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EXTENSION OF EIGENVALUE PROBLEMS ON GAUSS MAP OF RULED SURFACES

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Abstract. A finite-type immersion or smooth map is a nice tool to classify submanifolds of Euclidean space, which comes from eigenvalue problem of immersion. The notion of generalized 1-type is a natural generalization of those of 1-type in the usual sense and pointwise 1-type. We classify ruled surfaces with generalized 1-type Gauss map as part of a plane, a circular cylinder, a cylinder over a base curve of an infinite type, a helicoid, a right cone and a conical surface of G-type.

1. Introduction

Nash's embedding theorem enables us to study Riemannian manifolds extensively by regarding a Riemannian manifold as a submanifold of Euclidean space with sufficiently high codimension. By means of such a setting, we can have rich geometric information from the intrinsic and extrinsic properties of submanifolds of Euclidean space. Inspired by the degree of algebraic varieties, B.-Y. Chen introduced the notion of order and type of submanifolds of Euclidean space. Furthermore, he developed the theory of finite-type submanifolds and estimated the total mean curvature of compact submanifolds of Euclidean space in the late 1970s ([3]).

In particular, the notion of finite-type immersion is a direct generalization of eigenvalue problem relative to the immersion of a Riemannian manifold into a Euclidean space: Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a submanifold M into the Euclidean m-space $\mathbb{E}^m$ and Δ the Laplace operator of M in $\mathbb{E}^m$. The submanifold M is said to be of finite-type if x has a spectral decomposition by $x = x_0 + x_1 + ... + x_k$, where x_0 is a constant vector and x_i are the vector fields satisfying $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$ $(i = 1, 2, ..., k)$. If the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ are different, it is called k-type. Since this notion was introduced, many works have been made in this area (see [3, 5]). This notion of finite-type immersion was naturally extended to that of pseudo-Riemannian manifolds in pseudo-Euclidean space and it was also applied to

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smooth maps, particularly the Gauss map defined on submanifolds of Euclidean space or pseudo-Euclidean space ([1, 2, 3, 10, 11]).

Regarding the Gauss map of finite-type, B.-Y. Chen and P. Piccini ([6]) initiated to study submanifolds of Euclidean space with finite-type Gauss map and classified compact surfaces with 1-type Gauss map, that is, $\Delta G = \lambda(G + \mathbb{C})$, where $\mathbb{C}$ is a constant vector and $\lambda \in \mathbb{R}$. Since then, quite a few works on ruled surfaces and ruled submanifolds with finite-type Gauss map in Euclidean space or pseudo-Euclidean space have been established ([1, 2, 3, 4, 7, 8, 9, 12, 13, 14, 15]).

However, some surfaces including a helicoid and a right cone in Euclidean 3-space have an interesting property concerning the Gauss map: The helicoid in $\mathbb{E}^3$ parameterized by

$$x(u, v) = (u \cos v, u \sin v, av), \quad a \neq 0$$

has the Gauss map and its Laplacian respectively given by

$$G = \frac{1}{\sqrt{a^2 + u^2}}(a \sin v, -a \cos v, u)$$

and

$$\Delta G = \frac{2a^2}{(a^2 + u^2)^2} G.$$

The right (or circular) cone C_a with parametrization

$$x(u, v) = (u \cos v, u \sin v, au), \quad a \geq 0$$

has the Gauss map

$$G = \frac{1}{\sqrt{1 + a^2}}(a \cos v, a \sin v, -1)$$

which satisfies

$$\Delta G = \frac{1}{u^2}(G + (0, 0, \frac{1}{\sqrt{1 + a^2}})$$

(cf. [4, 8]). The Gauss maps above are similar to 1-type, but it is not of 1-type Gauss map in the usual sense. Based upon such cases, B.-Y. Chen and the present authors defined the notion of pointwise 1-type Gauss map ([4]).

Definition 1.1. A submanifold M in $\mathbb{E}^m$ is said to have *pointwise 1-type Gauss map* if the Gauss map G of M satisfies

$$\Delta G = f(G + \mathbb{C})$$

for some non-zero smooth function f and a constant vector $\mathbb{C}$. In particular, if $\mathbb{C}$ is zero, then the Gauss map is said to be of pointwise 1-type of the first kind. Otherwise, it is said to be of pointewise 1-type of the second kind.

