayley-Rosenfeld planes and to three series of Grassmannians.

**Keywords:**  $\text{Spin}(9)$ ; octonions; vector fields on spheres; Hopf fibration; locally conformally parallel; Clifford structure; Clifford system; symmetric spaces.

**MSC:** Primary 53C26, 53C27, 53C35, 57R25.

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## 24 1. Introduction

One of the oldest evidences of interest for the group  $\text{Spin}(9)$  in geometry goes back to the 1943 Annals of Mathematics paper by D. Montgomery and H. Samelson [37], classifying compact Lie groups that act transitively and effectively on spheres, and giving the list:

$$\text{SO}(n), \text{U}(n), \text{SU}(n), \text{Sp}(n), \text{Sp}(n) \cdot \text{U}(1), \text{Sp}(n) \cdot \text{Sp}(1), \text{G}_2, \text{Spin}(7), \text{Spin}(9).$$

25 In particular  $\text{Spin}(9)$  acts transitively on the sphere  $S^{15}$  through its  $\text{Spin}$  representation, and the  
26 stabilizer of the action is a subgroup  $\text{Spin}(7)$ .

27 In the following decade the above groups, with the only exception of  $\text{Sp}(n) \cdot \text{U}(1)$ , appeared  
28 in the celebrated M. Berger theorem [8] as the list of the possible holonomy groups of irreducible  
29 simply connected and non symmetric Riemannian manifolds. In the following further decade,  
30 D. Alekseevsky [3] proved that  $\text{Spin}(9)$  is the Riemannian holonomy only of symmetric spaces,  
31 namely of the Cayley projective plane and of its non-compact dual. Accordingly,  $\text{Spin}(9)$  started to  
32 be omitted in the statement of Berger theorem. Much later, a geometric proof of Berger theorem was  
33 given by C. Olmos [39], using submanifold geometry of orbits and still referring to possible transitive  
34 actions on spheres.

35 Moreover, in the last decades of the twentieth century, compact examples have been shown to  
36 exist for almost all classes of Riemannian manifolds related with the other holonomy groups in Berger  
37 list. References are the S. Salamon and the D. Joyce books [48], [32], [33]. For all of these reasons, the  
38 best known feature of the group  $\text{Spin}(9)$  seemed to be, around the year 2000, that of being a group  
39 that had been removed from an interesting list.

40 Coming to the new millennium, since its very beginning new interest appeared dealing with  
41 different aspects of octonionic geometry and mentioning features of structures and weakened  
42 holonomies related with  $\text{Spin}(9)$ . Among the references, there are certainly the J. Baez extensive  
43 Bulletin AMS paper on octonions [6] as well as the not less extensive discussions in his web page  
44 [7]. Next and from a more specific point of view, there is the Thomas Friedrich paper on "weak  
45  $\text{Spin}(9)$ -structures" [22], proposing a way of dealing with  $\text{Spin}(9)$  structures later recognized by  
46 A. Moroianu and U. Semmelmann [38] to fit in the broader context of Clifford structures. Also, the  
47 M. Atiyah and J. Berndt paper on Surveys in Differential Geometry [5] shows interesting connections  
48 with classical algebraic geometry. Coming to more recent contributions, it is worth to mention the  
49 very recent work by N. Hitchin [28], based on a talk for R. Penrose's 80th birthday, and dealing with  
50  $\text{Spin}(9)$  in relation to further groups of interest in octonionic geometry.

51 Aim of the present article is to give a survey on our recent work on  $\text{Spin}(9)$  and octonionic  
52 geometry, in part also with L. Ornea and V. Vuletescu, and mostly contained in the references [42],  
53 [43], [40], [44], [45], [46], [47].

54 Our initial motivation was to give a construction, as simple as possible, of a canonical octonionic  
55 8-form  $\Phi_{\text{Spin}(9)}$  that had been defined independently through different integrals by M. Berger [9] and  
56 by R. Brown and A. Gray [12]. Our construction of  $\Phi_{\text{Spin}(9)}$  uses the already mentioned definition of  
57 a  $\text{Spin}(9)$ -structure proposed by Thomas Friedrich and has a strong analogy with a construction of a  
58  $\text{Sp}(2) \cdot \text{Sp}(1)$ -structure in dimension 8, see Section 3 as well as [42]. By developing our construction  
59 of  $\Phi_{\text{Spin}(9)}$ , we realized that some features of the sphere  $S^{15}$  can be conveniently described through  
60 the same approach that we used. It is certainly related to the Friedrich point of view the fact that  
61  $S^{15}$  is the lowest dimensional sphere that admit more than 7 global linearly independent tangent  
62 vector field. Namely, by developing a convenient linear algebra we were able to prove that the full  
63 system of maximal linearly independent vector fields on any sphere  $S^n$  can be written in terms of  
64 the unit imaginary elements in  $\mathbb{C}, \mathbb{H}, \mathbb{O}$  and of the complex structures that Friedrich associates to  
65  $\text{Spin}(9)$ , see Section 5 and [43]. Another feature of  $S^{15}$  is of course that of being the total space of  
66 the octonionic Hopf fibration, whose group of symmetries is  $\text{Spin}(9) \subset \text{SO}(16)$ . Here the Friedrich  
67 approach to  $\text{Spin}(9)$  allows to recognize the non existence of nowhere zero vertical vector fields and

68 simple properties of locally conformally parallel Spin(9)-structures, see here Theorem 17, Section 7,  
 69 and reference [40]. We then discuss the broader contexts of Clifford structures and Clifford systems,  
 70 that allow to deal with the the complex Cayley projective plane, whose geometry and topology can  
 71 be studied by referring to its projective algebraic model known as the fourth Severi variety. With  
 72 similar methods one can also study structure and properties of the remaining two Cayley-Rosenfeld  
 73 projective planes, and for all of this see Sections 8, 9, 10, and [44], [45]. Finally, Clifford structures and  
 74 Clifford systems can be studied in relation with the exceptional symmetric spaces of compact type as  
 75 well as with some real, complex and quaternionic Grassmannians that carry a geometry very much  
 76 related with octonions, see Sections 11, 12, and [46], [47].

77 During the years of our work, we convinced ourselves that Spin(9) influences not only  
 78 16-dimensional Riemannian geometry, but also aspects, related with octonions, of some lower  
 79 dimensional and higher dimensional geometry. It is in fact our hope that the reader of this survey can  
 80 share with us the feeling of the beauty of Spin(9), that seems to have some rôle in geometry, besides  
 81 being a group that had been removed from an interesting list.

## 82 2. Preliminaries, Hopf fibrations and Friedrich's work

The multiplication in the algebra  $\mathbb{O}$  of octonions can be defined from the one in quaternions  $\mathbb{H}$   
 by the Cayley-Dickson process: if  $x = h_1 + h_2e$ ,  $x' = h'_1 + h'_2e \in \mathbb{O}$ , then

$$xx' = (h_1h'_1 - \bar{h}'_2h_2) + (h_2\bar{h}'_1 + h'_2h_1)e, \quad (2.1)$$

where  $\bar{h}'_1, \bar{h}'_2$  are the conjugates of  $h'_1, h'_2 \in \mathbb{H}$ . As for quaternions, the conjugation  $\bar{x} = \bar{h}_1 - h_2e$  relates  
 with the non-commutativity:  $xx' = \bar{x}'\bar{x}$ . The associator

$$[x, x', x''] = (xx')x'' - x(x'x'')$$

83 vanishes whenever two among  $x, x', x'' \in \mathbb{O}$  are equal or conjugate. For a survey on octonions and  
 84 their applications in geometry, topology, and mathematical physics, the excellent article [6] by J. Baez  
 85 is a basic reference.

The 16-dimensional real vector space  $\mathbb{O}^2$  decomposes into its *octonionic lines*

$$l_m \stackrel{\text{def}}{=} \{(x, mx) | x \in \mathbb{O}\} \quad \text{or} \quad l_\infty \stackrel{\text{def}}{=} \{(0, x) | x \in \mathbb{O}\},$$

that intersect each other only in the origin  $(0, 0) \in \mathbb{O}^2$ . Here  $m \in S^8 = \mathbb{O}P^1 = \mathbb{O} \cup \{\infty\}$  parametrizes  
 the set of octonionic lines  $l$ , whose volume elements  $v_l \in \Lambda^8 l$  allow to define the following *canonical*  
 8-form on  $\mathbb{O}^2 = \mathbb{R}^{16}$

$$\Phi_{\text{Spin}(9)} = \frac{110880}{\pi^4} \int_{\mathbb{O}P^1} p_l^* v_l \in \Lambda^8(\mathbb{R}^{16}), \quad (2.2)$$

86 where  $p_l$  denotes the orthogonal projection  $\mathbb{O}^2 \rightarrow l$ .

87 The definition of  $\Phi_{\text{Spin}(9)}$  through this integral was given by M. Berger [9], and we chose here the  
 88 proportionality factor in such a way to make integers and with no common factors all the coefficients  
 89 of  $\Phi_{\text{Spin}(9)}$  as exterior 8-form in  $\mathbb{R}^{16}$ . The notation is motivated by the following:

90 **Proposition 1.** [17] *The subgroup of  $GL(16, \mathbb{R})$  preserving  $\Phi_{\text{Spin}(9)}$  is the image of Spin(9) under its spin*  
 91 *representation into  $\mathbb{R}^{16}$ .*

92 Thus Spin(9)  $\subset$  SO(16), so that 16-dimensional oriented Riemannian manifolds are the natural  
 93 setting for Spin(9)-structures. The following definition was proposed by Th. Friedrich, [22].

94 **Definition 2.** Let  $(M, g)$  be a 16-dimensional oriented Riemannian manifold. A Spin(9) *structure* on  
 95  $M$  is the datum of any of the following equivalent alternatives.

1. A rank 9 subbundle  $E = E^9 \subset \text{End}(TM)$ , locally spanned by endomorphisms  $\{\mathcal{I}_\alpha\}_{\alpha=1,\dots,9}$  with

$$\mathcal{I}_\alpha^2 = \text{Id}, \quad \mathcal{I}_\alpha^* = \mathcal{I}_\alpha, \quad \text{and} \quad \mathcal{I}_\alpha \mathcal{I}_\beta = -\mathcal{I}_\beta \mathcal{I}_\alpha \quad \text{for} \quad \alpha \neq \beta, \quad (2.3)$$

96 where  $\mathcal{I}_\alpha^*$  denotes the adjoint of  $\mathcal{I}_\alpha$ .

97 2. A reduction  $\mathcal{R}$  of the principal bundle  $\mathcal{F}(M)$  of orthonormal frames from  $\text{SO}(16)$  to  $\text{Spin}(9)$ .

98 In particular, the existence of a  $\text{Spin}(9)$  structure depends only on the conformal class of the  
99 metric  $g$  on  $M$ .

100 We describe now the vector bundle  $E^9$  when  $M$  is the model space  $\mathbb{R}^{16}$ . Here  $\mathcal{I}_1, \dots, \mathcal{I}_9$  can be  
101 chosen as generators of the Clifford algebra  $\text{Cl}(9)$ , the endomorphisms' algebra of its 16-dimensional  
102 real representation  $\Delta_9 = \mathbb{R}^{16} = \mathbb{O}^2$ . Accordingly, unit vectors  $v \in S^8 \subset \mathbb{R}^9$  can be viewed, via the  
103 Clifford multiplication, as symmetric endomorphisms  $v : \Delta_9 \rightarrow \Delta_9$ .

The explicit way to describe this action is by  $v = u + r \in S^8$  ( $u \in \mathbb{O}, r \in \mathbb{R}, u\bar{u} + r^2 = 1$ ), acting on pairs  $(x, x') \in \mathbb{O}^2$ :

$$\begin{pmatrix} x \\ x' \end{pmatrix} \rightarrow \begin{pmatrix} r & R_{\bar{u}} \\ R_u & -r \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}, \quad (2.4)$$

104 where  $R_u, R_{\bar{u}}$  denote the right multiplications by  $u, \bar{u}$ , respectively (cf. [27, page 288]).

A basis of the standard  $\text{Spin}(9)$  structure on  $\mathbb{O}^2 = \mathbb{R}^{16}$  can be written by looking at the action (2.4) and at the nine vectors

$$(0, 1), (0, i), (0, j), (0, k), (0, e), (0, f), (0, g), (0, h) \quad \text{and} \quad (1, 0) \in S^8 \subset \mathbb{O} \times \mathbb{R} = \mathbb{R}^9.$$

This gives the symmetric endomorphisms:

$$\begin{aligned} \mathcal{I}_1 &= \left( \begin{array}{c|c} 0 & \text{Id} \\ \text{Id} & 0 \end{array} \right), & \mathcal{I}_2 &= \left( \begin{array}{c|c} 0 & -R_i \\ R_i & 0 \end{array} \right), & \mathcal{I}_3 &= \left( \begin{array}{c|c} 0 & -R_j \\ R_j & 0 \end{array} \right), \\ \mathcal{I}_4 &= \left( \begin{array}{c|c} 0 & -R_k \\ R_k & 0 \end{array} \right), & \mathcal{I}_5 &= \left( \begin{array}{c|c} 0 & -R_e \\ R_e & 0 \end{array} \right), & \mathcal{I}_6 &= \left( \begin{array}{c|c} 0 & -R_f \\ R_f & 0 \end{array} \right), \\ \mathcal{I}_7 &= \left( \begin{array}{c|c} 0 & -R_g \\ R_g & 0 \end{array} \right), & \mathcal{I}_8 &= \left( \begin{array}{c|c} 0 & -R_h \\ R_h & 0 \end{array} \right), & \mathcal{I}_9 &= \left( \begin{array}{c|c} \text{Id} & 0 \\ 0 & -\text{Id} \end{array} \right), \end{aligned} \quad (2.5)$$

where  $R_i, \dots, R_h$  are the right multiplications by the 7 unit octonions  $i, \dots, h$ . Their spanned subspace

$$E^9 \stackrel{\text{def}}{=} \langle \mathcal{I}_1, \dots, \mathcal{I}_9 \rangle \subset \text{End}(\mathbb{R}^{16}) \quad (2.6)$$

105 is such that:

106 **Proposition 3.** [17] *The subgroup of  $\text{SO}(16)$  preserving  $E^9$  is  $\text{Spin}(9)$ .*

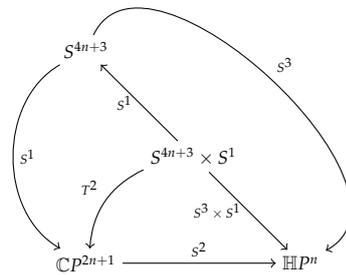
The projection  $\mathbb{O}^2 - \{0\} \rightarrow \mathbb{O}P^1$  associated with the decomposition into the octonionic lines  $l_m, l_\infty$  is a non compact version of the octonionic Hopf fibration

$$S^{15} \rightarrow \mathbb{O}P^1 \cong S^8,$$

unique surviving possibility when passing from quaternions to octonions from the series of quaternionic Hopf fibrations

$$S^{4n+3} \rightarrow \mathbb{H}P^n.$$

107 Recall that the latter enter in the diagram

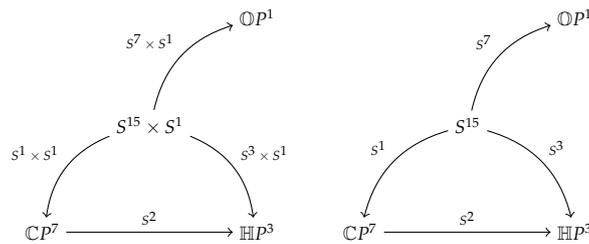


108

109 encoding prototypes of several structures of interest in quaternionic geometry.

110 Namely at the center of the diagram there is the locally conformally hyperkähler Hopf manifold  
 111  $S^{4n+3} \times S^1$ . All the other manifolds are leaf spaces of foliations on it: the 3-Sasakian sphere  $S^{4n+3}$ ,  
 112 the positive Kähler-Einstein twistor space  $CP^{2n+1}$ , and the positive quaternion Kähler  $HP^n$ . Most  
 113 of the foliations carry similar structures on their leaves; for example one has locally conformally  
 114 hyperkähler Hopf surfaces  $S^3 \times S^1$ . This prototype diagram is only an example, since when a compact  
 115 locally hyperkähler manifold has compact leaves of the four canonically defined vertical foliations,  
 116 our diagram still makes sense, although in the broader orbifolds' category, cf. [41].

117 When  $n = 3$  there are also the octonionic Hopf fibrations:



118

119 and no arrow connects  $O^1P^1$  with  $HP^3$  and with  $CP^7$ , since the complex and quaternionic Hopf  
 120 fibrations are not subfibrations of the octonionic one, cf. [36] as well as Theorem 17 in the following  
 121 Section 6.

122 Coming back to Spin(9), as the title of Th. Friedrich's article [22] suggests, there is a scheme for  
 123 "weak Spin(9) structures" to include some possibilities besides the very restrictive holonomy Spin(9)  
 124 condition. Although the original A. Gray proposal [25] to look at "weak holonomies" has been much  
 125 later shown by B. Alexandrov [4] not to produce new geometries for the series of groups quoted in  
 126 the Introduction, one can refer to symmetries of relevant tensors to understand the possibilities for  
 127 any  $G$ -structure. We briefly recall a unified scheme one can refer to, following the presentation of [1].

