

Analysis of the Angular Dependence of Time Delay in Gravitational Lensing

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Abstract :

We consider an alternative formula for time delay in gravitational lensing. Imposing a smoothness condition on the gravitationally deformed paths followed by the photons from the source to the observer, we show that our formula displays the same degrees of freedom of the standard one. In addition to this, it is shown that the standard expression for time delay is recovered when small angles are involved. These two features strongly support the claim that the formula for time delay studied in this paper is the generalization to arbitrary angles of the standard one, which is valid at small angles. This could therefore result in a useful tool in view of softening the known discrepancy between the various estimates of the Hubble constant. As an aside, two interesting consequences of our proposal for time delay are discussed: the existence of a constraint on the gravitational potential generated by the lens and a formula for the mass of the lens in the case of central potential.

1 Introduction

One of the first modern cosmological models [1] was proposed by Albert Einstein in 1917: a static, spatially closed and temporally infinite universe with positive spatial curvature. This model, known as Einstein static universe, although unstable, had the nice feature to be completely determined: the request of staticity fixes the content of the universe, which is composed by matter, curvature and a cosmological constant Λ . However, in 1929, Hubble showed that the universe is expanding, ruling out the Einstein model [2]. The value of the expansion speed of the universe, later characterized by the Hubble constant H_0 , was measured to be $500 \text{ km s}^{-1} \text{ Mpc}^{-1}$. The value of H_0 was then revised along the 20th century, with a controversy about the measurements made by Sandage [3] ($50 \text{ km s}^{-1} \text{ Mpc}^{-1}$) and de Vaucouleurs [4] ($100 \text{ km s}^{-1} \text{ Mpc}^{-1}$). Only in the early 2000s the HST project found a value of $(72 \pm 8) \text{ km/s/Mpc}$ [5].

After these works, the measurements of H_0 refined more and more, to reach the most recent direct estimate of the expansion rate of the Universe: $H_0 = (73.0 \pm 1.8) \text{ km s}^{-1} \text{ Mpc}^{-1}$ [6]. The most important modern techniques arise from the study of the perturbations of the Cosmic Microwave Background (CMB) [7], the Supernovae 1A (SN 1A) [6] and the effect of Gravitational Lensing (GL) [8, 13]. Nevertheless, the different measurements are not compatible one with each other, and still a slight tension about the correct value of H_0 does remain [14].

The discrepancy could be due to a statistical fluctuations or could be evidence of new physics. In order to solve this puzzle, different theoretical scenarios have been proposed. For instance, it has been showed on general grounds that dynamical dark energy [15, 16] or a specific quintessence model [17] can solve the tension. Other possibilities have also been considered, such as dark matter-neutrinos interactions [18, 19].

Our contribution towards an attempt to solve the disagreement [20] concerns the theoretical analysis of the time delay Δt and its connection to H_0 . In the standard analysis [21], the time delay is calculated adding two contributions: the Shapiro delay from the gravitational potential of the lens and the geometric delay due to the deformations of the ray paths, approximated by straight lines [21]. The formula we are considering in this work derives from a different approach which allows to compute the time delay directly in a single shot, rather than two [20].

The two formulas for time delay, the standard one [21] and the one we are proposing [20], at first sight look quite different. In particular, our proposal seems to depend on more degrees of freedom. In this paper we show that, imposing a reasonable condition of smoothness on the paths of the photons, the parameters of our formula collapse on those characterizing the standard one, and, most remarkably, we prove that the standard formula is the limit at small angles of the one we are proposing, which therefore turns out to be more general.

Furthermore, a weak point of the measurement of H_0 through GL is the choice of the gravitational potential Φ generated by the lens, which is a crucial quantity in GL, and which, in general is not a known quantity. The determination of the potential Φ is therefore an important issue [11], which motivated us to study a possible check of consistency for the supposed gravitational potential Φ .

The paper is organized as follows. In Section 2 we briefly summarize the model underlying our formula for time delay. In Section 3 we impose a smoothness condition to the paths followed by the photons and we derive the analytical expression of the

new parameters in terms of the ones already present in the standard formula. The main result achieved in this paper is contained in Section 4, where we show that our formula is the generalization of the standard formula to arbitrary angles. In Section 5 a consistency check is proposed in order to select a gravitational potential amongst different possibilities. In Section 6 we make the exercise of applying our formula to the simple case of central potential generated by the lens. In that case, we give a formula for the mass of the lens in the case of central potential. The mass of the lens, indeed, in general is unknown, being also possibly generated by dark matter. Phenomenological consequences concerning this point, as well as comparison with known results are also discussed in this Section. Our results are summarized in the concluding Section 7.