Let p be a point of $\mathbb{E}^3$ and $\beta = \beta(s)$ a unit speed curve such that p does not lie on β. A surface parametrized by

$$x(s, t) = p + t\beta(s)$$

is called a conical surface. A typical conical surface is a right cone and a plane.

Let us consider a following example of a conical surface.

Example 1.2. ([15]) Let M be a surface in $\mathbb{E}^3$ parameterized by

$$x(s,t) = (s\cos^2 t, s\sin t\cos t, s\sin t).$$

Then, the Gauss map G can be obtained by

$$G = \frac{1}{\sqrt{1+\cos^2 t}}(-\sin^3 t, (2-\cos^2 t)\cos t, -\cos^2 t).$$

After a considerably long computation, its Laplacian turns out to be

$$\Delta G = fG + g\mathbb{C}$$

for some non-zero smooth functions f, g and a constant vector $\mathbb{C}$. The surface M is a kind of conical surfaces generated by a spherical curve $\beta(t) = (\cos^2 t, \sin t\cos t, \sin t)$ on the unit sphere $\mathbb{S}^2(1)$ centered at the origin.

Inspired by such an example, we would like to generalize the notion of pointwise 1-type Gauss map as follows:

Definition 1.3. ([15]) The Gauss map G of a submanifold M in $\mathbb{E}^m$ is of *generalized 1-type* if the Gauss map G of M satisfies

$$\Delta G = fG + g\mathbb{C} \tag{1.1}$$

for some non-zero smooth functions f, g and a constant vector $\mathbb{C}$.

Especially we define a conical surface with generalized 1-type Gauss map.

Definition 1.4. A conical surface with generalized 1-type Gauss map is called a *conical surface of G-type.*

Remark 1.5. ([15]) A conical surface of G-type is constructed by the functions f, g and the constant vector $\mathbb{C}$ by solving the differential equations generated by (1.1).

In the present paper, we classify a ruled surface with generalized 1-type Gauss map in $\mathbb{E}^3$ as a plane, a circular cylinder, a cylinder over a base curve of an infinite type generated by the given function f, g and the constant vector $\mathbb{C}$, a helicoid, a right cone and a conical surface of G-type.

2. Preliminaries

Let M be a surface of the 3-dimensional Euclidean space $\mathbb{E}^3$. The map $G : M \to \mathbb{S}^2(1) \subset \mathbb{E}^3$ which maps each point p of M to a point G_p of $\mathbb{S}^2(1)$ by identifying the unit normal vector N_p to M at the point with G_p is called the Gauss map of the surface M, where $\mathbb{S}^2(1)$ is the unit sphere in $\mathbb{E}^3$ centered at the origin.

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For the matrix $\tilde{g} = (\tilde{g}_{ij})$ consisting of the components of the metric on M, we denote by $\tilde{g}^{-1} = (\tilde{g}^{ij})$ (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix $(\tilde{g}_{ij})$. Then the Laplacian Δ on M is in turn given by

$$\Delta = -\frac{1}{\sqrt{\mathcal{G}}} \sum_{i,j} \frac{\partial}{\partial x^i} \Big(\sqrt{\mathcal{G}}\, \tilde{g}^{ij} \frac{\partial}{\partial x^j} \Big). \tag{2.1}$$

Now, we define a ruled surface M in the 3-dimensional Euclidean space $\mathbb{E}^3$. Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$ defined on an open interval I and $\beta = \beta(s)$ a transversal vector field to $\alpha'(s)$ along α. Then the ruled surface M can be parameterized by

$$x(s,t) = \alpha(s) + t\beta(s), \quad s \in I, \quad t \in \mathbb{R}$$

satisfying $\langle \alpha', \beta \rangle = 0$ and $\langle \beta, \beta \rangle = 1$, where $'$ denotes d/ds. The curve α is called the *base curve* and β the *director vector field* or *ruling*. In particular, M is said to be *cylindrical* if β is constant, or, *non-cylindrical* otherwise.

Throughout this paper, we assume that all the geometric objects are smooth and all surfaces are connected unless otherwise stated.

3. Cylindrical ruled surfaces in $\mathbb{E}^3$ with generalized 1-type Gauss map

In this section, we study the cylindrical ruled surfaces with generalized 1-type Gauss map in $\mathbb{E}^3$.