By definition a  $G$ -structure on an oriented Riemannian manifold  $M^n$  is a reduction  $\mathcal{R} \subset \mathcal{F}(M^n)$   
 of the orthonormal frame bundle to the subgroup  $G \subset SO(n)$ . The Levi Civita connection  $Z$ ,  
 thought as a 1-form on  $\mathcal{F}(M^n)$  with values in the Lie algebra  $\mathfrak{so}(n)$ , restricts to a connection on  $\mathcal{R}$   
 decomposing, with respect to the Lie algebra splitting  $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ , as

$$Z|_{T(\mathcal{R})} = Z^* \oplus \Gamma.$$

128 Here  $Z^*$  is a connection in the principal  $G$ -bundle  $\mathcal{R}$  and  $\Gamma$  is a 1-form on  $M^n$  with values in the  
 129 associated bundle  $\mathcal{R} \times_G \mathfrak{m}$ , called the *intrinsic torsion* of the  $G$ -structure. Of course the condition  
 130  $\Gamma = 0$  is equivalent to the inclusion  $\text{Hol} \subset G$  for the Riemannian holonomy, and  $G$ -structures with  
 131  $\Gamma \neq 0$  are called non-integrable.

This scheme can be used in particular when  $G$  is the stabilizer of some tensor  $\eta$  in  $\mathbb{R}^n$ , so that the  $G$ -structure on  $M$  defines a global tensor  $\eta$ , and here  $\Gamma = \nabla\eta$  can be conveniently thought as section of the vector bundle

$$\mathcal{W} = T^* \otimes \mathfrak{m}.$$

132 Accordingly, the action of  $G$  splits  $\mathcal{W}$  into irreducible  $G$ -components:  $\mathcal{W} = \mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_k$ .

A prototype of such decompositions occurs when  $G = U(n) \subset SO(2n)$  and yields, when  $n \geq 3$ , the four irreducible components of the so-called Gray-Hervella classification [26]. It is a fact from representation theory that there are several further interesting cases yielding four irreducible components. This occurs when  $G = G_2 \subset SO(7)$ ,  $G = Sp(2) \cdot Sp(1) \subset SO(8)$ ,  $G = Spin(9) \subset SO(16)$  as computed in [20], [49, page 115], [22], respectively:

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4. \tag{2.7}$$

133 For the mentioned four situations, the last component  $\mathcal{W}_4$  is the "vectorial type" one, and gives rise  
 134 to a 1-form  $\theta$  on the manifold. A general theory of  $G$ -structures in this last class  $\mathcal{W}_4$ , typically locally  
 135 conformally parallel  $G$ -structures, has been developed in [2]. In Section 7 we will go back to  $\mathcal{W}_4$   
 136 in the  $G = Spin(9)$  case, following [22] as well as our work [40].

137 **3. The canonical 8-form  $\Phi_{Spin(9)}$**

The space  $\Lambda^2\mathbb{R}^{16}$  of 2-forms in  $\mathbb{R}^{16}$  decomposes under  $Spin(9)$  as

$$\Lambda^2\mathbb{R}^{16} = \Lambda_{36}^2 \oplus \Lambda_{84}^2 \tag{3.1}$$

(cf. [22, page 146]), where  $\Lambda_{36}^2 \cong \mathfrak{spin}(9)$  and  $\Lambda_{84}^2 = \mathfrak{m}$  is an orthogonal complement in  $\Lambda^2 \cong \mathfrak{so}(16)$ . Explicit bases of both subspaces can be written by looking at the nine generators (2.5) of the vector space  $E^9$  defining the  $Spin(9)$  structure. Namely, one has the compositions

$$J_{\alpha\beta} \stackrel{\text{def}}{=} \mathcal{I}_\alpha \mathcal{I}_\beta,$$

138 for  $\alpha < \beta$  as a basis of  $\Lambda_{36}^2 \cong \mathfrak{spin}(9)$ , and at the compositions  $J_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \mathcal{I}_\alpha \mathcal{I}_\beta \mathcal{I}_\gamma$ , for  $\alpha < \beta < \gamma$  as a basis  
 139 of  $\Lambda_{84}^2$ .

The Kähler 2-forms  $\psi_{\alpha\beta}$  of the complex structures  $J_{\alpha\beta}$ , by denoting the coordinates in  $\mathbb{O}^2 \cong \mathbb{R}^{16}$  by  $(1, \dots, 8, 1', \dots, 8')$ , are:

$$\begin{aligned} \psi_{12} &= (-12 + 34 + 56 - 78) - ( )' , & \psi_{13} &= (-13 - 24 + 57 + 68) - ( )' , & \psi_{14} &= (-14 + 23 + 58 - 67) - ( )' , \\ \psi_{15} &= (-15 - 26 - 37 - 48) - ( )' , & \psi_{16} &= (-16 + 25 - 38 + 47) - ( )' , & \psi_{17} &= (-17 + 28 + 35 - 46) - ( )' , \\ \psi_{18} &= (-18 - 27 + 36 + 45) - ( )' , & \psi_{23} &= (-14 + 23 - 58 + 67) + ( )' , & \psi_{24} &= (13 + 24 + 57 + 68) + ( )' , \\ \psi_{25} &= (-16 + 25 + 38 - 47) + ( )' , & \psi_{26} &= (15 + 26 - 37 - 48) + ( )' , & \psi_{27} &= (18 + 27 + 36 + 45) + ( )' , \\ \psi_{28} &= (-17 + 28 - 35 + 46) + ( )' , & \psi_{34} &= (-12 + 34 - 56 + 78) + ( )' , & \psi_{35} &= (-17 - 28 + 35 + 46) + ( )' , \\ \psi_{36} &= (-18 + 27 + 36 - 45) + ( )' , & \psi_{37} &= (+15 - 26 + 37 - 48) + ( )' , & \psi_{38} &= (16 + 25 + 38 + 47) + ( )' , \\ \psi_{45} &= (-18 + 27 - 36 + 45) + ( )' , & \psi_{46} &= (17 + 28 + 35 + 46) + ( )' , & \psi_{47} &= (-16 - 25 + 38 + 47) + ( )' , \\ \psi_{48} &= (15 - 26 - 37 + 48) + ( )' , & \psi_{56} &= (-12 - 34 + 56 + 78) + ( )' , & \psi_{57} &= (-13 + 24 + 57 - 68) + ( )' , \\ \psi_{58} &= (-14 - 23 + 58 + 67) + ( )' , & \psi_{67} &= (14 + 23 + 58 + 67) + ( )' , & \psi_{68} &= (-13 + 24 - 57 + 68) + ( )' , \\ \psi_{78} &= (12 + 34 + 56 + 78) + ( )' , \end{aligned} \tag{3.2}$$

where  $( )'$  denotes the ' of what appears before it, for instance

$$\psi_{12} = (-12 + 34 + 56 - 78) - (-1'2' + 3'4' + 5'6' - 7'8') .$$

Next:

$$\begin{aligned}\psi_{19} &= -11' - 22' - 33' - 44' - 55' - 66' - 77' - 88' , & \psi_{29} &= -12' + 21' + 34' - 43' + 56' - 65' - 78' + 87' , \\ \psi_{39} &= -13' - 24' + 31' + 42' + 57' + 68' - 75' - 86' , & \psi_{49} &= -14' + 23' - 32' + 41' + 58' - 67' + 76' - 85' , \\ \psi_{59} &= -15' - 26' - 37' - 48' + 51' + 62' + 73' + 84' , & \psi_{69} &= -16' + 25' - 38' + 47' - 52' + 61' - 74' + 83' , \\ \psi_{79} &= -17' + 28' + 35' - 46' - 53' + 64' + 71' - 82' , & \psi_{89} &= -18' - 27' + 36' + 45' - 54' - 63' + 72' + 81' ,\end{aligned}\quad (3.3)$$

and a computation gives:

**Proposition 4.** *The characteristic polynomial of the matrix  $\psi = (\psi_{\alpha\beta})_{\alpha,\beta=1,\dots,9}$ , of the Kähler forms explicitly listed in (3.2), (3.3), reduces to*

$$\det(tI - \psi) = t^9 + \tau_4(\psi)t^5 + \tau_8(\psi)t .$$

In particular  $\tau_2(\psi) = \sum_{\alpha < \beta} \psi_{\alpha\beta}^2 = 0$ , and the Spin(9)-invariant 8-form  $\tau_4(\psi)$  has to be proportional to  $\Phi_{\text{Spin}(9)}$ . The proportionality factor, computed by looking at any of the terms of  $\Phi_{\text{Spin}(9)}$  and  $\tau_4(\psi)$  turns out to be 360.

This can be rephrased in the context of Spin(9) structures on Riemannian manifolds  $M^{16}$  and gives the following two (essentially equivalent) algebraic expressions for the the 8-form  $\Phi_{\text{Spin}(9)}$ :

**Theorem 5.** [15] *The 8-form  $\Phi_{\text{Spin}(9)}$  associated with the Spin(9)-structure  $E^9 \rightarrow M^{16}$  and defined by the integral (2.2) coincides, up to a constant, with the global form*

$$\Omega_{\text{CGM}} = \sum_{\alpha,\beta,\alpha',\beta'=1,\dots,9} \psi_{\alpha,\beta} \wedge \psi_{\alpha,\beta'} \wedge \psi_{\alpha',\beta} \wedge \psi_{\alpha',\beta'} .$$

**Theorem 6.** [42] *The 8-form  $\Phi_{\text{Spin}(9)}$  associated with the Spin(9)-structure  $E^9 \rightarrow M^{16}$  coincides, up to a constant, with the coefficient*

$$\tau_4(\psi) = \sum_{1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 9} (\psi_{\alpha_1\alpha_2} \wedge \psi_{\alpha_3\alpha_4} - \psi_{\alpha_1\alpha_3} \wedge \psi_{\alpha_2\alpha_4} + \psi_{\alpha_1\alpha_4} \wedge \psi_{\alpha_2\alpha_3})^2$$

in the characteristic polynomial

$$\det(tI - \psi) = t^9 + \tau_4(\psi)t^5 + \tau_8(\psi)t ,$$

where  $\psi = (\psi_{\alpha\beta})_{\alpha,\beta=1,\dots,9}$  is any matrix of local associated Kähler 2-forms  $M$ . The proportionality factor is given by

$$360\Phi_{\text{Spin}(9)} = \tau_4(\psi) .$$

These two expressions of  $\Phi_{\text{Spin}(9)}$  have been shown to be proportional according to the following algebraic relation.

**Proposition 7.** [16] *Let  $\mathbb{R}[x_{12}, \dots, x_{89}]$  be the polynomial ring in the 36 variables  $x_{12}, \dots, x_{89}$ , and let  $x$  be the skew-symmetric matrix whose upper diagonal entries are  $x_{12}, \dots, x_{89}$ . Among the homogeneous polynomials*

$$\begin{aligned}F &= \sum_{\alpha,\beta,\alpha',\beta'=1,\dots,9} x_{\alpha,\beta} x_{\alpha,\beta'} x_{\alpha',\beta} x_{\alpha',\beta'} , & P &= \sum_{\alpha < \beta} x_{\alpha,\beta}^2 , \\ Q &= \tau_4(x) = \sum_{1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 9} (x_{\alpha_1\alpha_2} x_{\alpha_3\alpha_4} - x_{\alpha_1\alpha_3} x_{\alpha_2\alpha_4} + x_{\alpha_1\alpha_4} x_{\alpha_2\alpha_3})^2 .\end{aligned}$$

the following relation holds:  $F = 2P^2 - 4Q$ . Thus, since  $P(\psi) = 0$ ,

$$\Omega_{\text{CGM}} = -4\tau_4(\psi) .$$

**Corollary 8.** *The Kähler forms of the Spin(9)-structure of  $\mathbb{O}^2$  allow to compute the integral (2.2) as*

$$\int_{\mathbb{O}P^1} p_i^* v_i dl = \frac{\pi^4}{110880 \cdot 360} \tau_4(\psi).$$

148 When Spin(9) is the holonomy group of the Riemannian manifold  $M^{16}$ , the Levi-Civita  
 149 connection  $\nabla$  preserves the vector bundle  $E^9$ , and the local sections  $\mathcal{I}_1, \dots, \mathcal{I}_9$  of  $E^9$  induce the Kähler  
 150 forms  $\psi_{\alpha\beta}$  on  $M$  as local curvature forms.

**Corollary 9.** *Let  $M^{16}$  be a compact Riemannian manifold with holonomy Spin(9), i.e.  $M^{16}$  is either isometric to the Cayley projective plane  $\mathbb{O}P^2$  or to any compact quotient of the Cayley hyperbolic plane  $\mathbb{O}H^2$ . Then its Pontrjagin classes are given by*

$$p_1(M) = 0, \quad p_2(M) = -\frac{45}{2\pi^4} [\Phi_{\text{Spin}(9)}], \quad p_3(M) = 0, \quad p_4(M) = -\frac{13}{256\pi^8} [\tau_8(\psi)].$$

**Proof.** The Pontrjagin classes of the vector bundle  $E^9 \rightarrow M$  are given by:

$$p_1(E) = 0, \quad 16\pi^4 p_2(E) = \tau_4(\psi) = 360[\Phi_{\text{Spin}(9)}], \quad p_3(E) = 0, \quad 256\pi^8 p_4(E) = [\tau_8(\psi)].$$

For a compact  $M$  with a Spin(9)-structure, the following relations hold, [22, Page 138]:

$$\begin{aligned} p_1(M) &= 2p_1(E), \\ p_2(M) &= \frac{7}{4} p_1^2(E) - p_2(E), \\ p_3(M) &= \frac{1}{8} (7p_1^3(E) - 12p_1(E)p_2(E) + 16p_3(E)), \\ p_4(M) &= \frac{1}{128} (35p_1^4(E) - 120p_1^2(E)p_2(E) + 400p_1(E)p_3(E) - 1664p_4(E)). \end{aligned} \tag{3.4}$$

151 Thus,  $\tau_2(\psi) = \tau_6(\psi) = 0$  gives  $p_1(E) = p_3(E) = 0$ , so that  $p_1(M) = p_3(M) = 0$ ,  $p_2(M) = -p_2(E)$   
 152 and  $p_4(M) = -13p_4(E)$ .  $\square$

The Pontrjagin classes of  $\mathbb{O}P^2$  are known to be  $p_2(\mathbb{O}P^2) = 6u$  and  $p_4(\mathbb{O}P^2) = 39u^2$ , where  $u$  is the canonical generator of  $H^8(\mathbb{O}P^2; \mathbb{Z})$ , and Corollary 9 gives the representative forms:

$$u = [-\frac{15}{4\pi^4} \Phi_{\text{Spin}(9)}] = [-\frac{1}{96\pi^4} \tau_4(\psi)], \quad u^2 = [-\frac{1}{768\pi^8} \tau_8(\psi)].$$

**Remark 10.** Very recently an alternative way of writing the 8-form  $\Phi_{\text{Spin}(9)}$  in  $\mathbb{R}^{16}$  has been proposed by J. Korbátý [35]. This is in terms of the differentials of the octonionic coordinates  $x, y \in \mathbb{O}^2$ . If

$$\begin{aligned} dx &= dx_1 + idx_2 + jdx_3 + \dots + hdx_8, & \overline{dx} &= dx_1 - idx_2 - jdx_3 - \dots - hdx_8, \\ dy &= dy_1 + idy_2 + jdy_3 + \dots + hdy_8, & \overline{dy} &= dy_1 - idy_2 - jdy_3 - \dots - hdy_8, \end{aligned}$$

consider formally the "octonionic 4-forms"

$$\begin{aligned} \Psi_{40} &= ((\overline{dx} \wedge dx) \wedge \overline{dx}) \wedge dx, & \Psi_{31} &= ((\overline{dy} \wedge dx) \wedge \overline{dx}) \wedge dx \\ \Psi_{13} &= ((\overline{dx} \wedge dy) \wedge \overline{dy}) \wedge dy, & \Psi_{04} &= ((\overline{dy} \wedge dy) \wedge \overline{dy}) \wedge dy. \end{aligned}$$

Then, by defining their conjugates through

$$\overline{\alpha \wedge \beta} = (-1)^{kl} \overline{\beta} \wedge \overline{\alpha},$$

for  $\alpha \in \Lambda^k, \beta \in \Lambda^l$ , Kotrbatý shows that the real 8-form

$$\Psi_8 = \Psi_{40} \wedge \overline{\Psi_{40}} + 4\Psi_{31} \wedge \overline{\Psi_{31}} - 5(\Psi_{31} \wedge \Psi_{13} + \overline{\Psi_{13}} \wedge \overline{\Psi_{31}} + 4\Psi_{13} \wedge \overline{\Psi_{13}} + \Psi_{04} \wedge \overline{\Psi_{04}})$$

153 gives the proportionality relation  $\Phi_{\text{Spin}(9)} = -\frac{1}{4 \cdot 6!} \Psi_8$ , and recovers from this the table of the 702 non  
 154 zero monomials of  $\Phi_{\text{Spin}(9)}$  in  $\mathbb{R}^{16}$ .