2 The model

In [20] we have obtained a new formula to determine the Hubble constant H_0 using time delay Δt between multiple images of lensed objects¹

$$\Delta t = [b_2\mu_2 - b_1\mu_1] + \frac{1}{H_0} [(\mathcal{R}(z_{P_2}) - \mathcal{R}(z_{Q_2})) - (\mathcal{R}(z_{P_1}) - \mathcal{R}(z_{Q_1}))] + \frac{1}{H_0} \sum_{k=1}^{+\infty} \left[\frac{\mathcal{R}(z_S)\mathcal{R}(z_{Q_2})}{\mathcal{R}(z_S) - \mathcal{R}(z_{Q_2})} \left(\frac{c_k\gamma_2^{2k}}{2} - \psi_2 \right) - \frac{\mathcal{R}(z_S)\mathcal{R}(z_{Q_1})}{\mathcal{R}(z_S) - \mathcal{R}(z_{Q_1})} \left(\frac{c_k\gamma_1^{2k}}{2} - \psi_1 \right) \right]. \quad (2.1)$$

where b_i , P_i , Q_i , γ_i , μ_i are parameters defined in Figure 1, z_X is the redshift of the generic point X and

$$\mathcal{R}(z_X) \equiv \int_0^{z_X} \frac{dz'}{[\sum_i \Omega_{i0}(1+z)^{n_i}]^{1/2}}, \quad (2.2)$$

where Ω_{0i} are the four parameters corresponding to radiation, matter, curvature and vacuum and $n_i = 4, 3, 2, 0$ respectively. The GL potentials $\psi_i(\vec{\theta})$ ($i = 1, 2$) are defined as follows

$$\psi_i(\vec{\theta}) \equiv 2 \frac{d_A(LS)}{d_A(EL)d_A(ES)} \int \Phi(d_L\vec{\theta}, l), \quad (2.3)$$

where $d_A(XY)$ is the angular diameter distance of the point Y from the observer X . Φ is the gravitational potential generated by the lens and the integral is done over past directed geodesic paths emanating from the observer. Finally, c_k are the coefficients of the Taylor series, which can be easily computed and can be found in [20].

The formula (2.1) was built in the framework of a new theoretical model for GL whose geometry is described in Figure 1. Our approach is the reasonings which led to the well known time delay formula [21]

$$\Delta t_{old} = \frac{1}{H_0} \frac{\mathcal{R}(z_S)\mathcal{R}(z_L)}{\mathcal{R}(z_S) - \mathcal{R}(z_L)} \left[\frac{(\alpha_2^2 - \alpha_1^2)}{2} - (\psi(\vec{\theta}_2) - \psi(\vec{\theta}_1)) \right]. \quad (2.4)$$

The assumptions on which our model is built are:

¹Throughout this paper we adopt the notations of [22].

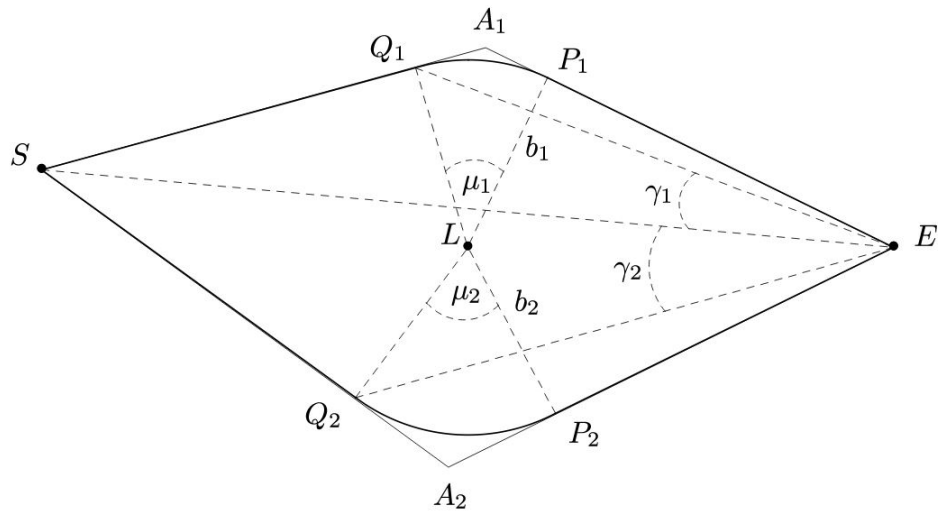


Figure 1: S = source. L = lens. E = observer. b_i = impact parameters. SQ_iP_iE = approximated paths of the photons from S to E , indexed by $i = 1, 2$.