Let M be a cylindrical ruled surface in $\mathbb{E}^3$. Without loss of generality, we assume that M is parameterized by

$$x(s,t) = \alpha(s) + t\beta,$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), 0)$ is a plane curve parameterized by the arc-length s and β a constant vector, namely $\beta = (0, 0, 1)$. In this case, the Gauss map G of M is given by

$$G = \alpha' \times \beta = (\alpha_2', -\alpha_1', 0) \tag{3.1}$$

and the Laplacian ΔG of the Gauss map G using (2.1) is obtained by

$$\Delta G = (-\alpha_2''', \alpha_1''', 0), \tag{3.2}$$

where $'$ stands for d/ds.

From now on, $'$ denotes the differentiation with respect to the parameter s relative to the base curve.

Suppose that the Gauss map G of M is of generalized 1-type, i.e., G satisfies equation (1.1). We now consider two cases for equation (1.1).

Case 1. $f = g$.

In this case, the Gauss map G is of pointwise 1-type described in Definition 1.1. According to Classification Theorem in [8] and [9], we have the ruled surface M is

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part of a plane, a circular cylinder or a cylinder over a base curve of an infinite-type satisfying

$$\sin^{-1}\left(\frac{c^2f^{-\frac{1}{3}}-1}{\sqrt{c_1^2+c_2^2}}\right)-\sqrt{c_1^2+c_2^2-\left(c^2f^{-\frac{1}{3}}-1\right)^2}=\pm c^3(s+k), \tag{3.3}$$

where $\mathbb{C}=(c_1,c_2,0)$, $c(\neq 0)$ and k are constants.

Case 2. $f\neq g$.

By a direct computation using (3.1) and (3.2), we see that the third component c_3 of the constant vector $\mathbb{C}$ is zero. We put $\mathbb{C}=(c_1,c_2,0)$. Then, we have the following system of ordinary differential equations

$$\begin{aligned} -\alpha_2''' &= f\alpha_2'+gc_1,\\ \alpha_1''' &= -f\alpha_1'+gc_2. \end{aligned} \tag{3.4}$$

Since α is a unit speed curve, that is, $(\alpha_1')^2+(\alpha_2')^2=1$, we may put

$$\alpha_1'(s)=\cos\theta(s) \quad \text{and} \quad \alpha_2'(s)=\sin\theta(s)$$

for a smooth function $\theta=\theta(s)$ of s. It enables equation (3.4) to be rewritten in the form

$$(\theta')^2\sin\theta-\theta''\cos\theta=f\sin\theta+gc_1,$$
$$(\theta')^2\cos\theta+\theta''\sin\theta=f\cos\theta-gc_2,$$

which give

$$(\theta')^2=f+g(c_1\sin\theta-c_2\cos\theta), \tag{3.5}$$
$$-\theta''=g(c_1\cos\theta+c_2\sin\theta). \tag{3.6}$$

Taking the derivative of (3.5), we have

$$2\theta'\theta''=f'+g'(c_1\sin\theta-c_2\cos\theta)+g(c_1\cos\theta+c_2\sin\theta)\theta'.$$

With the help of (3.5) and (3.6) it implies that

$$\frac{3}{2}(\theta'^2)'=f'+\frac{g'}{g}((\theta')^2-f).$$

Solving the above differential equation, we get

$$\theta'(s)^2=kg^{\frac{2}{3}}(s)+\frac{2}{3}g^{\frac{2}{3}}(s)\int g^{-\frac{2}{3}}(s)f(s)(\frac{f'}{f}-\frac{g'}{g})ds, \quad k(\neq 0)\in\mathbb{R}.$$

If we put

$$\theta'(s)=\pm\sqrt{p(s)}\,, \tag{3.7}$$

where $p(s)=|kg^{\frac{2}{3}}(s)+\frac{2}{3}g^{\frac{2}{3}}(s)\int g^{-\frac{2}{3}}(s)f(s)(\frac{f'}{f}-\frac{g'}{g})ds|$ for some non-zero constant k, we get a base curve α of M as follows:

$$\alpha(s)=\left(\int\cos\theta(s)ds,\int\sin\theta(s)ds,0\right), \tag{3.8}$$

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where $\theta(s) = \pm\int\sqrt{p(s)}\,ds$. In fact, θ' is the signed curvature of the base curve α which is precisely determined by the given functions f, g and the constant vector $\mathbb{C}$.