155 **4. The analogy with  $\text{Sp}(2) \cdot \text{Sp}(1)$**

In the previous Section we saw that the matrices  $\mathcal{I}_1, \dots, \mathcal{I}_9$  are the starting point for the construction of the canonical 8-form  $\Phi_{\text{Spin}(9)}$ . Of course  $\mathcal{I}_1, \dots, \mathcal{I}_9$  are the octonionic analogues of the classical Pauli matrices:

$$\mathcal{I}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{I}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathcal{I}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.1)$$

156 defined by using just the unit imaginary  $i \in \mathbb{C}$ , and belonging to  $\text{U}(2)$ . Their compositions  $J_{\alpha\beta} = \mathcal{I}_\alpha \mathcal{I}_\beta$ ,  
 157 for  $\alpha < \beta$  act on  $\mathbb{H} \cong \mathbb{C}^2$  as multiplication on the right by unit quaternions:  $J_{12} = R_i, J_{13} = R_j, J_{23} =$   
 158  $R_k$ .

159 Similarly, quaternionic analogues of the Pauli matrices are the  $8 \times 8$  real matrices:

$$\mathcal{I}_1 = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}, \quad \mathcal{I}_2 = \begin{pmatrix} 0 & -R_i \\ R_i & 0 \end{pmatrix}, \quad \mathcal{I}_3 = \begin{pmatrix} 0 & -R_j \\ R_j & 0 \end{pmatrix}, \quad (4.2)$$

$$\mathcal{I}_4 = \begin{pmatrix} 0 & -R_k \\ R_k & 0 \end{pmatrix}, \quad \mathcal{I}_5 = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix},$$

160 where  $R_i, R_j, R_k$  are the multiplication on the right on  $\mathbb{H}$  by  $i, j, k$ .

The ten compositions  $J_{\alpha\beta} \stackrel{\text{def}}{=} \mathcal{I}_\alpha \mathcal{I}_\beta$  ( $\alpha < \beta$ ) of these latter matrices are a basis of the term  $\mathfrak{sp}(2)$  in the decomposition

$$\Lambda^2 \mathbb{R}^8 \cong \mathfrak{so}(8) = \mathfrak{sp}(1) \oplus \mathfrak{sp}(2) \oplus \Lambda_{15}^2,$$

where  $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$ . Their Kähler forms  $\theta_{\alpha\beta}$  read:

$$\begin{aligned} \theta_{12} &= -12 + 34 + 56 - 78, & \theta_{13} &= -13 - 24 + 57 + 68, & \theta_{14} &= -14 + 23 + 58 - 67, \\ \theta_{23} &= -14 + 23 - 58 + 67, & \theta_{24} &= 13 + 24 + 57 + 68, & \theta_{34} &= -12 + 34 - 56 + 78, \\ \theta_{15} &= -15 - 26 - 37 - 48, & \theta_{25} &= -16 + 25 + 38 - 47, & \theta_{35} &= -17 - 28 + 35 + 46, & \theta_{45} &= -18 + 27 - 36 + 45. \end{aligned}$$

If  $\theta \stackrel{\text{def}}{=} (\theta_{\alpha\beta})$ , it follows

$$\tau_2(\theta) = \sum_{\alpha < \beta} \theta_{\alpha\beta}^2 = -121234 - 41256 - 41357 + 41368 - 41278 - 41467 - 41458 + \star = -2\Omega_L \quad (4.3)$$

where  $\star$  denotes the Hodge star of what appears before, and

$$\Omega_L = \omega_{L_i}^2 + \omega_{L_j}^2 + \omega_{L_k}^2$$

161 is the left quaternionic 4-form on  $\mathbb{H}^2 = \mathbb{R}^8$ .

On the other hand, matrices  $B = \begin{pmatrix} B' & B'' \\ B''' & B'''' \end{pmatrix} \in \text{SO}(8)$  that commute with each of the involutions  $\mathcal{I}_1, \dots, \mathcal{I}_5$  are the ones satisfying  $B'' = B''' = 0$  and  $B' = B'''' \in \text{Sp}(1) \subset \text{SO}(4)$ . Thus the

subgroup preserving each of the  $\mathcal{I}_1, \dots, \mathcal{I}_5$  is the diagonal  $\text{Sp}(1)_\Delta \subset \text{SO}(8)$ . Thus, the subgroup of  $\text{SO}(8)$  preserving the vector bundle  $E^5$  consists of matrices  $B$  satisfying

$$B\mathcal{I}_\alpha = \mathcal{I}'_\alpha B,$$

162 with  $\mathcal{I}_1, \dots, \mathcal{I}_5$  and  $\mathcal{I}'_1, \dots, \mathcal{I}'_5$  bases of  $E^5$  related by a  $\text{SO}(5)$  matrix. This group is thus recognized to  
163 be  $\text{Sp}(1) \cdot \text{Sp}(2)$ , and:

164 **Proposition 11.** *Let  $M^8$  be a 8-dimensional oriented Riemannian manifold and let  $E^5$  be a vector subbundle*  
165 *of  $\text{End}(TM)$ , locally spanned by self dual anti-commuting involutions  $\mathcal{I}_1, \dots, \mathcal{I}_5$  and related, on open sets*  
166 *covering  $M$ , by functions giving  $\text{SO}(5)$  matrices. Then the datum of such a  $E^5$  is equivalent to a (left) almost*  
167 *quaternion Hermitian structure on  $M$ , i.e. to a  $\text{Sp}(2) \cdot \text{Sp}(1)$ -structure on  $M$ .*

In the above discussions we looked at the standard  $\text{U}(2)$  and  $\text{Sp}(1) \cdot \text{Sp}(2)$ -structures on  $\mathbb{R}^4$  and  $\mathbb{R}^8$ , through the decompositions of 2-forms

$$\mathfrak{so}(4) = \mathfrak{u}(1) \oplus \mathfrak{so}(3) \oplus \Lambda_2^2, \quad \mathfrak{so}(8) = \mathfrak{sp}(1) \oplus \mathfrak{sp}(2) \oplus \Lambda_{15}^2,$$

168 and orthonormal frames in the component  $\mathfrak{so}(3)$  and  $\mathfrak{sp}(2)$ , respectively. The last components  $\Lambda_2^2$  and  
169  $\Lambda_{15}^2$  describe all the similar structures on the linear spaces  $\mathbb{R}^4$  and  $\mathbb{R}^8$ . Thus, such decompositions  
170 give rise to the spaces  $\text{SO}(4)/\text{U}(2)$  and  $\text{SO}(8)/\text{Sp}(1) \cdot \text{Sp}(2)$ , the spaces of all possible structures in  
171 the two cases.

172 Summarizing:

**Corollary 12.** *The actions  $\begin{pmatrix} x \\ x' \end{pmatrix} \longrightarrow \begin{pmatrix} r & R\bar{u} \\ R_u & -r \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$ , when  $u, x, x' \in \mathbb{C}, \mathbb{H}, \mathbb{O}$  (and in any case  
 $r \in \mathbb{R}$  and  $r^2 + u\bar{u} = 1$ ) generate the groups  $\text{U}(2), \text{Sp}(2) \cdot \text{Sp}(1), \text{Spin}(9)$  of symmetries of the Hopf fibrations*

$$S^3 \longrightarrow S^2, \quad S^7 \longrightarrow S^4, \quad S^{15} \longrightarrow S^8.$$

173 *The corresponding  $G$ -structures on Riemannian manifolds  $M^4, M^8, M^{16}$  can be described through vector*  
174 *subbundles  $E \subset \text{End } TM$  of rank 3, 5, 9, respectively. Any such  $E$  is locally generated by self-dual involutions*  
175  *$\mathcal{I}_\alpha$  satisfying  $\mathcal{I}_\alpha \mathcal{I}_\beta = -\mathcal{I}_\beta \mathcal{I}_\alpha$  for  $\alpha \neq \beta$  and related, on open neighborhoods covering  $M$ , by functions giving*  
176 *matrices in  $\text{SO}(3), \text{SO}(5), \text{SO}(9)$ .*

## 177 5. Vector fields on spheres

178 An application of  $\text{Spin}(9)$  structures is the possibility of writing a maximal orthonormal system  
179 of tangent vector fields on spheres of any dimension. We outline here this construction only on some  
180 "low-dimensional" cases (in fact up to the sphere  $S^{511}$ ), referring for the general case to the linear  
181 algebra formalism developed in [43].

Recall that the identifications  $\mathbb{R}^{2n} = \mathbb{C}^n, \mathbb{R}^{4n} = \mathbb{H}^n$  and  $\mathbb{R}^{8n} = \mathbb{O}^n$  allow to act on the normal vector field of the unit sphere by the imaginary units of  $\mathbb{C}, \mathbb{H}, \mathbb{O}$ , giving 1 3 and 7 tangent orthonormal vector fields on  $S^{2n-1}, S^{4n-1}, S^{8n-1}$ . These are in fact a maximal system of linearly independent vector fields on  $S^{m-1} \subset \mathbb{R}^m$ , provided the (even) dimension  $m$  of the ambient space is not divisible by 16. The maximal number  $\sigma(m)$  of linearly independent vector fields on any  $S^{m-1}$  is well-known to be expressed as

$$\sigma(m) = 2^p + 8q - 1,$$

where  $\sigma(m) + 1 = 2^p + 8q$  is the Hurwitz-Radon number, referring to the decomposition

$$m = (2k + 1)2^p 16^q, \quad \text{where } 0 \leq p \leq 3. \quad (5.1)$$

182 (cf. [43] for further informations and references therein on this classical subject).

183 The following Table A lists some of the lowest dimensional spheres  $S^{m-1} \subset \mathbb{R}^m$  admitting a  
 184 maximal number  $\sigma(m) > 7$  of linearly independent vector fields.

**Table A.** Some spheres  $S^{m-1}$  with more than 7 vector fields

$m - 1$	15	31	47	63	79	95	111	127	143	159	175	191	...	255	...	511	...
$\sigma(m)$	8	9	8	11	8	9	8	15	8	9	8	11	...	16	...	17	...

The first of them is  $S^{15} \subset \mathbb{R}^{16}$ , acted on by  $\text{Spin}(9) \subset \text{SO}(16)$ . To write the eight vector fields on  $S^{15}$ , look at the involutions  $\mathcal{I}_1, \dots, \mathcal{I}_9$ , and at the eight complex structures  $J_1, \dots, J_8$  on  $\mathbb{R}^{16}$ :

$$J_\alpha \stackrel{\text{def}}{=} \mathcal{I}_\alpha \mathcal{I}_9 : \mathbb{R}^{16} \longrightarrow \mathbb{R}^{16}, \quad \alpha = 1, \dots, 8.$$

Denote by

$$N \stackrel{\text{def}}{=} (x, y) \stackrel{\text{def}}{=} (x_1, \dots, x_8, y_1, \dots, y_8)$$

185 the (outward) unit normal vector field of  $S^{15} \subset \mathbb{R}^{16}$ . Then:

**Proposition 13.** *The vector fields*

$$\begin{aligned}
 J_1 N &= (-y_1, -y_2, -y_3, -y_4, -y_5, -y_6, -y_7, -y_8, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8), \\
 J_2 N &= (-y_2, y_1, y_4, -y_3, y_6, -y_5, -y_8, y_7, -x_2, x_1, x_4, -x_3, x_6, -x_5, -x_8, x_7), \\
 J_3 N &= (-y_3, -y_4, y_1, y_2, y_7, y_8, -y_5, -y_6, -x_3, -x_4, x_1, x_2, x_7, x_8, -x_5, -x_6), \\
 J_4 N &= (-y_4, y_3, -y_2, y_1, y_8, -y_7, y_6, -y_5, -x_4, x_3, -x_2, x_1, x_8, -x_7, x_6, -x_5), \\
 J_5 N &= (-y_5, -y_6, -y_7, -y_8, y_1, y_2, y_3, y_4, -x_5, -x_6, -x_7, -x_8, x_1, x_2, x_3, x_4), \\
 J_6 N &= (-y_6, y_5, -y_8, y_7, -y_2, y_1, -y_4, y_3, -x_6, x_5, -x_8, x_7, -x_2, x_1, -x_4, x_3), \\
 J_7 N &= (-y_7, y_8, y_5, -y_6, -y_3, y_4, y_1, -y_2, -x_7, x_8, x_5, -x_6, -x_3, x_4, x_1, -x_2), \\
 J_8 N &= (-y_8, -y_7, y_6, y_5, -y_4, -y_3, y_2, y_1, -x_8, -x_7, x_6, x_5, -x_4, -x_3, x_2, x_1)
 \end{aligned} \tag{5.2}$$

186 are tangent to  $S^{15}$  and orthonormal.

187 Indeed, by fixing any  $\beta, 1 \leq \beta \leq 9$ , and considering the 8 complex structures  $\mathcal{I}_\alpha \mathcal{I}_\beta$ , with  $\alpha \neq \beta$ ,  
 188 the 8 vector fields  $\mathcal{I}_\alpha \mathcal{I}_\beta N$  are still tangent to  $S^{15}$  and orthonormal.

189 Although, as well-known,  $\mathbb{C}, \mathbb{H}$  and  $\mathbb{O}$  are the only normed algebras over  $\mathbb{R}$ , to move to higher  
 190 dimension it is convenient to consider the algebra  $\mathbb{S}$  of sedenions, obtained from  $\mathbb{O}$  through the  
 191 Cayley-Dickson process. Denoting by  $1, e_1, \dots, e_{15}$  the canonical basis of  $\mathbb{S}$  over  $\mathbb{R}$ , one can write a  
 192 multiplication table, cf. [43]. One gets divisors of the zero in  $\mathbb{S}$ , for example by  $(e_2 - e_{11})(e_7 + e_{14}) =$   
 193 0.

The following remark helps in higher dimension. Consider the sphere  $S^{m-1} \subset \mathbb{R}^m$ , and decompose  $m$  as  $m = (2k + 1)2^p 16^q$ , where  $p \in \{0, 1, 2, 3\}$ . First, observe that a vector field  $B$  tangent to the sphere  $S^{2^p 16^q - 1} \subset \mathbb{R}^{2^p 16^q}$  induces a vector field

$$\underbrace{(B, \dots, B)}_{2k+1 \text{ times}} \tag{5.3}$$

194 tangent to the sphere  $S^{(2k+1)2^p 16^q - 1}$ . Thus, assume in what follows  $k = 0$ , i. e.  $m = 2^p 16^q$ . Whenever  
 195 we extend a vector field in this way, we call the vector field given by (5.3) the *diagonal extension* of  $B$ .

196 If  $q = 0$ , that is, if  $m$  is not divisible by 16, the vector fields on  $S^{m-1}$  are given by the complex,  
 197 quaternionic or octonionic multiplication for  $p = 1, 2$  or 3 respectively, so that the Spin(9) contribution  
 198 is when  $q \geq 1$ , that is,  $m = 16l$ , and we can denote the coordinates in  $\mathbb{R}^{16l}$  by  $(s^1, \dots, s^l)$  where each  
 199  $s^\alpha$ , for  $\alpha = 1, \dots, l$ , belongs to the sedenions  $\mathbb{S}$ , and can thus be identified with a pair  $(x^\alpha, y^\alpha)$  of  
 200 octonions.

The unit (outward) normal vector field  $N$  of  $S^{16l-1}$  can be denoted by using sedenions:

$$N \stackrel{\text{def}}{=} (s^1, \dots, s^l) \quad \text{where} \quad \|s^1\|^2 + \dots + \|s^l\|^2 = 1 .$$

201 Therefore, we can think of  $N$  as an element of  $\mathbb{S}^l = \mathbb{O}^{2l} = \mathbb{R}^{16l}$ .

Whenever  $l = 2, 4$  or 8, denote by  $D$  the following automorphism of  $\mathbb{S}^l = \mathbb{O}^{2l}$

$$D : ((x^1, y^1), \dots, (x^l, y^l)) \longrightarrow ((x^1, -y^1), \dots, (x^l, -y^l)) . \quad (5.4)$$

202 We will refer to  $D$  as a *conjugation*, due to its similarity with that in  $*$ -algebras.

Moreover, it is convenient to use formal notations as:

$$N = (s^1, s^2) \stackrel{\text{def}}{=} s^1 + is^2 \in S^{31} , \quad (5.5)$$

$$N = (s^1, s^2, s^3, s^4) \stackrel{\text{def}}{=} s^1 + is^2 + js^3 + ks^4 \in S^{63} , \quad (5.6)$$

$$N = (s^1, s^2, s^3, s^4, s^5, s^6, s^7, s^8) \stackrel{\text{def}}{=} s^1 + is^2 + js^3 + ks^4 + es^5 + fs^6 + gs^7 + hs^8 \in S^{127} , \quad (5.7)$$

203 allowing to define left multiplications  $\mathcal{L}$  in sedenionic spaces  $\mathbb{S}^2, \mathbb{S}^4$  and  $\mathbb{S}^8$  (like in  $\mathbb{C}, \mathbb{H}$  and  $\mathbb{O}$ ) as  
 204 follows.