1. The space is divided into two regions: the first, far from L , where $\Phi \approx 0$, and the second, close to L , where $\Phi \neq 0$.
2. We approximate the curve Q_iP_i by an arc of a circle centered in L . We stress that this is not a necessary condition: the arc of a circle is a good choice to represent Q_iP_i , but it is not the only one possible.
3. The universe is spatially flat, a fact compatible with observations [23].

According to assumption 1, we chose P_i and Q_i as the points which divide the photons trajectory SQ_iP_iE : from S to Q_i and from P_i to E photons are in the region where $\Phi = 0$, and so they run along straight lines (thanks to assumption 3); from Q_i to P_i , instead, photons are in the region where $\Phi \neq 0$, and so they moves on a curved trajectory, which we choose to describe with the arc of a circle Q_iP_i in Figure 1.

There are at least two different geodesics along which photons can move from S to E and this causes the time delay Δt [21].

These assumptions led us to the time delay formula (2.4), which relates Δt to H_0 .

3 Smoothness condition

In [20] we emphasized that the formula (2.1), as it stands, is not well suited for actual calculations of time delay, in the hope of softening the discrepancy among the existing estimates of H_0 . The reason is that the geometry is not uniquely defined because we have placed no constraints on Q_i , P_i , μ_i , b_i and γ_i which, consequently, are free parameters. It is easy to show that imposing a smoothness condition on of the paths SQ_iP_iE uniquely fixes the parameters appearing in the new formula (2.1)

where δ is defined as in Figure 2, from which it is clear that

$$\delta = \frac{\pi}{2} + \theta - \hat{\alpha}. \quad (3.7)$$

Intersecting the line (3.6) with the circle (3.2) we obtain

$$(x - x_L)^2 = x_L^2 \sin^2 \theta \cos^2 \delta, \quad (3.8)$$

from which

$$x_Q = (1 + \cos \delta \sin \theta) x_L \quad y_Q = \sin \delta \sin \theta x_L. \quad (3.9)$$

Now we are able to obtain the value of γ' as defined in Figure 2

$$\gamma' = \arctan \left(\frac{\sin \theta \sin \delta}{1 + \cos \delta \sin \theta} \right). \quad (3.10)$$

Now we observe that

$$\gamma' = \gamma + \alpha - \theta, \quad (3.11)$$

hence

$$\gamma = \arctan \left(\frac{\sin \theta \sin \delta}{1 + \cos \delta \sin \theta} \right) + \alpha - \theta, \quad (3.12)$$

and the angles α and $\hat{\alpha}$ are related by [22]

$$\alpha = \frac{d_A(LS)}{d_A(ES)} \hat{\alpha}. \quad (3.13)$$

We have now all the tools to write the formula (2.1) in terms of the parameters appearing in the usual one (2.4). Let us rewrite (2.1) as

$$\Delta t = \Delta\tau_1 + \Delta\tau_2 + \Delta\tau_3, \quad (3.14)$$

where

$$\Delta\tau_1 \equiv [b_2\mu_2 - b_1\mu_1], \quad (3.15)$$

$$\Delta\tau_2 \equiv \frac{1}{H_0} [(\mathcal{R}(z_{P_2}) - \mathcal{R}(z_{Q_2})) - (\mathcal{R}(z_{P_1}) - \mathcal{R}(z_{Q_1}))], \quad (3.16)$$

$$\Delta\tau_3 \equiv \frac{1}{H_0} \sum_{k=1}^{+\infty} \left[\frac{\mathcal{R}(z_S)\mathcal{R}(z_{Q_2})}{\mathcal{R}(z_S) - \mathcal{R}(z_{Q_2})} \left(\frac{c_k\gamma_2^{2k}}{2} - \psi_2 \right) - \frac{\mathcal{R}(z_S)\mathcal{R}(z_{Q_1})}{\mathcal{R}(z_S) - \mathcal{R}(z_{Q_1})} \left(\frac{c_k\gamma_1^{2k}}{2} - \psi_1 \right) \right]. \quad (3.17)$$