Note that if f and g are constant, the Gauss map G is of 1-type in the usual sense. In this case, the signed curvature of the base curve α is non-zero constant. So, the cylindrical ruled surface M is part of a circular cylinder.

Suppose that one of the functions f and g is not constant. Since a plane curve in $\mathbb{E}^3$ is of finite-type if and only if it is part of a straight line or a circle, the base curve defined by (3.8) is of an infinite-type ([5]). Thus, by putting together Cases 1 and 2, we have a classification theorem of cylindrical ruled surface with generalized 1-type Gauss map in $\mathbb{E}^3$.

Theorem 3.1. *Let M be a cylindrical ruled surface in $\mathbb{E}^3$. Suppose that M has generalized* 1*-type Gauss map. Then it is an open part of a plane, a circular cylinder or a cylinder over a base curve of an infinite-type satisfying* (3.3), (3.7) *and* (3.8).

4. Classification Theorem

In this section, we examine non-cylindrical ruled surfaces with generalized 1-type Gauss map in $\mathbb{E}^3$ and obtain a classification theorem.

Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$ parameterized by a base curve α and a director vector field β. Up to a rigid motion, its parametrization is given by

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle\alpha',\beta\rangle = 0$, $\langle\beta,\beta\rangle = 1$ and $\langle\beta',\beta'\rangle = 1$. Then, we have the natural frame $\{x_s, x_t\}$ given by $x_s = \alpha'(s) + t\beta'(s)$ and $x_t = \beta(s)$.

From this setting, we have an orthonormal frame $\{\beta, \beta', \beta\times\beta'\}$. For later use, we define the smooth functions q, u, Q and R as follows:

$$q = \langle x_s, x_s\rangle, \quad u = \langle\alpha',\beta'\rangle, \quad Q = \langle\alpha', \beta\times\beta'\rangle, \quad R = \langle\beta'', \beta\times\beta'\rangle.$$

In terms of the orthonormal frame $\{\beta, \beta', \beta\times\beta'\}$, we obtain

$$\begin{aligned} \alpha' &= u\beta' + Q\beta\times\beta', \\ \beta'' &= -\beta + R\beta\times\beta', \\ \alpha'\times\beta &= Q\beta' - u\beta\times\beta', \\ \beta\times\beta'' &= -R\beta', \end{aligned} \tag{4.1}$$

from which, the smooth function q is given by

$$q = t^2 + 2ut + u^2 + Q^2$$

and the Gauss map G of M is obtained by

$$G = \frac{x_s\times x_t}{||x_s\times x_t||} = q^{-1/2}\left(Q\beta' - (u+t)\beta\times\beta'\right). \tag{4.2}$$

Let H and K be the mean curvature and the Gaussian curvature of M respectively. By straightforward computation in using the first and second fundamental forms, they are given as follows:

$$\begin{aligned} H &= \frac{1}{2}q^{-3/2}(-Rt^2 - (2uR + Q')t + u'Q - Q^2R - u^2R - uQ'), \\ K &= -\frac{Q^2}{q^2}. \end{aligned} \tag{4.3}$$

Remark 4.1. If $R \equiv 0$, then the director vector field β is a plane curve.

The following formula is well known with respect to the Laplacian of the Gauss map of M in $\mathbb{E}^3$, which are easily obtained by applying the Gauss formula and the Weingarten formula:

$$\Delta G = 2\text{grad}H + (\text{tr}A^2)G,$$

where A denotes the shape operator of the surface M.

From (4.3), we get

$$\begin{aligned} 2\text{grad}H &= 2e_1(H)e_1 + 2e_2(H)e_2 \\ &= q^{-3}B_1e_1 + q^{-5/2}A_1e_2 \\ &= q^{-7/2}\left(qA_1\beta + (u+t)B_1\beta' + QB_1\beta \times \beta'\right), \end{aligned}$$

where $e_1 = \frac{x_s}{||x_s||}, \quad e_2 = \frac{x_t}{||x_t||}$,

$$\begin{aligned} A_1 =& Rt^3 + (3uR + 2Q')t^2 + (Q^2R - 3u'Q + 3u^2R + 4uQ')t \\ &+ (uQ^2R - 3uu'Q + u^3R + 2u^2Q' - Q^2Q'), \\ B_1 =& 3(u't + uu' + QQ')\{Rt^2 + (2uR + Q')t - u'Q + Q^2R + u^2R + uQ'\} \\ &+ (t^2 + 2ut + u^2 + Q^2)\{-R't^2 - (2u'R + 2uR' + Q'')t \\ &+ u''Q - 2QQ'R - Q^2R' - 2uu'R - u^2R' - uQ''\}. \end{aligned}$$