If  $l = 2$  the left multiplication is

$$\mathcal{L}_i(s^1, s^2) \stackrel{\text{def}}{=} -s^2 + is^1 , \quad (5.8)$$

whereas if  $l = 4$  we define

$$\begin{aligned} \mathcal{L}_i(s^1, \dots, s^4) &\stackrel{\text{def}}{=} -s^2 + is^1 - js^4 + ks^3 , \\ \mathcal{L}_j(s^1, \dots, s^4) &\stackrel{\text{def}}{=} -s^3 + is^4 + js^1 - ks^2 , \\ \mathcal{L}_k(s^1, \dots, s^4) &\stackrel{\text{def}}{=} -s^4 - is^3 + js^2 + ks^1 , \end{aligned} \quad (5.9)$$

and finally if  $l = 8$  we define

$$\begin{aligned} \mathcal{L}_i(s^1, \dots, s^8) &\stackrel{\text{def}}{=} -s^2 + is^1 - js^4 + ks^3 - es^6 + fs^5 + gs^8 - hs^7 , \\ \mathcal{L}_j(s^1, \dots, s^8) &\stackrel{\text{def}}{=} -s^3 + is^4 + js^1 - ks^2 - es^7 - fs^8 + gs^5 + hs^6 , \\ \mathcal{L}_k(s^1, \dots, s^8) &\stackrel{\text{def}}{=} -s^4 - is^3 + js^2 + ks^1 - es^8 + fs^7 - gs^6 + hs^5 , \\ \mathcal{L}_e(s^1, \dots, s^8) &\stackrel{\text{def}}{=} -s^5 + is^6 + js^7 + ks^6 + es^1 - fs^2 - gs^3 - hs^4 , \\ \mathcal{L}_f(s^1, \dots, s^8) &\stackrel{\text{def}}{=} -s^6 - is^5 + js^8 - ks^7 + es^2 + fs^1 + gs^4 - hs^3 , \\ \mathcal{L}_g(s^1, \dots, s^8) &\stackrel{\text{def}}{=} -s^7 - is^8 - js^5 + ks^6 + es^3 - fs^4 + gs^1 + hs^2 , \\ \mathcal{L}_h(s^1, \dots, s^8) &\stackrel{\text{def}}{=} -s^8 + is^7 - js^6 - ks^5 + es^4 + fs^3 - gs^2 + hs^1 . \end{aligned} \quad (5.10)$$

205 Note that in all three cases  $l = 2, 4$  and 8 the vector fields  $\mathcal{L}_i(N), \dots, \mathcal{L}_h(N)$  are tangent to  $S^{31}, S^{63}$   
 206 and  $S^{127}$ , respectively.

207 We can now write maximal systems of vector fields on  $S^{31}, S^{63}, S^{127}$ , as follows.

208 **Case  $p = 1$**

For  $S^{31}$ , whose maximal number of tangent vector fields is 9, we obtain 8 vector fields by writing the unit normal vector field as  $N = (s^1, s^2) = (x^1, y^1, x^2, y^2) \in S^{31} \subset S^2$ , where  $x^1, y^1, x^2, y^2 \in \mathbb{O}$ , and repeating Formulas (5.2) for each pair  $(x^1, y^1), (x^2, y^2)$ :

$$\begin{aligned}
 J_1 N &= (J_1 s^1, J_1 s^2) = (-y_1^1, -y_2^1, \dots, -y_7^1, -y_8^1, x_1^1, x_2^1, \dots, x_7^1, x_8^1, -y_1^2, -y_2^2, \dots, -y_7^2, -y_8^2, x_1^2, x_2^2, \dots, x_7^2, x_8^2) , \\
 J_2 N &= (J_2 s^1, J_2 s^2) = (-y_2^1, y_1^1, \dots, -y_8^1, y_7^1, -x_2^1, x_1^1, \dots, -x_8^1, x_7^1, -y_2^2, y_1^2, \dots, -y_8^2, y_7^2, -x_2^2, x_1^2, \dots, -x_8^2, x_7^2) , \\
 J_3 N &= (J_3 s^1, J_3 s^2) = (-y_3^1, -y_4^1, \dots, -y_5^1, -y_6^1, -x_3^1, -x_4^1, \dots, -x_5^1, -x_6^1, -y_3^2, -y_4^2, \dots, -y_5^2, -y_6^2, -x_3^2, -x_4^2, \dots, -x_5^2, -x_6^2) , \\
 J_4 N &= (J_4 s^1, J_4 s^2) = (-y_4^1, y_3^1, \dots, y_6^1, -y_5^1, -x_4^1, x_3^1, \dots, x_6^1, -x_5^1, -y_4^2, y_3^2, \dots, y_6^2, -y_5^2, -x_4^2, x_3^2, \dots, x_6^2, -x_5^2) , \\
 J_5 N &= (J_5 s^1, J_5 s^2) = (-y_5^1, -y_6^1, \dots, y_3^1, y_4^1, -x_5^1, -x_6^1, \dots, x_3^1, x_4^1, -y_5^2, -y_6^2, \dots, y_3^2, y_4^2, -x_5^2, -x_6^2, \dots, x_3^2, x_4^2) , \\
 J_6 N &= (J_6 s^1, J_6 s^2) = (-y_6^1, y_5^1, \dots, -y_4^1, y_3^1, -x_6^1, x_5^1, \dots, -x_4^1, x_3^1, -y_6^2, y_5^2, \dots, -y_4^2, y_3^2, -x_6^2, x_5^2, \dots, -x_4^2, x_3^2) , \\
 J_7 N &= (J_7 s^1, J_7 s^2) = (-y_7^1, y_8^1, \dots, y_1^1, -y_2^1, -x_7^1, x_8^1, \dots, x_1^1, -x_2^1, -y_7^2, y_8^2, \dots, y_1^2, -y_2^2, -x_7^2, x_8^2, \dots, x_1^2, -x_2^2) , \\
 J_8 N &= (J_8 s^1, J_8 s^2) = (-y_8^1, -y_7^1, \dots, y_2^1, y_1^1, -x_8^1, -x_7^1, \dots, x_2^1, x_1^1, -y_8^2, -y_7^2, \dots, y_2^2, y_1^2, -x_8^2, -x_7^2, \dots, x_2^2, x_1^2) .
 \end{aligned} \tag{5.11}$$

209

A ninth orthonormal vector field, completing the maximal system, is found by the formal left multiplication (5.8) and the automorphism D (5.4):

$$D(\mathcal{L}_i N) = D(-s^2, s^1) = (-x^2, y^2, x^1, -y^1) . \tag{5.12}$$

### 210 Case $p = 2$

The sphere  $S^{63}$  has a maximal number of 11 orthonormal vector fields. The normal vector field is in this case given by  $N = (s^1, \dots, s^4) = (x^1, y^1, \dots, x^4, y^4) \in S^{63} \subset S^4$ , and 8 vector fields arise as  $J_\alpha N$ , for  $\alpha = 1, \dots, 8$ . Three other vector fields are again given by the formal left multiplications 5.9 and the automorphism D (5.4):

$$\begin{aligned}
 D(\mathcal{L}_i N) &= (-x^2, y^2, x^1, -y^1, -x^4, y^4, x^3, -y^3) , \\
 D(\mathcal{L}_j N) &= (-x^3, y^3, x^4, -y^4, x^1, -y^1, -x^2, y^2) , \\
 D(\mathcal{L}_k N) &= (-x^4, y^4, -x^3, y^3, x^2, -y^2, x^1, -y^1) .
 \end{aligned} \tag{5.13}$$

### 211 Case $p = 3$

212 The sphere  $S^{127}$  has a maximal number of 15 orthonormal vector fields. Eight of them are still  
 213 given by  $J_\alpha N$ , for  $\alpha = 1, \dots, 8$ , whereas the formal left multiplications given in 5.10 yield the 7 tangent  
 214 vector fields  $D(\mathcal{L}_\alpha N)$ , for  $\alpha \in \{i, \dots, h\}$ .

### 215 The sphere $S^{255}$

To write a system of 16 orthonormal vector fields on  $S^{255} \subset \mathbb{R}^{256}$  look at the decomposition

$$\mathbb{R}^{256} = \mathbb{R}^{16} \oplus \dots \oplus \mathbb{R}^{16} , \tag{5.14}$$

into sixteen components. The unit outward normal vector field is

$$N = (s^1, \dots, s^{16}) ,$$

216 where  $s^1, \dots, s^{16}$  are sedenions.

The matrices in  $M_{16}(\mathbb{R})$  giving the complex structures  $J_1, \dots, J_8$  act on  $N$  not only separately on each of the 16-dimensional components of (5.14), but also formally on the (column) 16-plets of sedenions  $(s^1, \dots, s^{16})^T$ . According to which of the two actions of the same matrices are considered in  $\mathbb{R}^{256}$ , we use notations

$$J_1, \dots, J_8 \quad \text{or} \quad \text{block}(J_1), \dots, \text{block}(J_8)$$

in both cases all complex structures on  $\mathbb{R}^{256}$ . The following 16 vector fields are obtained:

$$J_1N, \dots, J_8N, \quad (5.15)$$

$$D(\text{block}(J_1)N), \dots, D(\text{block}(J_8)N), \quad (5.16)$$

where  $D$  has been defined in Formula (5.4). We call *level 1 vector fields* and *level 2 vector fields* the ones given by (5.15) and (5.16) respectively. Then:

**Proposition 14.** Formulas (5.15) and (5.16) give a maximal system of 16 orthonormal tangent vector fields on  $S^{255}$ .

**Proof.** Denote sedenions as pairs  $s^\alpha \stackrel{\text{def}}{=} (x^\alpha, y^\alpha)$  of octonions. The unit normal vector field is

$$N = (s^1, \dots, s^{16}) = (x^1, y^1, \dots, x^{16}, y^{16}) \in S^{255}, \quad (5.17)$$

and one gets the tangent vectors:

$$\begin{aligned} J_1N &= (J_1s^1, \dots, J_1s^{16}) = (-y^1, x^1, \dots, -y^{16}, x^{16}), \\ J_2N &= (J_2s^1, \dots, J_2s^{16}) = (R_iy^1, R_ix^1, \dots, R_iy^{16}, R_ix^{16}), \\ J_3N &= (J_3s^1, \dots, J_3s^{16}) = (R_jy^1, R_jx^1, \dots, R_jy^{16}, R_jx^{16}), \\ J_4N &= (J_4s^1, \dots, J_4s^{16}) = (R_ky^1, R_kx^1, \dots, R_ky^{16}, R_kx^{16}), \\ J_5N &= (J_5s^1, \dots, J_5s^{16}) = (R_ey^1, R_ex^1, \dots, R_ey^{16}, R_ex^{16}), \\ J_6N &= (J_6s^1, \dots, J_6s^{16}) = (R_fy^1, R_fx^1, \dots, R_fy^{16}, R_fx^{16}), \\ J_7N &= (J_7s^1, \dots, J_7s^{16}) = (R_gy^1, R_gx^1, \dots, R_gy^{16}, R_gx^{16}), \\ J_8N &= (J_8s^1, \dots, J_8s^{16}) = (R_hy^1, R_hx^1, \dots, R_hy^{16}, R_hx^{16}), \end{aligned} \quad (5.18)$$

that are easily checked to be orthonormal.

Moreover, one obtains eight further vector fields:

$$\begin{aligned} D(\text{block}(J_1)N) &= D(-s^9, -s^{10}, -s^{11}, -s^{12}, -s^{13}, -s^{14}, -s^{15}, -s^{16}, s^1, s^2, s^3, s^4, s^5, s^6, s^7, s^8), \\ D(\text{block}(J_2)N) &= D(-s^{10}, s^9, s^{12}, -s^{11}, s^{14}, -s^{13}, -s^{16}, s^{15}, -s^2, s^1, s^4, -s^3, s^6, -s^5, -s^8, s^7), \\ D(\text{block}(J_3)N) &= D(-s^{11}, -s^{12}, s^9, s^{10}, s^{15}, s^{16}, -s^{13}, -s^{14}, -s^3, -s^4, s^1, s^2, s^7, s^8, -s^5, -s^6), \\ D(\text{block}(J_4)N) &= D(-s^{12}, s^{11}, -s^{10}, s^9, s^{16}, -s^{15}, s^{14}, -s^{13}, -s^4, s^3, -s^2, s^1, s^8, -s^7, s^6, -s^5), \\ D(\text{block}(J_5)N) &= D(-s^{13}, -s^{14}, -s^{15}, -s^{16}, s^9, s^{10}, s^{11}, s^{12}, -s^5, -s^6, -s^7, -s^8, s^1, s^2, s^3, s^4), \\ D(\text{block}(J_6)N) &= D(-s^{14}, s^{13}, -s^{16}, s^{15}, -s^{10}, s^9, -s^{12}, s^{11}, -s^6, s^5, -s^8, s^7, -s^2, s^1, -s^4, s^3), \\ D(\text{block}(J_7)N) &= D(-s^{15}, s^{16}, s^{13}, -s^{14}, -s^{11}, s^{12}, s^9, -s^{10}, -s^7, s^8, s^5, -s^6, -s^3, s^4, s^1, -s^2), \\ D(\text{block}(J_8)N) &= D(-s^{16}, -s^{15}, s^{14}, s^{13}, -s^{12}, -s^{11}, s^{10}, s^9, -s^8, -s^7, s^6, s^5, -s^4, -s^3, s^2, s^1), \end{aligned} \quad (5.19)$$

similarly verified to be orthonormal.

To see that each vector  $J_\alpha N$  is orthogonal to each  $D(\text{block}(J_\beta)N)$ , for  $\alpha, \beta = 1, \dots, 8$ , look at the matrix representations of  $R_i, \dots, R_h$  and write the octonionic coordinates as  $x^\lambda = h_1^\lambda + h_2^\lambda e$ ,  $y^\mu = k_1^\mu + k_2^\mu e$ . Then the scalar product  $\langle J_\alpha N, D(\text{block}(J_\beta)N) \rangle$  can be computed by using Formula (2.1) for product of octonions. For example, recall from Formulas (5.18) that

$$J_8N = (y^1h, x^1h, \dots, y^8h, x^8h, y^9h, x^9h, \dots, y^{16}h, x^{16}h),$$

so that the computation of  $\langle J_8 N, D(\text{block}(J_1)N) \rangle$  gives rise to pairs of terms like in

$$\begin{aligned} \langle J_8 N, D(\text{block}(J_1)N) \rangle &= \Re(-(R_h y^1) \bar{x}^9 - (R_h x^9) \bar{y}^1 + \dots) = \\ &= \Re(\underline{-kk_2^1 \bar{h}_1^9} - \underline{\bar{h}_2^9 k k_1^1} - \underline{kh_2^9 \bar{k}_1^1} - \underline{\bar{k}_2^1 kh_1^9} + \dots) . \end{aligned}$$

223 To conclude, observe that the real part  $\Re$  of the sums of each of the corresponding underlined terms  
224 is zero. This is due to the identity  $\Re(hh'h'') = \Re(h'h''h)$ , that holds for all  $h, h', h'' \in \mathbb{H}$ .  $\square$

225 More generally:

**Proposition 15.** Fix any  $\beta$ ,  $1 \leq \beta \leq 9$ , and consider the 8 complex structures  $\mathcal{I}_\alpha \mathcal{I}_\beta$ , with  $\alpha \neq \beta$ , defined on  $\mathbb{R}^{256} = \mathbb{R}^{16} \oplus \dots \oplus \mathbb{R}^{16}$  by acting with the corresponding matrices on the listed 16-dimensional components, that is, by the diagonal extension of  $\mathcal{I}_\alpha \mathcal{I}_\beta$ . Consider also the further 8 complex structures  $D(\text{block}(\mathcal{I}_\alpha \mathcal{I}_\beta))$ , for  $\alpha \neq \beta$ , defined by the same matrices, now acting on the column matrix  $(s^1, \dots, s^{16})^T$  of sedenions. Then

$$\{\mathcal{I}_\alpha \mathcal{I}_\beta N, D(\text{block}(\mathcal{I}_\alpha \mathcal{I}_\beta)N)\}_{\alpha \neq \beta}$$

226 is a maximal system of 16 orthonormal tangent vector fields on  $S^{255}$ .

The dimension  $m = 2 \cdot 16^2$ , that is the sphere  $S^{511}$ , is the lowest dimensional case where the last ingredient of our construction enters. To define here the additional vector field we need to extend the formal left multiplication defined by Formula (5.8). Consider the decomposition

$$\mathbb{R}^{2 \cdot 16^2} = \mathbb{R}^{16^2} \oplus \mathbb{R}^{16^2}$$

and denote now by  $s^1, s^2$  elements in  $\mathbb{R}^{16^2}$ . With the notation

$$N = (s^1, s^2) \stackrel{\text{def}}{=} s^1 + is^2 \in S^{2 \cdot 16^2 - 1} , \quad (5.20)$$

227 define a formal left multiplication  $\mathcal{L}_i$  in  $\mathbb{R}^{2 \cdot 16^2}$  using Formula (5.8). One could then expect that  $D(\mathcal{L}_i N)$   
228 be orthogonal to  $\{J_\alpha N, D(\text{block}(J_\alpha)N)\}_{\alpha=1, \dots, 8}$ , but this is not the case. In fact,  $D(\mathcal{L}_i N)$  appears to be  
229 orthogonal to the first 8 vector fields, but not to the second ones.