Using (??) and (3.3) we can write the first contribution to Δt as

$$\Delta\tau_1 = x_L(\sin \theta_2 \hat{\alpha}_2 - \sin \theta_1 \hat{\alpha}_1). \quad (3.18)$$

Let us now focus on the $\Delta\tau_2$ term. From [20]

$$\mathcal{R}(z_X) = H_0 r_X, \quad (3.19)$$

and (3.9), we have

$$\mathcal{R}(z_Q) = \mathcal{R}(z_L)\sqrt{1 + \sin^2 \theta - 2 \sin(\theta - \hat{\alpha}) \sin \theta} \quad (3.20)$$

$$\mathcal{R}(z_P) = H_0 r_P = H_0 \cos \theta x_L = \mathcal{R}(z_L) \cos \theta. \quad (3.21)$$

We are thus able to write $\Delta\tau_2$ in (3.14) as follows

$$\begin{aligned} \Delta\tau_2 = & \frac{1}{H_0} \mathcal{R}(z_L) (\cos \theta_2 - \sqrt{1 + \sin^2 \theta_2 - 2 \sin(\theta_2 - \hat{\alpha}_2) \sin \theta_2} \\ & - \cos \theta_1 + \sqrt{1 + \sin^2 \theta_1 - 2 \sin(\theta_1 - \hat{\alpha}_1) \sin \theta_1}). \end{aligned} \quad (3.22)$$

Finally, using (3.12) and (3.9), we have

$$\begin{aligned} \Delta\tau_3 = & \frac{1}{H_0} \sum_{k=1}^{+\infty} \left(\frac{\mathcal{R}(z_S)\mathcal{R}(z_L)\sqrt{1 + \sin^2 \theta_2 - 2 \sin(\theta_2 - \hat{\alpha}_2) \sin \theta_2}}{\mathcal{R}(z_S) + \mathcal{R}(z_L)\sqrt{1 + \sin^2 \theta_2 - 2 \sin(\theta_2 - \hat{\alpha}_2) \sin \theta_2}} \right. \\ & \left. \frac{c_k}{2} \left(\arctan \left(\frac{\sin \theta_2 \cos(\hat{\alpha}_2 - \theta_2)}{1 - \sin(\theta_2 - \hat{\alpha}_2) \sin \theta_2} \right) + \alpha_2 - \theta_2 \right)^2 + \right. \\ & \frac{\mathcal{R}(z_S)\mathcal{R}(z_L)\sqrt{1 + \sin^2 \theta_1 - 2 \sin(\theta_1 - \hat{\alpha}_1) \sin \theta_1}}{\mathcal{R}(z_S) + \mathcal{R}(z_L)\sqrt{1 + \sin^2 \theta_1 - 2 \sin(\theta_1 - \hat{\alpha}_1) \sin \theta_1}} \\ & \left. \frac{c_k}{2} \left(\arctan \left(\frac{\sin \theta_1 \cos(\hat{\alpha}_1 - \theta_1)}{1 - \sin(\theta_1 - \hat{\alpha}_1) \sin \theta_1} \right) + \alpha_1 - \theta_1 \right)^2 \right). \end{aligned} \quad (3.23)$$

The exact expression for the time delay (2.1), using (3.14), is then given by the sum of the three terms $\Delta\tau_i$ (3.18), (3.22) and (3.23). We stress that the two expression for time delay Δt (2.1) and Δt_{old} (2.4) are expressed by means of the same parameters: no new degrees of freedom have been introduced.

4 Small angles limit

In this Section we show that the standard formula for time delay (2.4) is recovered in the small angles limit of our formula (2.1). This is a remarkable result, because it proves that the formula (2.1) is the generalization of the standard formula to arbitrary angles. In order to get this result, the crucial step is that the formula Δt , written in the form (3.14), is expressed in terms of the same degrees of freedom appearing in the standard formula (2.4).