We also have

$$\text{tr}A^2 = q^{-3}D_1,$$

where

$$D_1 = \{-Rt^2 - (2uR + Q')t - u(uR + Q') + Q(u' - QR)\}^2 + 2Q^2(t^2 + 2ut + u^2 + Q^2).$$

Thus we obtain the Laplacian ΔG of the Gauss map G of M given by

$$\Delta G = q^{-7/2}[qA_1\beta + ((u+t)B_1 + D_1Q)\,\beta' + (QB_1 - D_1(u+t))\,\beta \times \beta']. \tag{4.4}$$

Suppose that M has generalized 1-type Gauss map G. Then, with the help of (1.1), (4.2) and (4.4), we obtain

$$\begin{aligned} &q^{-7/2}[qA_1\beta + \{(u+t)B_1 + D_1Q\}\beta' + \{QB_1 - D_1(u+t)\}\beta \times \beta'] \\ &= fq^{-1/2}\{Q\beta' - (u+t)\beta \times \beta'\} + g\mathbb{C} \end{aligned} \tag{4.5}$$

for some non-zero smooth functions f, g and a constant vector $\mathbb{C}$.

If we take the inner product to equation (4.5) with β, β' and $\beta \times \beta'$ respectively, then we get the following:

$$q^{-5/2}A_1 = g\,\langle \mathbb{C}, \beta\rangle, \tag{4.6}$$

$$q^{-7/2}\{(u+t)B_1 + D_1Q\} = fq^{-1/2}Q + g\,\langle \mathbb{C}, \beta'\rangle, \tag{4.7}$$

$$q^{-7/2}\{QB_1 - (u+t)D_1\} = -fq^{-1/2}(u+t) + g\,\langle \mathbb{C}, \beta\times\beta'\rangle. \tag{4.8}$$

Combining equations (4.6), (4.7) and (4.8), we have

$$qA_1\omega_2 - \{(u+t)B_1 + D_1Q\}\omega_1 + fq^3Q\omega_1 = 0, \tag{4.9}$$

$$qA_1\omega_3 - \{QB_1 - (u+t)D_1\}\omega_1 - fq^3(u+t)\omega_1 = 0, \tag{4.10}$$

$$\{(u+t)B_1 + D_1Q\}\omega_3 - \{QB_1 - (u+t)D_1\}\omega_2 - fq^3\{Q\omega_3 + (u+t)\omega_2\} = 0, \tag{4.11}$$

where we have put $\omega_1 = \langle \mathbb{C}, \beta\rangle$, $\omega_2 = \langle \mathbb{C}, \beta'\rangle$ and $\omega_3 = \langle \mathbb{C}, \beta\times\beta'\rangle$.

On the other hand, differentiating a constant vector $\mathbb{C} = \omega_1\beta + \omega_2\beta' + \omega_3\beta\times\beta'$ with respect to the parameter s and using (4.1), we get

$$\begin{aligned} &\omega_1' - \omega_2 = 0,\\ &\omega_3' + \omega_2 R = 0,\\ &\omega_1 + \omega_2' - \omega_3 R = 0. \end{aligned} \tag{4.12}$$

Combining equations (4.9) and (4.10), we obtain

$$A_1\{\omega_2(u+t) + \omega_3 Q\} - B_1\omega_1 = 0. \tag{4.13}$$

First of all, we consider the case of $R \equiv 0$.

Theorem 4.2. *Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$ with generalized 1-type Gauss map. If $R \equiv 0$, then M is part of a plane or a helicoid.*

Proof. If the constant vector $\mathbb{C}$ is zero in the definition given by (1.1), then the Gauss map G is nothing but of pointwise 1-type Gauss map of the first kind. By Characterization Theorem of a ruled surface with pointwise 1-type Gauss map of the first kind, M is part of a helicoid ([8]).