To make things work, we need to extend not only  $\mathcal{L}_i$ , but also the conjugation  $D$ . To this aim, split elements  $s^\alpha \in \mathbb{R}^{16^2}$  as  $(x^\alpha, y^\alpha)$  where  $x^\alpha, y^\alpha \in \mathbb{R}^{16^2/2}$ , and define a conjugation  $D_2$  on  $\mathbb{R}^{16^2}$  using Formula (5.4):

$$D_2 : ((x^1, y^1), (x^2, y^2)) \longrightarrow ((x^1, -y^1), (x^2, -y^2)) . \quad (5.21)$$

230 The additional vector field is then  $D(D_2(\mathcal{L}_i N))$ , and:

**Theorem 16.** A maximal orthonormal system of tangent vector fields on  $S^{2 \cdot 16^2 - 1}$  is given by the following  $8 \cdot 2 + 1$  vector fields:

$$\begin{aligned} &J_1 N , \dots , J_8 N , \\ &D(\text{block}(J_1)N) , \dots , D(\text{block}(J_8)N) , \\ &D(D_2(\mathcal{L}_i N)) . \end{aligned} \quad (5.22)$$

## 231 6. Back to the octonionic Hopf fibration

232 As we saw in the previous Section,  $S^{15}$  is the the lowest dimensional sphere with more than 7  
233 linearly independent vector fields. There are further features that distinguish  $S^{15}$  among spheres.  
234 For example,  $S^{15}$  is the only sphere that admits three homogeneous Einstein metrics, and the only  
235 sphere that appears as regular orbit in three cohomogeneity one actions on projective spaces, namely

236 of  $SU(8)$ ,  $Sp(4)$  and  $Spin(9)$  on  $\mathbb{C}P^8$ ,  $\mathbb{H}P^4$  and  $\mathbb{O}P^2$  respectively (see [10], [34]). All of these features  
 237 can be traced back to the transitive action of  $Spin(9)$  on the octonionic Hopf fibration  $S^{15} \rightarrow S^8$ .

238 We outline here the proof of the following:

239 **Theorem 17.** Any global vector field on  $S^{15}$  which is tangent to the fibers of the octonionic Hopf fibration  
 240  $S^{15} \rightarrow S^8$  has at least one zero.

**Proof.** For any  $(x, y) \in S^{15} \subset \mathbb{O}^2 = \mathbb{R}^{16}$ , we already denoted by

$$N = (x, y) = (x_1, \dots, x_8, y_1, \dots, y_8)$$

the (outward) unit normal vector field of  $S^{15}$  in  $\mathbb{R}^{16}$ . Identifying the tangent spaces  $T_{(x,y)}(\mathbb{R}^{16})$  with  
 $\mathbb{R}^{16}$ , note that the involutions  $\mathcal{I}_1, \dots, \mathcal{I}_9$  define the following sections of  $T(\mathbb{R}^{16})|_{S^{15}}$  of length one:

$$\begin{aligned} \mathcal{I}_1 N &= (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8), \\ \mathcal{I}_2 N &= (y_2, -y_1, -y_4, y_3, -y_6, y_5, y_8, -y_7, -x_2, x_1, x_4, -x_3, x_6, -x_5, -x_8, x_7), \\ \mathcal{I}_3 N &= (y_3, y_4, -y_1, -y_2, -y_7, -y_8, y_5, y_6, -x_3, -x_4, x_1, x_2, x_7, x_8, -x_5, -x_6), \\ \mathcal{I}_4 N &= (y_4, -y_3, y_2, -y_1, -y_8, y_7, -y_6, y_5, -x_4, x_3, -x_2, x_1, x_8, -x_7, x_6, -x_5), \\ \mathcal{I}_5 N &= (y_5, y_6, y_7, y_8, -y_1, -y_2, -y_3, -y_4, -x_5, -x_6, -x_7, -x_8, x_1, x_2, x_3, x_4), \\ \mathcal{I}_6 N &= (y_6, -y_5, y_8, -y_7, y_2, -y_1, y_4, -y_3, -x_6, x_5, -x_8, x_7, -x_2, x_1, -x_4, x_3), \\ \mathcal{I}_7 N &= (y_7, -y_8, -y_5, y_6, y_3, -y_4, -y_1, y_2, -x_7, x_8, x_5, -x_6, -x_3, x_4, x_1, -x_2), \\ \mathcal{I}_8 N &= (y_8, y_7, -y_6, -y_5, y_4, y_3, -y_2, -y_1, -x_8, -x_7, x_6, x_5, -x_4, -x_3, x_2, x_1), \\ \mathcal{I}_9 N &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, -y_1, -y_2, -y_3, -y_4, -y_5, -y_6, -y_7, -y_8). \end{aligned} \quad (6.1)$$

241

Their span:

$$EN \stackrel{\text{def}}{=} \langle \mathcal{I}_1 N, \dots, \mathcal{I}_9 N \rangle,$$

242 is at any point a 9-plane in  $\mathbb{R}^{16}$ , not tangent to the sphere  $S^{15}$ . Observe that the 9-plane  $EN$  is invariant  
 243 under  $Spin(9)$ . This is certainly the case for the single vector field  $N$ , since  $Spin(9) \subset SO(16)$ . On the  
 244 other hand, the endomorphisms  $\mathcal{I}_\alpha$  are rotating under the  $Spin(9)$  action inside their vector bundle  
 245  $E^9 \subset \text{End}(\mathbb{R}^{16})$ .

Next, note that  $EN$  contains  $N$ :

$$N = \lambda_1 \mathcal{I}_1 N + \lambda_2 \mathcal{I}_2 N + \dots + \lambda_8 \mathcal{I}_8 N + \lambda_9 \mathcal{I}_9 N,$$

where the coefficients  $\lambda_\alpha$  are computed from 6.1 in terms of the inner products of vectors

$$\vec{x} = (x_1, \dots, x_8), \quad \vec{y} = (y_1, \dots, y_8) \in \mathbb{R}^8$$

and of the right translations  $R_i, \dots, R_h$ , as follows:

$$\lambda_1 = 2\vec{x} \cdot \vec{y}, \quad \lambda_2 = -2\vec{x} \cdot R_i \vec{y}, \quad \dots, \quad \lambda_8 = -2\vec{x} \cdot R_h \vec{y}, \quad \lambda_9 = |\vec{x}|^2 - |\vec{y}|^2.$$

246 In particular, at points with  $\vec{x} = \vec{0}$ , that is on the octonionic line  $l_\infty$ , the vector fields  $\mathcal{I}_1 N, \dots, \mathcal{I}_9 N$   
 247 are orthogonal to the unit sphere  $S^7 \subset l_\infty$ . This latter is the fiber of the Hopf fibration  $S^{15} \rightarrow S^8$  over the  
 248 north pole  $(0, \dots, 0, 1) \in S^8$ , and the mentioned orthogonality of this fiber  $S^7$ , is immediate from 6.1  
 249 for  $\mathcal{I}_1 N, \dots, \mathcal{I}_8 N$ .

250 Also, at these points, we have  $\mathcal{I}_9 N = N$ , so  $\mathcal{I}_9 N$  is orthogonal to this fiber  $S^7$  as well. Now, the  
 251 invariance under  $Spin(9)$  of the octonionic Hopf fibration shows that all its fibers are characterized as  
 252 orthogonal in  $\mathbb{R}^{16}$  to the vector fields  $\mathcal{I}_1 N, \dots, \mathcal{I}_9 N$ .

Now, assume that  $X$  is a vertical vector field of  $S^{15} \rightarrow S^8$ . By the previous characterization we have the following orthogonality relations in  $\mathbb{R}^{16}$ :

$$\langle X, \mathcal{I}_\alpha N \rangle = 0, \quad \text{for } \alpha = 1, \dots, 9,$$

and it follows that  $\langle \mathcal{I}_\alpha X, N \rangle = 0$ . But from the definition of a  $\text{Spin}(9)$  structure we see that if  $\alpha \neq \beta$ , then  $\langle \mathcal{I}_\alpha X, \mathcal{I}_\beta X \rangle = 0$ . Thus if  $X$  is a nowhere zero vertical vector field, and we would obtain in this way 9 pairwise orthogonal vector fields  $\mathcal{I}_1 X, \dots, \mathcal{I}_9 X$ , all tangent to  $S^{15}$ . But  $S^{15}$  is known to admit at most 8 linearly independent vector fields. Thus  $X$  cannot be vertical and nowhere zero.  $\square$

One gets as a consequence the following alternative proof of a result in [36]:

**Corollary 18.** *The octonionic Hopf fibration  $S^{15} \rightarrow S^8$  does not admit any  $S^1$  subfibration.*

**Proof.** In fact, any  $S^1$  subfibration would give rise to a real line subbundle  $L \subset T_{\text{vert}}(S^{15})$  of the vertical subbundle of  $T(S^{15})$ . Such line bundle  $L$  is necessarily trivial, due to the vanishing of its first Stiefel-Whitney class  $w_1(L) \in H^1(S^{15}; \mathbb{Z}_2) = 0$ . It follows that  $L$  would admit a nowhere zero section, thus a global vertical nowhere zero vector field.  $\square$

## 7. Locally conformally parallel $\text{Spin}(9)$ manifolds

Let  $G \subset \text{SO}(d)$ . Recall that a *locally conformally parallel  $G$ -structure* on a manifold  $M^d$  is the datum of a Riemannian metric  $g$  on  $M$ , a covering  $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$  of  $M$ , and for each  $\alpha \in A$  a metric  $g_\alpha$  defined on  $U_\alpha$  which has holonomy contained in  $G$  such that the restriction of  $g$  to each  $U_\alpha$  is conformal to  $g_\alpha$ :

$$g|_{U_\alpha} = e^{f_\alpha} g_\alpha$$

for some smooth map  $f_\alpha$  defined on  $U_\alpha$ .

Some of the possible cases here are:

- $G = \text{U}(n)$ , where we have the *locally conformally Kähler metrics*;
- $G = \text{Sp}(n) \cdot \text{Sp}(1)$ , yielding the *locally conformally quaternion Kähler metrics*;
- $G = \text{Spin}(9)$ , which is the case we are dealing with.

In any of the cases above, for each overlapping  $U_\alpha, U_\beta$  the functions  $f_\alpha, f_\beta$  differ by a constant:

$$f_\alpha - f_\beta = ct_{\alpha,\beta} \text{ on } U_\alpha \cap U_\beta.$$

This implies that  $df_\alpha = df_\beta$  on  $U_\alpha \cap U_\beta \neq \emptyset$ , hence defining a global, closed 1-form, usually denoted by  $\theta$  and called *the Lee form*. Its metric dual with respect to  $g$  is denoted by  $N$ ,

$$N = \theta^\sharp$$

and is called *the Lee vector field*.

Case  $G = \text{U}(n)$  of locally conformally Kähler metrics has been extensively studied in the last decades, see for instance [18].

Choosing  $G$  to be  $\text{Sp}(n)$  or  $\text{Sp}(n) \cdot \text{Sp}(1)$ , there are close relations to 3-Sasakian geometry: see [41] or the surveys [11], [14]. Finally, locally conformally parallel  $G_2$  and  $\text{Spin}(7)$  structures have been studied in [31], and they relate to nearly parallel  $\text{SU}(3)$  and  $G_2$  geometries, respectively.

As mentioned in the Introduction, holonomy  $\text{Spin}(9)$  is only possible on manifolds that are either flat or locally isometric to  $\mathbb{O}P^2$  or to the hyperbolic Cayley plane  $\mathbb{O}H^2$ . Weakened holonomy conditions give rise to several classes of  $\text{Spin}(9)$  structures (cf. [22] and Section 2). One of these classes consists of structures of *vectorial type* (see [2] and [22, page 148]). According to the following Remark, this class fits into the locally conformally parallel scheme.

**Remark 19.** In [22] and [2] the class of locally conformally parallel  $\text{Spin}(9)$  structures has been identified and studied, under the name of  $\text{Spin}(9)$  structures of *vectorial type*. We outline now a proof that, for  $\text{Spin}(9)$  structures, vectorial type is equivalent to locally conformally parallel. As already in Section 2, the splitting of the Levi-Civita connection, viewed as a connection in the principal bundle of orthonormal frames on  $M$ :

$$Z = Z^* \oplus \Gamma$$

where  $Z^*$  is the connection in the induced bundle of  $\text{Spin}(9)$ -frames and  $\Gamma$  is its orthogonal complement. Thus  $\Gamma$  is 1-form with values in the orthogonal complement  $\mathfrak{m}$  in the splitting of Lie algebras  $\mathfrak{so}(16) = \mathfrak{spin}(9) \oplus \mathfrak{m}$  and, under the identification  $\Lambda_{\mathfrak{g}_4}^2 = \mathfrak{m} = \Lambda^3(E^9)$ , cf. beginning of Section 3,  $\Gamma$  can be seen as a 1-form with values in  $\Lambda^3(E^9)$ . Under the action of  $\text{Spin}(9)$  the space  $\Lambda^1(M) \otimes \Lambda^3(E)$  decomposes as a direct sum of 4 irreducible components:

$$\Lambda^1(M) \otimes \Lambda^3(E) = P_0 \oplus P_1 \oplus P_2 \oplus P_3,$$

284 and, looking at all the possible direct sums, this yields 16 types of  $\text{Spin}(9)$  structures. The component  
285  $P_0$  identifies with  $\Lambda^1(M)$ , and thus with the component  $\mathcal{W}_4$  in formula (2.7).

286 **Definition 20.** [2] A  $\text{Spin}(9)$  structure is of *vectorial type* if  $\Gamma$  lives in  $P_0$ .

287 Now, let  $(M^{16}, g)$  be a Riemannian manifold endowed with a  $\text{Spin}(9)$  structure of vectorial type.  
288 Let  $\Gamma$  be as above, and let  $\Phi$  be its  $\text{Spin}(9)$ -invariant 8-form. Now,  $\Gamma = 0$  implies that the holonomy  
289 of  $M$  is contained in  $\text{Spin}(9)$  (cf. [22, page 21]).

From [2, page 5] we know that the following relations hold:

$$d\Phi = \theta \wedge \Phi, \quad d\theta = 0. \quad (7.1)$$

290 Let  $(M, \tilde{g})$  be the Riemannian universal cover of  $(M, g)$  and let  $\tilde{\Phi}, \tilde{\theta}$  be the lifts of  $\Phi, \theta$  respectively.  
291 Then relations (7.1) hold as well for  $\tilde{\Phi}$  and  $\tilde{\theta}$ . Since  $\tilde{M}$  is simply connected, then  $\tilde{\theta} = df$ , for some  
292  $f : \tilde{M} \rightarrow \mathbb{R}$ . Then, defining  $g_0 \stackrel{\text{def}}{=} e^{-f} \tilde{g}$  and  $\Phi_0 \stackrel{\text{def}}{=} e^{-4f} \tilde{\Phi}$ , we have  $d\Phi_0 = 0$ , that is, the  $\theta$ -factor of  $\Phi_0$  is  
293 zero. Hence  $g_0$  has holonomy contained in  $\text{Spin}(9)$ , and on the other hand it is locally conformal to  $g$ .  
294 Thus  $M$  can be covered by open subsets on which the metric is conformal to a metric with holonomy  
295 in  $\text{Spin}(9)$ .

296 The conformal flatness of metrics with holonomy  $\text{Spin}(9)$  has the following consequences (cf.  
297 [40] for the proofs).

298 **Theorem 21.** Let  $M^{16}$  be a compact manifold equipped with a locally, non globally, conformally parallel  
299  $\text{Spin}(9)$  metric  $g$ . Then:

- 300 1. The Riemannian universal covering  $(\tilde{M}, \tilde{g})$  of  $M$  is conformally equivalent to the euclidean  $\mathbb{R}^{16} \setminus \{0\}$ ,  
301 the Riemannian cone over  $S^{15}$ , and  $M$  is locally isometric, up to homotheties, to  $S^{15} \times \mathbb{R}$ .
- 302 2.  $M$  is equipped with a canonical 8-dimensional foliation.
- 303 3. If all the leaves of  $\mathcal{F}$  are compact, then  $M$  fibers over an orbifold  $\mathcal{O}^8$  finitely covered by  $S^8$  and all fibers  
304 are finitely covered by  $S^7 \times S^1$ .

305 **Theorem 22.** Let  $(M, g)$  be a compact Riemannian manifold. Then  $(M, g)$  is locally, non globally, conformally  
306 parallel  $\text{Spin}(9)$  if and only if the following three properties are satisfied:

1.  $M$  is the total space of a fiber bundle

$$M \xrightarrow{\pi} S_r^1$$

307 where  $\pi$  is a Riemannian submersion over the circle of radius  $r$ .

- 308 2. The fibers of  $\pi$  are spherical space forms  $S^{15}/K$ , where  $K$  is a finite subgroup of  $\text{Spin}(9)$ .

309 3. The structure group of  $\pi$  is contained in the normalizer of  $K$  in  $\text{Spin}(9)$ .

310 **8. Clifford systems and Clifford structures**

311 The self dual anti-commuting involutions  $\mathcal{I}_1, \dots, \mathcal{I}_9$  that define the standard  $\text{Spin}(9)$ -structure  
 312 on  $\mathbb{R}^{16}$  are an example of *Clifford system*. The definition, formalized in 1981 by D. Ferus, H. Karcher  
 313 and H. F. Münzner, in the study of isometric hypersurfaces of spheres [21], is the following.

**Definition 23.** A *Clifford system* on the Euclidean vector space  $\mathbb{R}^N$  is the datum

$$C_m = (P_0, \dots, P_m)$$

of a  $(m + 1)$ -ple of symmetric endomorphisms  $P_\alpha$  such that:

$$P_\alpha^2 = \text{Id} \text{ for all } \alpha, \quad P_\alpha P_\beta = -P_\beta P_\alpha \text{ for all } \alpha \neq \beta.$$

314 A Clifford system on  $\mathbb{R}^N$  is said to be *irreducible* if  $\mathbb{R}^N$  is not direct sum of two positive dimensional  
 315 subspaces that are invariant under all the  $P_\alpha$ .