Let us consider the equation (3.12) and let us suppose that all the angles involved are very small. Up to the second non vanishing order, we get

$$\gamma \simeq \alpha - \frac{1}{2} \theta \hat{\alpha}^2. \quad (4.1)$$

We will now expand $\mathcal{R}(z_Q)$ and $\mathcal{R}(z_P)$, respectively given by (3.20) and (3.21). We obtain

$$\mathcal{R}(z_Q) = H_0 r_Q \simeq H_0 \left(1 + \hat{\alpha}\theta - \frac{\theta^2}{2}\right) x_L, \quad (4.2)$$

$$\mathcal{R}(z_P) = H_0 r_P \simeq H_0 \left(1 - \frac{\theta^2}{2}\right) x_L. \quad (4.3)$$

Now we are able to expand in the small angles limit the time delay formula (2.1) term by term, according to (3.14). For what concerns $\Delta\tau_1$ (3.18), we have

$$\Delta\tau_1 \simeq x_L(\hat{\alpha}_2\theta_2 - \hat{\alpha}_1\theta_1). \quad (4.4)$$

Let us consider the $\Delta\tau_2$ term given by (3.16). Using (4.2) and (4.3) we obtain

$$\Delta\tau_2 \simeq -x_L(\hat{\alpha}_2\theta_2 - \hat{\alpha}_1\theta_1). \quad (4.5)$$

Finally, using (4.1) and (4.2) we obtain

$$\frac{\mathcal{R}(z_S)\mathcal{R}(z_{Q_i})}{\mathcal{R}(z_S) + \mathcal{R}(z_{Q_i})} \gamma_i^2 \simeq \frac{\mathcal{R}(z_S)\mathcal{R}(z_L)}{\mathcal{R}(z_S) + \mathcal{R}(z_L)} \alpha_i^2. \quad (4.6)$$

Plugging (4.6) and (4.1) in (3.17) we obtain for the last contribution $\Delta\tau_3$

$$\Delta\tau_3 \simeq \frac{1}{H_0} \frac{\mathcal{R}(z_S)\mathcal{R}(z_L)}{\mathcal{R}(z_S) - \mathcal{R}(z_L)} \left[\frac{(\alpha_2^2 - \alpha_1^2)}{2} - \left(\psi(\vec{\theta}_2) - \psi(\vec{\theta}_1)\right) \right]. \quad (4.7)$$

Using (4.4), (4.5) and (4.7) in the equation (3.14) we obtain the following formula

$$\Delta t = \frac{1}{H_0} \frac{\mathcal{R}(z_S)\mathcal{R}(z_L)}{\mathcal{R}(z_S) - \mathcal{R}(z_L)} \left[\frac{(\alpha_2^2 - \alpha_1^2)}{2} - \left(\psi(\vec{\theta}_2) - \psi(\vec{\theta}_1)\right) \right] + O(\theta^3), \quad (4.8)$$

which is precisely the formula already present in literature (2.4) and currently used in the determination of H_0 through GL. We can therefore conclude that

$$\lim_{\alpha, \theta \rightarrow 0} \Delta t = \Delta t_{old} + O(\theta^3). \quad (4.9)$$

This remarkable fact justifies our claim: our time delay formula (3.14), which is equivalent to (2.1) once that the smoothness condition is imposed, is the generalization at arbitrary angles of the standard expression (2.4), and, differently from (2.4), it has been obtained in one shot only without the need of considering two distinct steps (geometrical and Shapiro).

5 A constraint on the lens gravitational potential

The time delay formula (2.4) or, equivalently, (2.1), allows to determine H_0 . The crucial physical quantity for the time delay is the gravitational potential Φ originated by the lens L . In most cases, Φ is not known but rather is assumed to belong to a certain class, or inferred by some theoretical considerations [11]. We will denote with Φ_{hp} the assumed potential, which, hopefully, should not be very far from the real one, which we shall call Φ_{phys} . Our aim is to provide a consistency relation for Φ_{hp} , by means of which it will be possible to check whether Φ_{hp} represents a realistic assumption for the real gravitational potential Φ_{phys} of the lens or not. From now on we will use the subscript “ hp ” for the quantities deduced from Φ_{hp} and the subscript “ $phys$ ” for their actual values. Once that the potential Φ_{hp} has been assumed, from [22]

$$\hat{\alpha} = 2 \int \nabla_{\perp} \Phi ds, \quad (5.1)$$

where $\nabla_{\perp} \Phi$ is the transverse gradient of the potential with respect to the path, we can compute the angle $\hat{\alpha}_{hp}$ which, consequently, not necessarily coincides with $\hat{\alpha}_{phys}$. From the experimental knowledge of θ_i and of z_L , and once that $(\hat{\alpha}_i)_{hp}$ are determined, we can predict the position of the source $(\vec{r}_S)_{hp}$. An obvious check for Φ_{hp} would be

$$(\vec{r}_S)_{hp} = (\vec{r}_S)_{phys}, \quad (5.2)$$

where $(\vec{r}_S)_{phys}$ identifies the actual position of the the source S . Unfortunately, the exact position of the source $(\vec{r}_S)_{phys}$ rarely is known. What in general is known, rather, is its redshift z_S , from which, using (2.2), the distance $|(\vec{r}_S)_{phys}| \equiv (r_S)_{phys}$ can be computed. Therefore, the check of consistency reduces to

$$(r_S)_{hp} = (r_S)_{phys}, \quad (5.3)$$

which is a necessary condition for Φ_{hp} . We can compute $(r_S)_{phys}$ explicitly as follows.