We now assume that the constant vector $\mathbb{C}$ is non-zero. In this case, we will show $Q \equiv 0$ on M and thus M is part of a plane due to (4.3).

Suppose that the open subset $U = \{s \in \mathrm{dom}(\alpha) | Q(s) \neq 0\}$ of $\mathbb{R}$ is not empty. Then, on a component U_C of U, we have from (4.12) that ω_3 is a constant and $\omega_1'' = -\omega_1$. Observing equation (4.13), the left side is a polynomial in t with functions of s as the coefficients. Hence the leading coefficient must vanish and $\omega_1^2 Q'$ is a constant on U_C with the help of (4.12).

Next, by examining the coefficient of the term involving t^2 in (4.13), we obtain

$$3\omega_2 u'Q - 2\omega_3 QQ' + 3\omega_1 u'Q' + \omega_1 u''Q = 0. \tag{4.14}$$

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Similarly as above, from the coefficient of the linear term in t of (4.13) with the help of (4.14), we get

$$\omega_2 QQ' + \omega_3 u'Q - \omega_1 (u')^2 + \omega_1 (Q')^2 = 0. \tag{4.15}$$

Also, the constant term in (4.13) with respect to the parameter t is automatically zero. If we make use of (4.14), we obtain

$$Q[\omega_1\{3u(u')^2 + 3u'QQ' - 3u(Q')^2 - u''Q^2\} - 3\omega_2 uQQ' \\ - \omega_3(3uu'Q + Q^2Q')] = 0.$$

Hence, on U_C, we have

$$\omega_1\{3u(u')^2 + 3u'QQ' - 3u(Q')^2 - u''Q^2\} - 3\omega_2 uQQ' \\ - \omega_3(3uu'Q + Q^2Q') = 0. \tag{4.16}$$

Using (4.14) and (4.15), equation (4.16) can be reduced to

$$2\omega_1 u'Q' + \omega_2 u'Q - \omega_3 QQ' = 0. \tag{4.17}$$

Suppose that there is a point $s_0 \in U_C$ such that $u'(s_0) \neq 0$. Then, $u'(s) \neq 0$ everywhere on an open interval I containing s_0. So, (4.15) yields

$$\omega_3 Q = \frac{1}{u'}\{\omega_1(u')^2 - \omega_1(Q')^2 - \omega_2 QQ'\}. \tag{4.18}$$

Putting (4.18) into (4.17), $(u'^2 + Q'^2)(\omega_2 Q + \omega_1 Q') = 0$, which implies $\omega_2 Q + \omega_1 Q' = 0$. Since $\omega_2 = \omega_1'$, we see that $\omega_1 Q$ is constant on I.

If $\omega_1 \equiv 0$ on some subinterval J in I, $\omega_2 = 0$ on J. (4.15) gives $\omega_3 = 0$ on J. Since $\mathbb{C}$ is a constant vector, $\mathbb{C}$ is zero vector, which is a contradiction. Thus, without loss of generality we may assume that $\omega_1 \neq 0$ everywhere on I and it is of the form $\omega_1 = k_1 \cos(s + s_1)$ for some non-zero constant k_1 and $s_1 \in \mathbb{R}$. Since $\omega_1^2 Q'$ is constant and $\omega_1 Q$ is constant on I, ω_1 must be zero on I, which contradicts $\omega_1 = k_1 \cos(s + s_1)$ for some non-zero constant k_1. Therefore, the open interval I is empty and thus $u' = 0$ on U_C. If we take it into account of (4.15) and (4.17), we get $Q'(\omega_2 Q + \omega_1 Q') = 0$ and $\omega_3 Q' = 0$, respectively.

Suppose that $Q'(s_2) \neq 0$ at some point $s_2 \in U_C$. Then $\omega_3 = 0$ and $\omega_1 Q$ is a constant on an open interval J_1 containing s_2. Similarly as above, since $\omega_1^2 Q'$ and $\omega_1 Q$ are constant on J_1, it follows that $\omega_1 = 0$. By (4.12), ω_2 is zero. Hence the constant vector $\mathbb{C}$ is zero, a contradiction. Thus J_1 is empty. Therefore, Q is constant on U_C. By continuity, Q is either a non-zero constant or zero on M. Because of (4.3), M is minimal and it is an open part of a helicoid, which means that the Gauss map is of pointwise 1-type of the first kind. Therefore, the open subset U is empty. Consequently, Q is zero on M. Hence, M is an open part of a plane. □

Now, without loss of generality we may assume that the function R is not vanishing everywhere.