From representation theory of Clifford algebras one recognizes (cf. [21], p. 483, [29], p. 163) that  $\mathbb{R}^N$  admits an irreducible Clifford system  $C = (P_0, \dots, P_m)$  if and only if

$$N = 2\delta(m),$$

316 where  $\delta(m)$  is given by the following

**Table B.** Clifford systems

$m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...	$8 + h$
$\delta(m)$	1	2	4	4	8	8	8	8	16	32	64	64	128	128	128	128	...	$16\delta(h)$

317 One can discuss uniqueness as follows. Given on  $\mathbb{R}^N$  two Clifford systems  $C_m = (P_0, \dots, P_m)$   
 318 and  $C'_m = (P'_0, \dots, P'_m)$ , they are said to be *equivalent* if there exists  $A \in O(N)$  such that  $P'_\alpha = A^t P_\alpha A$  for  
 319 all  $\alpha$ . Then for  $m \not\equiv 0 \pmod{4}$  there is a unique equivalence class of irreducible Clifford systems, and  
 320 for  $m \equiv 0 \pmod{4}$  there are two, classified by the two possible values of trace  $P_0 P_1 \dots P_m = \pm 2\delta(m)$ .

321 In [45] we outlined the following inductive construction for the irreducible Clifford systems on  
 322 real Euclidean vector spaces  $\mathbb{R}^N$ . taking as starting point the basic Clifford systems  $C_1, C_2, C_4, C_8$   
 323 associated with structures given by  $U(1), U(2), \text{Sp}(2) \cdot \text{Sp}(1), \text{Spin}(9)$ . All the cases appearing in Table  
 324 B make sense in the natural context of Riemannian manifolds. We get the following (cf. [45] for  
 325 details):

**Theorem 24.** (Procedure to write new Clifford systems from old). Let  $C_m = (P_0, P_1, \dots, P_m)$  be the last (or unique) Clifford system in  $\mathbb{R}^N$ . Then the first (or unique) Clifford system

$$C_{m+1} = (Q_0, Q_1, \dots, Q_m, Q_{m+1})$$

in  $\mathbb{R}^{2N}$  has as first and as last endomorphisms respectively

$$Q_0 = \left( \begin{array}{c|c} 0 & \text{Id} \\ \hline \text{Id} & 0 \end{array} \right), \quad Q_{m+1} = \left( \begin{array}{c|c} \text{Id} & 0 \\ \hline 0 & -\text{Id} \end{array} \right),$$

where the blocks are  $N \times N$ . The remaining matrices are

$$Q_\alpha = \left( \begin{array}{c|c} 0 & -P_{0\alpha} \\ \hline P_{0\alpha} & 0 \end{array} \right) \quad (\alpha = 1, \dots, m).$$

326 Here  $P_{0\alpha}$  are the complex structures given by compositions  $P_0P_\alpha$  in the Clifford system  $C_m$ . When the complex  
 327 structures  $P_{0\alpha}$  can be viewed as (possibly block-wise) right multiplications by some of the unit quaternions  
 328  $i, j, k$  or unit octonions  $i, j, k, e, f, g, h$ , and if the dimension permits, similarly defined further endomorphisms  
 329  $Q_\beta$  can be added by using some others among  $i, j, k$  or  $i, j, k, e, f, g, h$ .

**Table C.** Clifford systems  $C_m$  and  $G$ -structures on Riemannian manifolds  $M^N$

$m$	1	2	3	4	5	6	7	8	9	10	11	12
$N$	2	4	8	8	16	16	16	16	32	64	128	128
$G$	U(1)	U(2)	Sp(1) <sup>3</sup>	Sp(2)Sp(1)	SU(4)Sp(1)	Spin(7)U(1)	Spin(8)	Spin(9)	Spin(10)	Spin(11)	Spin(12)	Spin(13)

330 The notion of even Clifford structure, a kind of unifying notion proposed by A Moroianu and U.  
 331 Semmelmann [38], is instead given by the following datum on a Riemannian manifold  $(M, g)$ .

332 **Definition 25.** An even Clifford structure on  $(M, g)$  is a real oriented Euclidean vector bundle  
 333  $(E, h)$ , together with an algebra bundle morphism  $\varphi : \text{Cl}^0(E) \rightarrow \text{End}(TM)$  mapping  $\Lambda^2 E$  into  
 334 skew-symmetric endomorphisms.

335 By definition a Clifford system gives always rise to an even Clifford structure, but there are some  
 336 even Clifford structures on manifolds that cannot be constructed, even locally, from Clifford systems.  
 337 An example of this is given by Spin(7) structure on any oriented 8-dimensional Riemannian manifold,  
 338 as a consequence of the following observations (cf. [42] for further details).

339 **Proposition 26.** Let  $C_m = \{P_0, \dots, P_m\}$  be a Clifford system in  $\mathbb{R}^n$ . The compositions  $J_{\alpha\beta} \stackrel{\text{def}}{=} P_\alpha P_\beta$ , for  $\alpha < \beta$ ,  
 340 and  $J_{\alpha\beta\gamma} \stackrel{\text{def}}{=} P_\alpha P_\beta P_\gamma$ , for  $\alpha < \beta < \gamma$ , are linearly independent complex structures on  $\mathbb{R}^n$ .

341 **Proof.** One easily recognize that  $J_{\alpha\beta}$  and  $J_{\alpha\beta\gamma}$  are complex structures. On the other hand, for any  $\alpha =$   
 342  $0, \dots, m$ , one sees that  $\text{tr}(P_\alpha^* P_\alpha) = 1$ , and for  $\alpha < \beta$ , and  $\text{tr}(P_\alpha^* P_\beta) = 0$  so that the  $P_\alpha$  are orthonormal  
 343 and symmetric. By a similar argument,  $\text{tr}(J_{\alpha\beta}^* J_{\alpha\beta}) = 1$  and  $\text{tr}(J_{\alpha\beta}^* J_{\gamma\delta}) = \text{tr}(P_\beta P_\alpha P_\gamma P_\delta) = 0$  if any of  
 344  $\gamma, \delta$  equals  $\alpha$  or  $\beta$ . Also, for  $\alpha \neq \gamma$  and  $\beta \neq \delta$ , the  $J_{\alpha\beta}^* J_{\gamma\delta}$  is the composition of the skew-symmetric  
 345  $J_{\beta\alpha\gamma}$  and the symmetric  $P_\delta$  and as such its trace is necessarily zero. Similar arguments show that the  
 346  $J_{\alpha\beta\gamma}$ , for  $\alpha < \beta < \gamma$ , are orthonormal.  $\square$

347 **Corollary 27.** The Spin(7)-structures on  $\mathbb{R}^8$  cannot be defined through a Clifford system  $C_6$ .

**Proof.** For any choice of such a Clifford system  $C_6 = (P_0, \dots, P_6)$  in  $\mathbb{R}^8$ , the complex structures  
 $J_{\alpha\beta\gamma}$ , for  $\alpha < \beta < \gamma$ , would give rise to 35 linearly independent skew-symmetric endomorphisms,  
 contradicting the decomposition

$$\Lambda^2 \mathbb{R}^8 = \Lambda_7^2 \oplus \Lambda_{21}^2, \tag{8.1}$$

348 of 2-forms in  $\mathbb{R}^8$  under Spin(7).  $\square$

349 Nevertheless, the right multiplications by  $i, j, k, e, f, g, h \in \mathbb{O}$  span the vector bundle  $E^7 \subset$   
 350  $\text{End}^- \mathbb{R}^8$ , and this identifies Spin(7) structures among even Clifford structure.

351 We will see in the following Sections further examples of such *essential Clifford structures*, i. e.  
352 Clifford structures not coming from Clifford systems.

353 On Riemannian manifolds  $(M, g)$  it is natural to consider the following class of even Clifford  
354 structures.

**Definition 28.** The even Clifford structure  $E^r$  on  $(M, g)$  is said to be *parallel* if there exists a metric connection  $\nabla^E$  on  $E$  such that  $\varphi$  is connection preserving, i.e.

$$\varphi(\nabla_X^E \sigma) = \nabla_X^g \varphi(\sigma),$$

355 for every tangent vector  $X \in TM$  and section  $\sigma$  of  $Cl^0 E$ , and where  $\nabla^g$  is the Levi Civita connection.

356 The following Table D summarizes the non-flat parallel even Clifford structure, as classified in  
357 [38]. The non compact duals of the appearing symmetric spaces have to be added. A good part of the  
358 listed manifolds will appear in the following Sections.

**Table D.** Parallel non-flat even Clifford structures (cf. [38])

$r$	type of $E^r$	$M$	dimension of $M$
2		Kähler	$2m, m \geq 1$
3	projective if $M \neq \mathbb{H}P^q$	quaternion Kähler (qK)	$4q, q \geq 1$
4	projective if $M \neq \mathbb{H}P^{q^+} \times \mathbb{H}P^{q^-}$	product of two qK	$4(q^+ + q^-)$
5		qK	8
6	projective if $M$ non-spin	Kähler	8
7		Spin(7) holonomy	8
8	projective if $M$ non-spin	Riemannian	8
5		$Gr_2(\mathbb{H}^{n+2})$	$8n$
6	projective for $n$ odd	$Gr_4(\mathbb{C}^{n+4})$	$8n$
8	projective for $n$ odd	$Gr_8(\mathbb{R}^{n+8})$	$8n$
9		F II	16
10		E III	32
12		E VI	64
16		E VIII	128

## 359 9. The complex Cayley projective plane

This Section will deal with

$$E \text{ III} = E_6 / \text{Spin}(10) \cdot U(1) \cong V_{16}^{78} \subset \mathbb{C}P^{26},$$

360 the second, after the Cayley projective plane FII, of the exceptional symmetric spaces appearing in  
361 the classification table D.

362 A remarkable feature of E III, one of the two exceptional Hermitian symmetric spaces of compact  
363 type, is its model as a smooth projective algebraic variety of complex dimension 16 and degree 78, the  
364 so-called fourth Severi variety  $V_{16}^{78} \subset \mathbb{C}P^{26}$ . This name was proposed by F. Zak [50], who classified  
365 the smooth projective algebraic varieties  $V_n$  in a  $\mathbb{C}P^N$  that, in spite of their critical dimension  $n =$   
366  $\frac{2}{3}(N - 2)$ , are unable to fill  $\mathbb{C}P^N$  through their chords.

On the other hand E III admits a construction that is very similar to the one of the Cayley projective plane F II. One can in fact look at complex octonionic Hermitian matrices

$$Z = \begin{pmatrix} c_1 & x_1 & x_2 \\ \bar{x}_1 & c_2 & x_3 \\ \bar{x}_2 & \bar{x}_3 & c_3 \end{pmatrix} \in \text{Herm}_3(\mathbb{C} \otimes \mathbb{O}) \cong \mathbb{C}^{27}, \quad c_\alpha \in \mathbb{C}; \quad x_\alpha \in \mathbb{C} \otimes \mathbb{O},$$

acted on by  $E_6$  with three orbits on  $\mathbb{C}P^{26}$ . The closed one consists of matrices  $Z$  of rank one:

$$Z^2 = (\text{trace } Z)Z$$

and as such can be thought as (virtual) "projectors on complex octonionic lines in  $(\mathbb{C} \otimes \mathbb{O})^3$ ", thus points of the complex projective Cayley plane  $E \text{ III} = E_6/\text{Spin}(10) \cdot U(1) \subset \mathbb{C}P^{26}$ .

The projective algebraic geometry of  $E \text{ III} \subset \mathbb{C}P^{26}$  has been studied in details in the article [30]. Similarly to Corollary 27 we have:

**Proposition 29.** *The complex space  $\mathbb{C}^{16}$  does not admit any family of ten endomorphisms  $P_0, \dots, P_9$ , satisfying the properties of a Clifford system and compatible with respect to the standard Hermitian scalar product  $g$ .*

**Proof.** Such a family  $P_0, \dots, P_9$  would define (after multiplying each of them by  $i$ ) a representation of the complex Clifford algebra  $\mathbb{C}l_{10} \cong \mathbb{C}(32)$  (the order 32 complex matrix algebra) on the vector space  $\mathbb{C}^{16}$ .  $\square$

Note however that the Euclidean space  $\mathbb{R}^{32}$  admits a Clifford system  $C_9$ , cf. Table B. The parallel even Clifford structure on E III can be defined through the following one, here described on the model space  $\mathbb{C}^{16}$ .

For this, observe that  $\mathfrak{spin}(10) \subset \mathfrak{su}(16)$  is generated as Lie algebra by  $\mathfrak{spin}(9)$  and  $\mathfrak{u}(1)$ , with  $\mathfrak{u}(1)$  spanned by

$$\begin{pmatrix} i \text{Id}_8 & 0 \\ 0 & -i \text{Id}_8 \end{pmatrix} = \begin{pmatrix} i \text{Id}_8 & 0 \\ 0 & i \text{Id}_8 \end{pmatrix} \cdot \begin{pmatrix} \text{Id}_8 & 0 \\ 0 & -\text{Id}_8 \end{pmatrix} = \mathcal{I}_0 \cdot \mathcal{I}_9 = J_{09}$$

where  $\mathcal{I}_0 = \mathcal{J} =$  complex structure of  $\mathbb{C}^{16}$  and  $\mathcal{I}_1, \dots, \mathcal{I}_9$  are the octonionic Pauli matrices.

The rank 10 even Clifford structure on  $\mathbb{C}^{16}$  is then given by the vector bundle

$$E^{10} = \langle \mathcal{I}_0 \rangle \oplus \langle \mathcal{I}_1, \dots, \mathcal{I}_9 \rangle = \langle \mathcal{J} \rangle \oplus \langle \mathcal{I}_1, \dots, \mathcal{I}_9 \rangle \subset \text{End}(TM)$$

and

$$\mathfrak{spin}(10) = \text{lie}\{J_{09}, J_{19}, \dots, J_{89}\} = \text{span}\{J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta\}_{0 \leq \alpha < \beta \leq 9}.$$

We get (cf. Proposition 4 and Theorem 7):

**Theorem 30.** [44] *Let  $E^{10}$  be the even Clifford structure on E III. With the previous notations the characteristic polynomials*

$$t^{10} + \tau_2(\psi)t^8 + \tau_4(\psi)t^6 + \dots$$

of the matrix

$$\psi = (\psi_{\alpha\beta}) \in \Lambda^2 \otimes \mathfrak{so}(10)$$

of Kähler 2-forms of the  $J_{\alpha\beta}$  give:

i)  $\tau_2(\psi) = -3\omega^2$  where  $\omega =$  Kähler 2-form of E III,

ii)  $[\tau_4(\psi)] \in H^8$ , is the primitive generator of the cohomology ring  $H^*(E \text{ III})$ .

383 In analogy with the Spin(9) situation (Section 3), we call  $\tau_4(\psi) = \Phi_{\text{Spin}(10) \cdot \text{U}(1)}$  the *canonical*  
 384 *8-form* on E III.

385 Moreover:

386 **Theorem 31.** *Let  $\omega$  be the Kähler form and let  $\Phi_{\text{Spin}(10)} = \tau_4(\psi)$  be the 8-form on E III previously defined.*  
 387 *Then:*

388 (i) *The de Rham cohomology algebra  $H^*(\text{E III})$  is generated by (the classes of)  $\omega \in \Lambda^2$  and  $\Phi_{\text{Spin}(10)} \in$*   
 389  *$\Lambda^8$ .*

(ii) *By looking at E III as the fourth Severi variety  $V_{16}^{78} \subset \mathbb{C}P^{26}$ , the de Rham dual of the basis represented in  $H^8(\text{E III}; \mathbb{Z})$  by the forms  $(\frac{1}{(2\pi)^4} \Phi_{\text{Spin}(10)}, \frac{1}{(2\pi)^4} \omega^4)$  is given by the pair of algebraic cycles*

$$(\mathbb{C}P^4 + 3(\mathbb{C}P^4)', \mathbb{C}P^4 + 5(\mathbb{C}P^4)'),$$

390 where  $\mathbb{C}P^4, (\mathbb{C}P^4)'$  are maximal linear subspaces, belonging to the two different families ruling a totally  
 391 geodesic non-singular quadric  $Q_8$  contained in  $V_{16}^{78}$ .

### 392 10. Cayley-Rosenfeld planes

Besides the real and the complex Cayley projective planes F II and E III, there are two further exceptional symmetric spaces of compact type usually referred to as the *Cayley-Rosenfeld projective planes*, namely the following "projective plane over the quaternionic octonions"

$$\text{E VI} = E_7/\text{Spin}(12) \cdot \text{Sp}(1) = (\mathbb{H} \otimes \mathbb{O})P^2,$$

and the "projective plane over the octonionic octonions"

$$\text{E VIII} = E_8/\text{Spin}(16)^+ = (\mathbb{O} \otimes \mathbb{O})P^2.$$

393 Referring to the inclusions  $\mathbb{O} \hookrightarrow \mathbb{C} \otimes \mathbb{O} \hookrightarrow \mathbb{H} \otimes \mathbb{O} \hookrightarrow \mathbb{O} \otimes \mathbb{O}$ , the projective geometry of the projective  
 394 planes, well present on the first two steps, becomes weaker at the third and fourth Cayley-Rosenfeld  
 395 projective planes, cf. [6]. However, at least the dimensions of the four exceptional symmetric spaces is  
 396 coherent with this terminology. According to the Table D, these four spaces have the highest possible  
 397 ranks for non-flat parallel even Clifford structures.