From (3.9) and (3.7) we have

$$x_{Q_i} = (1 + \sin((\hat{\alpha}_i)_{hp} - \theta_i) \sin \theta_i) x_L \quad y_{Q_i} = \sin \theta_i \cos((\hat{\alpha}_i)_{hp} - \theta_i) x_L, \quad (5.4)$$

and the equations of the straight lines passing through Q_i and S are

$$y - y_{Q_i} = -\tan((\hat{\alpha}_i)_{hp} - \theta_i)(x - x_{Q_i}), \quad (5.5)$$

where, again, $i = 1, 2$. From (5.5) we get the coordinates of the source S

$$(x_S)_{hp} = \frac{y_{Q_1} - y_{Q_2} + \tan((\hat{\alpha}_1)_{hp} - \theta_1)x_{Q_1} + \tan((\hat{\alpha}_2)_{hp} - \theta_2)x_{Q_2}}{\tan((\hat{\alpha}_1)_{hp} - \theta_1) + \tan((\hat{\alpha}_2)_{hp} - \theta_2)}, \quad (5.6)$$

$$(y_S)_{hp} = y_{Q_1} + \tan((\hat{\alpha}_1)_{hp} - \theta_1)x_L - \tan((\hat{\alpha}_1)_{hp} - \theta_1)x_S. \quad (5.7)$$

The limit at small angles $\hat{\alpha}_i$ and θ_i gives

$$(x_S)_{hp} = \frac{(\hat{\alpha}_1)_{hp} + (\hat{\alpha}_2)_{hp}}{(\hat{\alpha}_1)_{hp} + (\hat{\alpha}_2)_{hp} - \theta_1 - \theta_2} x_L, \quad (5.8)$$

$$(y_S)_{hp} = \frac{(\hat{\alpha}_2)_{hp}\theta_1 - (\hat{\alpha}_1)_{hp}\theta_2}{(\hat{\alpha}_1)_{hp} + (\hat{\alpha}_2)_{hp} - \theta_1 - \theta_2} x_L, \quad (5.9)$$

and hence the distance of the source $(r_S)_{hp}$ is

$$(r_S)_{hp} = \frac{\sqrt{(\hat{\alpha}_1)_{hp}^2(1 + \theta_2^2) + (\hat{\alpha}_2)_{hp}^2(1 + \theta_1^2) + 2(\hat{\alpha}_1)_{hp}(\hat{\alpha}_2)_{hp}(1 - \theta_1\theta_2)}}{(\hat{\alpha}_1)_{hp} + (\hat{\alpha}_2)_{hp} - \theta_1 - \theta_2} x_L. \quad (5.10)$$

This quantity depends on the assumed choice of the gravitational potential Φ_{hp} generated by the lens L through the angles $(\hat{\alpha}_i)_{hp}$. Therefore, once that the expression (5.10) for $(r_S)_{hp}$ is given, the validity of the constraint (5.3) can be checked.

6 Determination of the lens mass for central potential

In order to give a simple example of how our formula for time delay can be used for phenomenological calculations, let us consider the textbook case of spherically symmetric potential Φ , being aware that this is not a realistic assumption for actual gravitational potentials generated by the lens. It is a known result [22] that in this case the angles are given by

$$\hat{\alpha}_i = \frac{4GM}{d_{EL}\theta_i}, \quad (6.1)$$

where M is the mass of the lens L and d_{EL} is the angular diameter distance between L and E . Inserting (6.1) in (5.10) we obtain

$$r_S = \frac{4GM\sqrt{(\theta_1 + \theta_2)^2 + (\theta_1^2 - \theta_2^2)^2}}{(4GM - d_{EL}\theta_1\theta_2)(\theta_1 + \theta_2)} x_L, \quad (6.2)$$

and hence

$$M = \frac{\theta_1\theta_2 d_{EL} r_S}{4G \left(r_{ES} - \left(1 + \frac{(\theta_1 - \theta_2)^2}{2} \right) x_L \right)}. \quad (6.3)$$