If $f = g$, the non-cylindrical ruled surface M has pointwise 1-type Gauss map which is characterized as an open part of a right cone including the case that M is a plane or a helicoid depending upon whether the constant vector $\mathbb{C}$ is non-zero or zero ([7]).

From now on, we may assume the constant vector $\mathbb{C}$ is non-zero and $f \neq g$ unless otherwise stated. Similarly as before, the leading coefficient of the polynomial in the left side of equation (4.13) in t with functions of s as the coefficients is zero and we get

$$\omega_2 R + \omega_1 R' = 0. \tag{4.19}$$

Since $\omega_1' = \omega_2$ in (4.12), we see that $\omega_1 R$ is constant. Also, the coefficient of the term involving t^3 in (4.13) must be zero. With the help of (4.19), we get

$$2\omega_2 Q' + \omega_3 QR - \omega_1 u' R + \omega_1 Q'' = 0. \tag{4.20}$$

If we examine the coefficient of the term involving t^2 in (4.13), using (4.19) and (4.20) we obtain

$$\omega_1 Q^2 R' - 3\omega_2 u' Q + 2\omega_3 QQ' - \omega_1 QQ' R - 3\omega_1 u' Q' - \omega_1 u'' Q = 0. \tag{4.21}$$

Furthermore, from the coefficient of the linear term in t in (4.13) with the help of (4.19), (4.20) and (4.21), we also get

$$Q\{\omega_2 QQ' + \omega_3 u' Q - \omega_1 (u')^2 + \omega_1 (Q')^2\} = 0. \tag{4.22}$$

Consider an open set $V = \{s \in \mathrm{dom}(\alpha) | Q(s) \neq 0\}$. Suppose that V is not empty. Equation (4.22) gives that

$$\omega_2 QQ' + \omega_3 u' Q - \omega_1 (u')^2 + \omega_1 (Q')^2 = 0. \tag{4.23}$$

Moreover, considering the constant term with respect to t in (4.13) and using (4.19), (4.20) and (4.21), we obtain

$$\begin{aligned} Q[\omega_1\{3u(u')^2 + 3u'QQ' - Q^2Q'R - 3u(Q')^2 - u''Q^2 + Q^3R'\} \\ - 3\omega_2 uQQ' - \omega_3(3uu'Q + Q^2Q')] = 0. \end{aligned}$$

Hence, on the open subset V in $\mathbb{R}$,

$$\begin{aligned} \omega_1\{3u(u')^2 + 3u'QQ' - Q^2Q'R - 3u(Q')^2 - u''Q^2 + Q^3R'\} \\ - 3\omega_2 uQQ' - \omega_3(3uu'Q + Q^2Q') = 0. \end{aligned} \tag{4.24}$$

Applying (4.21) and (4.23) to (4.24), we have

$$2\omega_1 u' Q' + \omega_2 u' Q - \omega_3 QQ' = 0. \tag{4.25}$$

On the other hand, since $\omega_3 R = \omega_1 + \omega_2'$ in (4.12), (4.20) becomes

$$(\omega_1 Q)'' + \omega_1 Q - \omega_1 u' R = 0. \tag{4.26}$$

Now, suppose the open subset $V_1 = \{s \in V | u'(s) \neq 0\}$ is not empty. Then (4.23) yields

$$\omega_3 Q = \frac{1}{u'}\{\omega_1 (u')^2 - \omega_1 (Q')^2 - \omega_2 QQ'\}. \tag{4.27}$$

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Putting (4.27) into (4.25), $(u'^2 + Q'^2)(\omega_2 Q + \omega_1 Q') = 0$ and thus $\omega_2 Q + \omega_1 Q' = 0$. Therefore, $\omega_1 Q$ is constant on a component $\mathcal{C}$ of V_1. From (4.26), we get $\omega_1 Q = \omega_1 u' R$.