398 The following Table summarizes the even Clifford structures on the four Cayley-Rosenfeld  
 399 planes. Only the one on F II is given by a Clifford systems, the other three are essential. Concerning  
 400 the cohomology generators, the one in dimension 8 can be constructed, also for E VI and E VIII, via  
 401 the fourth coefficient  $\tau_4(\psi)$  of the matrices  $\psi$  of Kähler 2-forms, associated with the even Clifford  
 402 structures that we are now listing.

**Table E.** Even Clifford structures on the Cayley-Rosenfeld projective planes

Model	Symm. space	Even Clifford structure	Cohomology gen.
$\mathbb{R}^{16}$	F II	$E^9 = \langle \mathcal{I}_1, \dots, \mathcal{I}_9 \rangle$	in $H^8$
$\mathbb{C}^{16}$	E III	$E^{10} = \langle \mathfrak{J} \rangle \oplus \langle \mathcal{I}_1, \dots, \mathcal{I}_9 \rangle$	in $H^2, H^8$
$\mathbb{H}^{16}$	E VI	$E^{12} = \langle \mathfrak{J}, \mathfrak{J}, \mathfrak{K} \rangle \oplus \langle \mathcal{I}_1, \dots, \mathcal{I}_9 \rangle$	in $H^4, H^8, H^{12}$
$\mathbb{O}^{16}$	E VIII	$E^{16} = \langle \mathfrak{J}, \dots, \mathfrak{J} \rangle \oplus \langle \mathcal{I}_1, \dots, \mathcal{I}_9 \rangle$	in $H^8, H^{12}, H^{16}, H^{20}$

**Remark 32.** The matrices of Kähler 2-forms associated with the even Clifford structures go through

$$\mathfrak{spin}(9) \subset \mathfrak{spin}(10) \subset \mathfrak{spin}(12) \subset \mathfrak{spin}(16),$$

and the last Lie algebra decomposes as

$$\mathfrak{spin}(16) = \mathfrak{so}(16) = \mathfrak{spin}(9) \oplus \Lambda_{\mathbb{R}^4}^2.$$

By recalling the identification ( $1 \leq \alpha < \beta < \gamma \leq 9$ )

$$\Lambda_{\mathbb{R}^4}^2 = \langle J_{\alpha\beta\gamma} = \mathcal{I}_\alpha \mathcal{I}_\beta \mathcal{I}_\gamma \rangle$$

403 (cf. the observation after (3.1) as well as [45]), this identification takes back the even Clifford structure  
404 of the 128-dimensional Cayley-Rosenfeld plane E VIII to the Spin(9)-structures we started with.

#### 405 11. Exceptional symmetric spaces

A good number of symmetric spaces appearing in the classification Table D belong to the list of the exceptional Riemannian symmetric spaces of compact type:

$$E I, E II, E III, E IV, E V, E VI, E VII, E VIII, E IX, F I, F II, G I,$$

that are part of the E. Cartan classification. Among them, the two exceptional Hermitian symmetric spaces

$$E III = \frac{E_6}{(\text{Spin}(10) \cdot U(1))} \quad \text{and} \quad E VII = \frac{E_7}{(E_6 \cdot U(1))}$$

are Kähler and therefore equipped with a non-flat parallel even Clifford structure of rank  $r = 2$ . Next, the five Wolf spaces

$$E II = \frac{E_6}{(\text{SU}(6) \cdot \text{Sp}(1))}, \quad E VI = \frac{E_7}{(\text{Spin}(12) \cdot \text{Sp}(1))}, \quad E IX = \frac{E_8}{(E_7 \cdot \text{Sp}(1))},$$

$$F I = \frac{F_4}{(\text{Sp}(3) \cdot \text{Sp}(1))}, \quad G I = \frac{G_2}{\text{SO}(4)},$$

406 examples of positive quaternion Kähler manifolds, carry a rank  $r = 3$  non-flat parallel Clifford  
407 structure.

408 Thus, seven of the twelve exceptional Riemannian symmetric spaces of compact type are  
409 either Kähler or quaternion Kähler. Accordingly, one of their de Rham cohomology generators is  
410 represented by a Kähler or quaternion Kähler form, and any further cohomology generators can be  
411 looked as primitive in the sense of the Lefschetz decomposition.

As seen in the previous Sections, the four Cayley-Rosenfeld projective planes

$$E III, E VI, E VIII, F II$$

412 carry a similar structure with  $r = 10, 12, 16, 9$ .

Thus, among the exceptional symmetric spaces of compact type, there are two spaces admitting two distinct even Clifford structures. Namely, the Hermitian symmetric E III has both a rank 2 and a rank 10 even Clifford structure, and the quaternion Kähler E VI has both a rank 3 and a rank 12 even Clifford structure. For simplicity, we will call *octonionic Kähler* the parallel even Clifford

structure defined by the vector bundles  $E^{10}, E^{12}, E^{16}, E^9$  on the Cayley-Rosenfeld projective planes E III, E VI, E VIII, F II. In conclusion, and with the exceptions of

$$E I = \frac{E_6}{Sp(4)}, \quad E IV = \frac{E_6}{F_4}, \quad E V = \frac{E_6}{SU(8)},$$

413 nine of the twelve exceptional Riemannian symmetric spaces of compact type admit at least one  
 414 parallel even Clifford structure. Any of such structures gives rise to a canonical differential form:  
 415 the Kähler 2-form  $\omega$  for the complex Kähler one, the quaternion Kähler 4-form  $\Omega$  for the five Wolf  
 416 space, and a canonical octonionic Kähler 8-form  $\Psi$  for the four Cayley-Rosenfeld projective planes.  
 417 Their classes are always one of the cohomology generators, and the following Table F collects some  
 418 informations on the exceptional symmetric spaces of compact type. For each of them the real  
 419 dimension, the existence of torsion in the integral cohomology, the Kähler or quaternion Kähler or  
 420 octonionic Kähler (K/qK/oK) property, the Euler characteristic  $\chi$ , and the Poincaré polynomial (up  
 421 to mid dimension) are listed.

**Table F.** Exceptional symmetric spaces of compact type

	dim	torsion	K/qK/oK	$\chi$	Poincaré polynomial $P(t) = \sum_{i=0, \dots} b_i t^i$
E I	42	yes		4	$1 + t^8 + t^9 + t^{16} + t^{17} + t^{18} + \dots$
E II	40	yes	qK	36	$1 + t^4 + t^6 + 2t^8 + t^{10} + 3t^{12} + 2t^{14} + 3t^{16} + 2t^{18} + 4t^{20} + \dots$
E III	32	no	K/oK	27	$1 + t^2 + t^4 + t^6 + 2(t^8 + t^{10} + t^{12} + t^{14}) + 3t^{16} + \dots$
E IV	26	no		0	$1 + t^9 + \dots$
E V	70	yes		72	$1 + t^6 + t^8 + t^{10} + t^{12} + 2(t^{14} + t^{16} + t^{18} + t^{20}) + 3(t^{22} + t^{24} + t^{26} + t^{28}) + 4(t^{30} + t^{32}) + 3t^{34} + \dots$
E VI	64	yes	qK/oK	63	$1 + t^4 + 2t^8 + 3t^{12} + 4t^{16} + 5t^{20} + 6(t^{24} + t^{28}) + 7t^{32} + \dots$
E VII	54	no	K	56	$1 + t^2 + t^4 + t^6 + t^8 + 2(t^{10} + t^{12} + t^{14} + t^{16}) + 3(t^{18} + t^{20} + t^{22} + t^{24} + t^{26}) + \dots$
E VIII	128	yes	oK	135	$1 + t^8 + t^{12} + 2(t^{16} + t^{20}) + 3(t^{24} + t^{28}) + 5t^{32} + 4t^{36} + 6(t^{40} + t^{44}) + 7(t^{48} + t^{52}) + 8t^{56} + 7t^{60} + 9t^{64} + \dots$
E IX	112	yes	qK	120	$1 + t^4 + t^8 + 2(t^{12} + t^{16}) + 3t^{20} + 4(t^{24} + t^{28}) + 5t^{32} + 6(t^{36} + t^{40}) + 7(t^{44} + t^{48} + t^{52}) + 8t^{56} + \dots$
F I	28	yes	qK	12	$1 + t^4 + 2(t^8 + t^{12}) + \dots$
F II	16	no	oK	3	$1 + t^8 + \dots$
G I	8	yes	qK	3	$1 + t^4 + \dots$

Next, we have Table G, containing the primitive Poincaré polynomials

$$\tilde{P}(t) = \sum_{i=0, \dots} \tilde{b}_i t^i$$

of the nine exceptional Riemannian symmetric spaces that admit an even parallel Clifford structures. Here "primitive" has a different meaning, according to the considered K/qK/oK structure. Thus, for the Hermitian symmetric spaces E III and E VII, they are simply the polynomials with coefficients the primitive Betti numbers

$$\tilde{b}_i = \dim(\ker[L_\omega^{n-i+1} : H^i \rightarrow H^{2n-i+2}]),$$

422 where  $L_\omega$  is the Lefschetz operator, the multiplication of cohomology classes with that of complex  
 423 Kähler form  $\omega$ , and  $n$  is the complex dimension.

In the positive quaternion Kähler setting, one has the vanishing of odd Betti numbers and the injectivity of the Lefschetz operator  $L_\Omega : H^{2k-4} \rightarrow H^{2k}$ ,  $k \leq n$ , now with  $\Omega$  the quaternion 4-form and  $n$  the quaternionic dimension. A remarkable aspect of the primitive Betti numbers

$$\tilde{b}_{2k} = \dim(\text{coker}[L_\Omega : H^{2k-4} \rightarrow H^{2k}])$$

424 for positive quaternion Kähler manifolds is their coincidence with the ordinary Betti numbers of the  
 425 associated Konishi bundle, the 3-Sasakian manifold fibering over it, cf. [23, page 56].

426 Finally, on the four Cayley-Rosenfeld planes, one still has the vanishing of odd Betti numbers  
 427 and the injectivity of the map  $L_\Phi : H^{2k-8} \rightarrow H^{2k}$ , defined by multiplication with the octonionic  
 428 8-form  $\Phi$ , and with  $k \leq 2n$ ,  $n$  now the octonionic dimension.

**Table G.** Primitive Poincaré polynomials  $\tilde{P}(t) = \sum_{i=0, \dots} \tilde{b}_i t^i$

Hermitian symmetric spaces	Kähler primitive Poincaré polynomial
E III	$1 + t^8 + t^{16}$
E VII	$1 + t^{10} + t^{18}$
Wolf spaces	Quaternion Kähler primitive Poincaré polynomial
E II	$1 + t^6 + t^8 + t^{12} + t^{14} + t^{20}$
E VI	$1 + t^8 + t^{12} + t^{16} + t^{20} + t^{24} + t^{32}$
E IX	$1 + t^{12} + t^{20} + t^{24} + t^{32} + t^{36} + t^{44} + t^{56}$
FI	$1 + t^8$
GI	1
Cayley-Rosenfeld projective planes	Octonionic Kähler primitive Poincaré polynomial
E III	$1 + t^2 + t^4 + t^6 + t^8 + t^{10} + t^{12} + t^{14} + t^{16}$
E VI	$1 + t^4 + t^8 + 2(t^{12} + t^{16} + t^{20}) + 3(t^{24} + t^{28} + t^{32})$
E VIII	$1 + t^{12} + t^{16} + t^{20} + t^{24} + t^{28} + t^{32} + t^{36} + t^{40} + t^{44} + t^{48} + t^{52} + t^{56} + t^{60} + t^{64}$
F II	1
Even Clifford exceptional symmetric spaces	Fully primitive Poincaré polynomial
E II	$1 + t^6 + t^8 + t^{12} + t^{14} + t^{20}$
E III	1
E VI	$1 + t^{12} + t^{24}$
E VII	$1 + t^{10} + t^{18}$
E VIII	$1 + t^{12} + t^{16} + t^{20} + t^{24} + t^{28} + t^{32} + t^{36} + t^{40} + t^{44} + t^{48} + t^{52} + t^{56} + t^{60} + t^{64}$
E IX	$1 + t^{12} + t^{20} + t^{24} + t^{32} + t^{36} + t^{44} + t^{56}$
FI	$1 + t^8$
F II	1
GI	1

429 **12. Grassmannians**

The classification Table D contains the following three series of Grassmannians:

$$Gr_8(\mathbb{R}^{n+8}) = \frac{SO(n+8)}{SO(n) \times SO(8)}, \quad Gr_4(\mathbb{C}^{n+4}) = \frac{SU(n+4)}{S(U(n) \times U(4))}, \quad Gr_2(\mathbb{H}^{n+2}) = \frac{Sp(n+2)}{Sp(n) \times Sp(2)}, \tag{12.1}$$

430 that carry an even Clifford structure of rank  $r = 8, 6, 5$ , respectively.

To define them, recall that the subgroup  $Spin(8) \subset SO(16) \subset Cl \mathbb{O}$  is generated by matrices

$$m_{u,v} = \begin{pmatrix} -R_u \circ R_{\bar{v}} & 0 \\ 0 & -R_{\bar{u}} \circ R_v \end{pmatrix} = m_u \circ m_v, \tag{12.2}$$

where

$$m_u = \begin{pmatrix} 0 & R_u \\ -R_{\bar{u}} & 0 \end{pmatrix}, \quad m_v = \begin{pmatrix} 0 & R_v \\ -R_{\bar{v}} & 0 \end{pmatrix},$$

(cf. [13]). For orthonormal  $u, v \in S^7 \subset \mathbb{O}$  the  $m_{u,v}$  satisfy the properties

$$m_{v,u} = -m_{u,v}, \quad m_{u,v}^2 = -\text{Id}.$$

On the other hand, recall that

$$TGr \cong W \otimes W^\perp,$$

431 where  $W$  is the tautological vector bundle and  $W^\perp$  its orthogonal complement in the ambient linear  
432 space.

433 Finally, it is convenient to recall that the complex structure and the local compatible  
434 hypercomplex structures of the following complex Kähler and quaternion Kähler Grassmannians:

$$Gr_2(\mathbb{R}^{n+2}) \cong Q_n \subset \mathbb{C}P^{n+1}, \quad Gr_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n) \times SO(4)}, \quad Gr_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n) \times U(2))}$$

435 come visibly from their elements, respectively oriented 2-planes, oriented 4-planes and complex  
436 2-planes.

Look first at Grassmannians in the first of the three series (12.1), namely at  $Gr_8(\mathbb{R}^{2m+8})$ , referring for simplicity to the case of an even dimensional ambient space  $\mathbb{R}^{2m+8}$ , thus insuring the spin property of the Grassmannian. From local orthonormal bases  $w_1, \dots, w_8$  and  $w_1^\perp, w_2^\perp, \dots, w_{2m-1}^\perp, w_{2m}^\perp$  of sections respectively of  $W$  and of  $W^\perp$ , one gets the following local basis of tangent vectors of  $Gr_8(\mathbb{R}^{2m+8})$ :

$$\begin{aligned} x_{1,1} &= w_1 \otimes w_1^\perp, & \dots & \dots & x_{8,1} &= w_8 \otimes w_1^\perp, \\ x_{1,2} &= w_1 \otimes w_2^\perp, & \dots & \dots & x_{8,2} &= w_8 \otimes w_2^\perp, \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{1,2m-1} &= w_1 \otimes w_{2m-1}^\perp, & \dots & \dots & x_{8,2m-1} &= w_8 \otimes w_{2m-1}^\perp, \\ x_{1,2m} &= w_1 \otimes w_{2m}^\perp, & \dots & \dots & x_{8,2m} &= w_8 \otimes w_{2m}^\perp. \end{aligned} \tag{12.3}$$

The listed 8-plets of sections can be written formally as octonions, i.e. for  $\alpha = 1, \dots, 2m$ :

$$\vec{x}_\alpha = (x_{1,\alpha}, x_{2,\alpha}, \dots, x_{8,\alpha}) = x_{1,\alpha} + ix_{2,\alpha} + \dots + hx_{8,\alpha}, \tag{12.4}$$

and can be ordered as a  $n$ -ple of pairs of octonions:

$$((\vec{x}_1, \vec{x}_2), \dots, (\vec{x}_{2m-1}, \vec{x}_{2m})) \in (\mathbb{O} \oplus \mathbb{O})^m.$$

437 The even Clifford structure on  $Gr_8(\mathbb{R}^{2m+8})$  can then be defined by looking at a rank 8  
 438 Euclidean vector bundle  $E \subset \text{End}^-(TGr_8(\mathbb{R}^{2m+8}))$ , satisfying the conditions to be locally generated  
 439 by anti-commuting orthogonal complex structures, that we denote here by  $m_1, m_2, \dots, m_8$ , in  
 440 correspondence with the  $m_1, m_i, \dots, m_h$ . The existence of such a  $E$  is insured by the holonomy  
 441 structure  $SO(2m) \times SO(8)$  of the Grassmannian, by its spin property, and by the given description  
 442 of  $\text{Spin}(8)$ . Accordingly, if  $u, v$  are local sections of  $E$ , we can look at them as octonions in the basis  
 443  $m_1, m_2, \dots, m_8$ .