Using the relation between the angular diameter distance d and the proper distance r [23]

$$d_{EL} = a(t_L) r_{EL}, \quad (6.4)$$

we can conclude that

$$M = \frac{\theta_1 \theta_2 d_{EL} d_{ES}}{4G \left(d_{ES} - \frac{1+z_L}{1+z_S} \left(1 + \frac{(\theta_1 - \theta_2)^2}{2} \right) d_{EL} \right)}. \quad (6.5)$$

This is a general result, valid for all spherically symmetric lenses. The lesson from this simple example, is that in principle it is possible to estimate the mass of a lens with central gravitational potential, like a star or a black hole, for example, once the angular diameter distances d_{ES} , d_{EL} , the observed angle $\hat{\theta}_i$ and the redshifts z_L , z_S are known. It is interesting to compare the results coming from the mass formula (6.5) with those obtained using the corresponding formula existing in literature.

If a circularly symmetric lens is considered, and if the lens, source and observer are collinear, as a consequence of the the rotational symmetry of the lens system the source is imaged as a ring. The radius of the ring, called Einstein radius, is given by

$$\theta_E = [4GM(\theta_E) \frac{d_{LS}}{d_{EL}d_{ES}}]^{1/2} \quad (6.6)$$

where d_{EL} , d_{ES} and d_{LS} are the angular diameter distances. From (6.6), it is possible to estimate the mass inside the Einstein ring, which is given by

$$M(\theta_E) = \frac{1}{4G} \frac{d_{EL}d_{ES}}{d_{LS}} \theta_E^2 \quad (6.7)$$

This is the standard formula provided in literature (see for example [24]). If the source and the lens are misaligned, multiple images can be observed. In this case it is possible to measure the relative distances between each image and the lens, called θ_1 and θ_2 for a two-images lens system, but it is also possible to extract an “effective” Einstein radius by performing a fit assuming a particular mass distribution of the lens. It has been shown that the assumed mass distribution model only partly affects the Einstein ring extraction [25, 26], and since the Einstein angle is an average between the lens center and multiple images, in the case of a two images system the θ_E angle is usually replaced with $\theta_E = \frac{(\theta_1 + \theta_2)}{2}$.

This is usually a good approximation for systems with nearly symmetric image configuration ($\theta_1/\theta_2 \lesssim 2$) where the mass enclosed within θ_E is accurate to within $\sim 5\%$ as reported in [26]. The equation (6.5) is indeed a generalisation of the standard formula and reduces to the standard formula if $\theta_1 = \theta_2$.

In order to test the validity of the lens mass estimation, a comparison between the standard formula (6.6) and the formula for the mass of the lens obtained in this paper (6.5), which uses explicitly θ_1 and θ_2 , is provided. To have a fair comparison and to verify the applicability of (6.5), the formula is applied to a subset of quasar lenses listed in the CASTLES webpage [27]

satisfying the following requirements: first of all, the source and the lens redshifts have to be known. Complex lens systems such as multiple galaxies or cluster of galaxies are excluded and only quasar lenses with a planar geometry with two lens images are chosen. Finally, the observed angles θ_1 and θ_2 must be known data. Five quasar lenses have been identified to fulfill the previous criteria. Our study shows that in the case, $\theta_1/\theta_2 \lesssim 2$, the standard formula is a good approximation of the formula (6.5) and as it can be seen in Table 2: for QJ0158-4325 the correction is negligible, while for SDSS1226-0006 the correction is of the order of 10%.

For $\theta_1/\theta_2 > 2$, the standard formula starts to show a significant discrepancy with respect to the formula obtained in this paper. The limit of the standard formula is overcome by equation (6.5) that can indeed be applied for any angle configuration.

An ideal outcome of our study would be the determination of the dark matter fraction in the lens galaxy in a more precise way. To do that, a more appropriate potential has of course to be used to take into account the density distribution of dark matter in a galaxy: this additional study represents a future development of our work.