If $\omega_1 \equiv 0$ on an open interval $\tilde{I} \subset \mathcal{C}$, the constant vector $\mathbb{C}$ is zero on M, a contradiction. Thus, $\omega_1 \neq 0$ and so $Q = u'R$ on $\mathcal{C}$. The fact that $\omega_1 Q$ and $\omega_1 R$ are constant on $\mathcal{C}$ implies that u' is a non-zero constant on $\mathcal{C}$. Then, (4.21) and (4.25) are simplified as follows:

$$\omega_1 Q^2 R' + 2\omega_3 QQ' - \omega_1 QQ'R = 0, \tag{4.28}$$

$$\omega_1 u'Q' - \omega_3 QQ' = 0. \tag{4.29}$$

Putting $Q = u'R$ into (4.28), $\omega_3 Q' = 0$ is derived. Thus, (4.29) implies that $\omega_1 Q' = 0$ and so $Q' = 0$ on $\mathcal{C}$. Hence, Q and R are both non-zero constants on $\mathcal{C}$.

On the other hand, without difficulty, we can show that the torsion of the director vector field $\beta = \beta(s)$ viewing as a curve is zero and so β is part of a plane curve which is a small circle on the unit sphere centered at the origin with the normal curvature -1 and the geodesic curvature R on $\mathcal{C}$. Without loss of generality, we may put

$$\beta(s) = \frac{1}{p}(\cos ps, \sin ps, R)$$

on $\mathcal{C}$, where we have put $p = \sqrt{1 + R^2}$. Then, $u = \langle \alpha', \beta' \rangle = -\alpha_1' \sin ps + \alpha_2' \cos ps$, where $\alpha'(s) = (\alpha_1'(s), \alpha_2'(s), \alpha_3'(s))$. Therefore, on $\mathcal{C}$, we get

$$u' = -(\alpha_1'' + \alpha_2' p) \sin ps + (\alpha_2'' - \alpha_1' p) \cos ps,$$

from which, we see that $u' = 0$ on $\mathcal{C} \subset V_1$, contradiction. Hence, V_1 is empty and so $u' = 0$ on V. Then, (4.20), (4.23) and (4.25) can be respectively reduced to

$$2\omega_2 Q' + \omega_3 QR + \omega_1 Q'' = 0, \tag{4.30}$$

$$\omega_2 QQ' + \omega_1 (Q')^2 = 0, \tag{4.31}$$

$$\omega_3 QQ' = 0. \tag{4.32}$$

Suppose that $Q'(\tilde{s}_0) \neq 0$ at a point $\tilde{s}_0$ in V. From (4.31) and (4.32), $\omega_3 = 0$ and $\omega_1 Q$ is a constant on an open interval $\tilde{J} \subset V$ containing $\tilde{s}_0$. Hence, ${\omega_2}' Q = 0$ is derived from (4.30). Therefore, ${\omega_2}' = 0$ on $\tilde{J}$. The third equation of (4.12) yields $\omega_1 = 0$. It follows that $\omega_2 = 0$. Since $\mathbb{C}$ is a constant vector, $\mathbb{C}$ is zero on M, a contradiction. So, $Q' = 0$ on V. Thus, Q is non-zero constant on each component of V. If we consider (4.20) and (4.21), we have

$$\omega_3 R = 0 \quad \text{and} \quad \omega_1 R' = 0.$$

Since $R \neq 0$, $\omega_3 = 0$ on each component of V. By (4.19), $\omega_2 R = 0$, which yields that $\mathbb{C}$ is zero on M. It is a contradiction. Hence, the open subset V of $\mathbb{R}$ is empty and the function Q is vanishing on M. Thus, M is flat due to (4.3). Since the ruled surface M is non-cylindrical, M is one of an open part of a tangent developable surface or a conical surface. One of the authors proved that tangential developable surfaces do not have generalized 1-type Gauss map and a conical surface of G-type can be constructed by the given functions f, g and the constant vector $\mathbb{C}$ ([15]).

Consequently, we have

Theorem 4.3. *Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$ with generalized 1-type Gauss map. Then, M is an open part of a plane, a helicoid, a right cone or a conical surface of G-type.*

Summing up our results, we obtain the following classification theorem.

Theorem 4.4. *(Classification) Let M be a ruled surface in $\mathbb{E}^3$ with generalized 1-type Gauss map. Then, M is an open part of a plane, a circular cylinder, a cylinder over a base curve of an infinite-type satisfying* (3.3), (3.7) *and* (3.8)*, a helicoid, a right cone or a conical surface of G-type.*

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