For any such orthonormal pair  $(u, v)$ , look at  $u \wedge v$  as a section of  $\Lambda^2 E$ , and define:

$$\varphi : \Lambda^2 E \rightarrow \text{End}^-(TGr_8(\mathbb{R}^{2m+8}))$$

by

$$\varphi(u \wedge v)((\vec{x}_1, \vec{x}_2), \dots, (\vec{x}_{2m-1}, \vec{x}_{2m})) = (m_{u,v}(\vec{x}_1, \vec{x}_2), \dots, m_{u,v}(\vec{x}_{2m-1}, \vec{x}_{2m})), \quad (12.5)$$

i.e. by applying diagonally the matrix (12.2). Extending by Clifford composition this gives the Clifford morphism

$$\varphi : \text{Cl}^0 E \rightarrow \text{End}(TGr_8(\mathbb{R}^{2m+8})).$$

444 Thus:

**Theorem 33.** *There is a rank 8 vector sub-bundle  $E \subset \text{End}^-(TGr_8(\mathbb{R}^{2m+8}))$  locally generated by anti-commuting orthogonal complex structures  $m_1, m_2, \dots, m_8$ , and  $E$  defines on  $Gr_8(\mathbb{R}^{2m+8})$  an even non-flat parallel Clifford structure of rank 8. The morphism*

$$\varphi : \text{Cl}^0 E \rightarrow \text{End}(TGr_8(\mathbb{R}^{2m+8}))$$

is given by the Clifford extension of the map:

$$u \wedge v \in \Lambda^2 E \longrightarrow [m_{u,v} : (\mathbb{O} \oplus \mathbb{O})^m \rightarrow (\mathbb{O} \oplus \mathbb{O})^m],$$

defined by applying diagonally the matrix  $m_{u,v}$ . Here  $u, v$  are local orthonormal sections of  $E$ , thus unitary orthogonal octonions in the basis  $m_1, m_2, \dots, m_8$ , so that  $m_{u,v}$  acts diagonally on the  $m$ -ples of pairs of local tangent vectors:

$$((\vec{x}_1, \vec{x}_2), \dots, (\vec{x}_{2m-1}, \vec{x}_{2m})),$$

445 that can be looked at as elements of  $(\mathbb{O} \oplus \mathbb{O})^m$ .

A similar statement holds for the second series of Grassmannians in (12.1), assuming again an even dimensional ambient space  $\mathbb{C}^{2m+4}$ . Let  $w_1, w_2, w_3, w_4$  and  $w_1^\perp, w_2^\perp, \dots, w_{2m-1}^\perp, w_{2m}^\perp$  be local orthonormal bases of  $W$  and  $W^\perp$ , respectively. Define the following local tangent vector fields, local sections of  $TGr_4(\mathbb{C}^{2m+4}) \cong W \otimes W^\perp$ :

$$\begin{aligned} z_{1,1} &= w_1 \otimes w_1^\perp, & \dots & & z_{4,1} &= w_4 \otimes w_1^\perp, \\ z_{1,2} &= w_1 \otimes w_2^\perp, & \dots & & z_{4,2} &= w_4 \otimes w_2^\perp, \\ \dots & & \dots & & \dots & \\ z_{1,2m-1} &= w_1 \otimes w_{2m-1}^\perp, & \dots & & z_{4,2m-1} &= w_4 \otimes w_{2m-1}^\perp, \\ z_{1,2m} &= w_1 \otimes w_{2m}^\perp, & \dots & & z_{4,2m} &= w_4 \otimes w_{2m}^\perp. \end{aligned} \quad (12.6)$$

Again, look at the above lines as

$$\vec{z}_\alpha = (z_{1,\alpha}, z_{2,\alpha}, z_{3,\alpha}, z_{4,\alpha}) \in \mathbb{C}^4, \quad (12.7)$$

( $\alpha = 1, \dots, 2m$ ), and order them as an  $m$ -ples of pairs:

$$((\vec{z}_1, \vec{z}_2), \dots, (\vec{z}_{2m-1}, \vec{z}_{2m})) \in (\mathbb{C}^4 \oplus \mathbb{C}^4)^m.$$

Consider now the vector sub-space  $F = \langle 1, i, j, k, e, f \rangle \subset \mathbb{O}$ , and note that the corresponding operators  $m_u$ , with  $u \in F$ , act on the complex vector space  $\mathbb{C}^4$ . Similarly to what described for the real Grassmannians, there is a vector sub-bundle  $E^6 \subset \text{End}^-(T \text{Gr}_4(\mathbb{C}^{2m+4}))$ , locally generated by anti-commuting orthogonal complex structures  $m_1, m_2, \dots, m_6$ , and corresponding to the  $m_1, m_i, m_j, m_k, m_e, m_f$ . This is due to the holonomy  $S(U(2m) \times U(4))$  of the Grassmannian and its spin property. If  $(u, v)$  is an orthonormal pair of sections of  $E^6$ , then  $u \wedge v$  is a section of  $\Lambda^2 E$ , and the map

$$\varphi : \Lambda^2 E \rightarrow \text{End}^-(T \text{Gr}_4(\mathbb{C}^{2m+4})),$$

given by

$$\varphi(u \wedge v)((\vec{z}_1, \vec{z}_2), \dots, (\vec{z}_{2m-1}, \vec{z}_{2m})) = (m_{u,v}(\vec{z}_1, \vec{z}_2), \dots, m_{u,v}(\vec{z}_{2m-1}, \vec{z}_{2m})), \quad (12.8)$$

extends by Clifford composition to the Clifford morphism

$$\varphi : \text{Cl}^0 E \rightarrow \text{End}(T \text{Gr}_4(\mathbb{C}^{2m+4})).$$

Note that the holonomy group  $S(U(2m) \times U(4))$  acts on the model tangent space  $\mathbb{C}^{8m}$ , and the orthogonal representation

$$S(U(2m) \times U(4)) \rightarrow \text{SU}(8m)$$

446 defines an equivariant algebra morphism  $\varphi : \text{Cl}_6^0 \rightarrow \text{End}(\mathbb{C}^{8m})$  mapping  $\mathfrak{su}(4) = \mathfrak{spin}(6) \subset \text{Cl}_6^0$   
 447 into  $\mathfrak{su}(8m) \subset \text{End}(\mathbb{C}^{8m})$ . The parallel non-flat feature of  $\varphi$  follows again from the holonomy based  
 448 construction. This gives the following:

**Theorem 34.** *There is a rank 6 vector sub-bundle  $E \subset \text{End}^-(T \text{Gr}_4(\mathbb{C}^{2m+4}))$  locally generated by anti-commuting orthogonal complex structures  $m_1, m_2, \dots, m_6$ , and  $E$  defines on  $\text{Gr}_4(\mathbb{C}^{2m+4})$  an even non-flat parallel Clifford structure of rank 6. The morphism*

$$\varphi : \text{Cl}^0 E \rightarrow \text{End}(T \text{Gr}_4(\mathbb{C}^{2m+4}))$$

is given by Clifford extension of the map:

$$u \wedge v \in \Lambda^2 E \longrightarrow [m_{u,v} : (\mathbb{C}^4 \oplus \mathbb{C}^4)^n \rightarrow (\mathbb{C}^4 \oplus \mathbb{C}^4)^m],$$

defined by applying diagonally the matrix  $m_{u,v}$ . Here  $u, v$  are local orthonormal sections of  $E$ , thus unitary orthogonal in the basis  $m_1, m_2, \dots, m_6$ , so that  $m_{u,v}$  acts diagonally on the  $m$ -ples of pairs of local tangent vectors:

$$((\vec{z}_1, \vec{z}_2), \dots, (\vec{z}_{2m-1}, \vec{z}_{2m})),$$

449 that can be looked at as elements of  $(\mathbb{C}^4 \oplus \mathbb{C}^4)^m$ .

450 **Remark 35.** . When the ambient linear spaces have odd dimensional, similar statements hold, but  
 451 with the Clifford vector bundles  $E^8$  and  $E^6$  defined only locally. This fact is due to the spin/non-spin  
 452 property of the two series of Grassmannians  $\text{Gr}_8(\mathbb{R}^{n+8})$  and  $\text{Gr}_4(\mathbb{C}^{n+4})$ , whose second Stiefel  
 453 Whitney class satisfies  $w_2(\text{Gr}) = nu$ , where  $0 \neq u \in H^2(\text{Gr}; \mathbb{Z}_2)$ . Cf. Table D, where in the non-spin  
 454 cases the even Clifford structure is referred to as "projective".

For Grassmannians in the last series, the spin property of  $\text{Gr}_2(\mathbb{H}^{n+2})$  holds good for all values of  $n$ , due to the vanishing of  $H^2(\text{Gr}; \mathbb{Z}_2)$ . The Clifford morphism  $\varphi$  is here constructed as follows. Let

$w_1, w_2$  and  $w_1^\perp, \dots, w_n^\perp$  be local orthonormal bases of  $W$  and  $W^\perp$ , respectively. Define the following local tangent vector fields, local sections of  $TGr_2(\mathbb{H}^{n+2}) \cong W \otimes W^\perp$ :

$$\begin{aligned} h_{1,1} &= w_1 \otimes w_1^\perp, & h_{2,1} &= w_2 \otimes w_1^\perp, \\ h_{1,2} &= w_1 \otimes w_2^\perp, & h_{2,2} &= w_2 \otimes w_2^\perp, \\ &\dots\dots\dots & &\dots\dots\dots \\ h_{1,n-1} &= w_1 \otimes w_{n-1}^\perp, & h_{2,n-1} &= w_2 \otimes w_{n-1}^\perp, \\ h_{1,n} &= w_1 \otimes w_n^\perp, & h_{2,n} &= w_2 \otimes w_n^\perp. \end{aligned} \tag{12.9}$$

Next, let  $u, v$  be local orthonormal sections of  $E^5$ , the sub-bundle of  $\text{End}^+(TGr_2(\mathbb{H}^{n+2}))$  locally generated by a Clifford system  $C_4$ , whose existence is insured by the holonomy of this spin Grassmannian. The composition  $uv$  acts diagonally on the  $n$ -ples:

$$(\vec{h}_1, \vec{h}_2, \dots, \vec{h}_{n-1}, \vec{h}_n) = ((h_{1,1}, h_{2,1}), (h_{1,2}, h_{2,2}), \dots, (h_{1,n}, h_{2,n})) \in (\mathbb{H} \oplus \mathbb{H})^n,$$

and

$$\varphi : \text{Cl}^0 E \rightarrow \text{End}(TGr_2(\mathbb{H}^{n+2}))$$

is given by the Clifford extension of the map:

$$u \wedge v \in \Lambda^2 E \longrightarrow [uv : (\mathbb{H} \oplus \mathbb{H})^n \rightarrow (\mathbb{H} \oplus \mathbb{H})^n].$$

455 This gives:

**Theorem 36.** *There is a rank 5 vector sub-bundle  $E \subset \text{End}^+(TGr_2(\mathbb{H}^{n+2}))$  locally generated by anti-commuting orthogonal self-dual involutions  $\sigma_1, \sigma_2, \dots, \sigma_5$ , and  $E$  gives rise to an even non-flat parallel Clifford structure of rank 5 on  $Gr_2(\mathbb{H}^{n+2})$ . The Clifford morphism  $\varphi$  is constructed as follows. Let  $u, v$  be local orthonormal sections of  $E$ , and let the composition  $uv$  act diagonally on the  $n$ -ples:*

$$(\vec{h}_1, \vec{h}_2, \dots, \vec{h}_{n-1}, \vec{h}_n),$$

that can be looked at as elements of  $(\mathbb{H} \oplus \mathbb{H})^n$ . Then the morphism

$$\varphi : \text{Cl}^0 E \rightarrow \text{End}(TGr_2(\mathbb{H}^{n+2}))$$

is given by the Clifford extension of the map:

$$u \wedge v \in \Lambda^2 E \longrightarrow [uv : (\mathbb{H} \oplus \mathbb{H})^n \rightarrow (\mathbb{H} \oplus \mathbb{H})^n].$$

**Example 37.** We briefly list here some properties of the 16-dimensional examples carrying the just described even Clifford structures. More details are in [47]. From the first series of Grassmannians, one has the "complex octonionic projective line"

$$(\mathbb{C} \otimes \mathbb{O})P^1 \cong Gr_8(\mathbb{R}^{10}) \cong Q_8 \subset \mathbb{C}P^9,$$

totally geodesic in E III. Two parallel even Clifford structures, of rank 2 (complex Kähler) and of rank 8. Accordingly

$$\text{Poin}_{Gr_8(\mathbb{R}^{10})} = 1 + t^2 + t^4 + t^6 + 2t^8 + t^{10} + t^{12} + t^{14} + t^{16}.$$

Next, from the second series one has the "third Severi variety"

$$Gr_4(\mathbb{C}^6) \cong V_8^{14} \subset \mathbb{C}P^{14}.$$

Three parallel even Clifford structures, of rank 2 (complex Kähler), of rank 3 (quaternion Kähler) and of rank 6. Here:

$$\text{Poin}_{Gr_4(\mathbb{C}^6)} = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16}.$$

Finally, from the third series one gets  $Gr_2(\mathbb{H}^4)$ , with its two families of 2-planes  $\mathbb{H}P^2$  lying on the Grassmannian and satisfying the classical intersection properties of the Klein quadric. Finally:

$$\text{Poin}_{Gr_2(\mathbb{H}^4)} = 1 + t^4 + 2t^8 + t^{12} + t^{16}.$$

**Example 38.** Finally, we mention some higher dimensional examples. First, the 32-dimensional Wolf space  $Gr_8(\mathbb{R}^{12})$ , that has three non-flat even Clifford structures: two of rank 3, corresponding to the two quaternion Kähler structures (in correspondence with two different ways to define hypercomplex structures on the planes on any  $Gr_4(\mathbb{R}^{n+4})$ ), and the one of rank 8, described in this Section. Indeed,  $Gr_8(\mathbb{R}^{12})$  can be looked at as the "quaternion octonionic" projective line  $(\mathbb{H} \otimes \mathbb{O})P^1$ , totally geodesic submanifold of the exceptional symmetric space  $(\mathbb{H} \otimes \mathbb{O})P^2 \cong E VI$ , cf. [19]. Its Poincaré polynomial

$$\text{Poin}_{Gr_4(\mathbb{R}^{12})} = 1 + 2t^4 + 4t^8 + 5t^{12} + 6t^{16} + 5t^{20} + 4t^{24} + 2t^{28} + t^{32}$$

456 exhibits the presence of two quaternion Kähler 4-forms and of an "octonionic Kähler" 8-form  $\Psi$ . This  
457 latter  $\Psi$  is related to one that is defined on E VI through its holonomy group  $\text{Spin}(12) \cdot \text{Sp}(1)$ , cf. [46].

Next, the 64-dimensional Grassmannian

$$Gr_8(\mathbb{R}^{16}) = \frac{\text{SO}(16)}{\text{SO}(8) \times \text{SO}(8)}$$

supports, besides the parallel even Clifford structure of rank 8 just described, another similar structure obtained by interchanging the rôles of the two vector bundles  $W$  and  $W^\perp$ , i. e. by operating through the  $m_{u,v}$  on elements of  $W^\perp$ . The real cohomology

$$H^*(Gr_8(\mathbb{R}^{16})) \cong \frac{\mathbb{R}[e, p_1, p_2, p_3, e^\perp, p_1^\perp, p_2^\perp, p_3^\perp]}{ee^\perp = 0, (1 + p_1 + p_2 + p_3)(1 + p_1^\perp + p_2^\perp + p_3^\perp) = 1}, \quad (12.10)$$

in terms of Pontrjagin classes  $p_\alpha, p_\alpha^\perp$  and Euler classes  $e, e^\perp$  of  $W$  and  $W^\perp$ , gives rise to the Poincaré polynomial

$$\text{Poin}_{Gr_8(\mathbb{R}^{16})} = 1 + t^4 + 4t^8 + 5t^{12} + 9t^{16} + 11t^{20} + 15t^{24} + 15t^{28} + 18t^{32} + \dots$$

These two mentioned even Clifford structures descend to a unique even parallel Clifford structure of rank 8 on the smooth  $\mathbb{Z}_2$ -quotient

$$Gr_8^\perp(\mathbb{R}^{16}) = Gr_8(\mathbb{R}^{16}) / \perp$$

by the orthogonal complement involution  $\perp$ . The quotient  $Gr_8^\perp(\mathbb{R}^{16})$  turns out to be a totally geodesic half dimensional submanifold of E VIII, and can be looked at as the "projective line"  $(\mathbb{O} \otimes \mathbb{O})P^1$  over the "octonionic octonions" [19]. For the computation of the cohomology of  $Gr_8^\perp(\mathbb{R}^{16})$ , just note that the involution  $\perp$  identifies  $p_1 \rightarrow p_1^\perp, p_2 \rightarrow p_2^\perp, p_3 \rightarrow p_3^\perp, e \rightarrow e^\perp$ . This, due to the relations in (12.10), allows to survive, up to dimension 32, only the classes  $p_1^2, e, p_1^4, p_1^2 e, p_1^6, p_1^4 e, p_1^8, p_1^6 e$ . This gives the Poincaré polynomial

$$\text{Poin}_{Gr_8^\perp(\mathbb{R}^{16})} = 1 + 2t^8 + 2t^{16} + 2t^{24} + 2t^{32} + \dots$$

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