The parameters for the five considered lenses are reported in Table 1. We

Lens	z_L	z_S	θ_1 [10^{-6} rad]	θ_2 [10^{-6} rad]	θ_E [10^{-6} rad]
QJ0158-4325	0.317	1.29	3.95 ± 0.07	1.99 ± 0.07	2.8
J1004+1229	0.95	2.65	6.156 ± 0.034	1.309 ± 0.037	4.02
HE1104-1805	0.73	2.32	10.216 ± 0.021	5.269 ± 0.015	6.8
SDSS1226-0006	0.52	1.12	3.992 ± 0.021	2.120 ± 0.021	2.76
HE2149-2745	0.5	2.03	6.563 ± 0.027	1.670 ± 0.031	4.1

Table 1

considered a flat Λ CDM universe, with $\Omega_M = 0.3$, $\Omega_\Lambda = 0.7$ and $H_0 = 70$ km/s/Mpc. The calculated masses using the parameters reported in Table 1 are reported in Table 2, where M_{our} and M_{std} are calculated using (6.5) and (6.7), respectively.

Lens	M_{std} [M_\odot]	M_{our} [M_\odot]	$100 \cdot (M_{\text{std}} - M_{\text{our}})/M_{\text{std}}$
QJ0158-4325	$5.80 \cdot 10^{10}$	$5.81 \cdot 10^{10}$	-0.17
J1004+1229	$2.97 \cdot 10^{11}$	$1.48 \cdot 10^{11}$	50.13
HE1104-1805	$6.81 \cdot 10^{11}$	$7.93 \cdot 10^{11}$	-16.41
SDSS1226-0006	$1.14 \cdot 10^{11}$	$1.26 \cdot 10^{11}$	-11.11
HE2149-2745	$1.75 \cdot 10^{11}$	$1.14 \cdot 10^{11}$	34.82

Table 2

7 Conclusions

In this paper we improved the analysis of the time delay contained in a previous work [20], where we obtained a new expression for Δt , characterized, however, by the presence of a number of apparently free parameters, whose presence renders that expression unsuitable for phenomenological considerations, although, as we showed, its validity is enforced by the fact that, for certain values of its parameters, it reduces to the standard one [21]. Both in the standard approach and in ours, the paths followed by the photons from the source to the observer, modified by the presence of the lens, were approximated either by straight lines (as in [21]) or by straight lines and an arc of a circle (as in [20]), not joining in a smooth way. Two are the main results presented in this paper. The first is that, imposing a smoothness condition on the photons rays, the number of free parameters drastically reduces, with the outcome that no new degrees of freedom are introduced with respect to the standard formula for time delay. The second outcome is represented by the fact that we were able to show that our formula exactly reduces to the standard one in the limit of small angles, which means that our expression for time delay generalizes the standard one to generic angles. In addition, we gave a consistency check for the gravitational potential generated by the lens. Although this physical quantity is crucial for the GL effect, and in particular for the determination of the Hubble constant H_0 by means of the time delay formula, its exact expression is rarely known, and all the considerations are made on the basis of conjectured potentials. For instance, the GL effect could be generated also by dark matter distributions, of which, at most, only hypothetical maps exist. A criterion for selecting amongst different guesses on the gravitational potential generating the GL effect is therefore useful. Finally, to give a taste of the possible applications of our formula, we applied our results to the case of a central gravitational potential, obtaining an expression for the mass of the lens, which, again, in general is unknown. Although different situations exist, for which the spherical symmetry is a good approximation, realistic gravitational potentials generated by the lenses are in general far more complicated. Our aim here is only to show which might be the phenomenological applications of our formula. The signal of the particular case of a spherically symmetric gravitational potential is the presence of only two images, which are aligned with the lens. We analyzed five situations where our formula (6.5) for the mass can be successfully applied, and we compared our results to those obtained with the mass formula traditionally used in case of central potentials (6.7). The promising outcome is that our formula reproduces the results obtained in the usual way, but, quite remarkably, also extends the range of application of the standard one to any angle configuration. This approach could give hints on the dark matter presence in high z galaxies: from their luminosity it is possible to estimate the value of their mass, which could be

compared to our theoretical prediction. This would be interesting in situations where it is not possible to use standard methods, such as the study of the galaxy rotations curve, due to the high distances involved.

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References

- [1] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1917**, 142 (1917).
- [2] E. Hubble, Proc. Nat. Acad. Sci. **15**, 168 (1929). doi:10.1073/pnas.15.3.168.
- [3] Sandage, A., Tammann, G. A. 1976, ApJ, 210, 7.
- [4] de Vaucouleurs, G., & Peters, W. L. 1981, ApJ, 248, 395.
- [5] W. L. Freedman *et al.* [HST Collaboration], Astrophys. J. **553**, 47 (2001) doi:10.1086/320638 [astro-ph/0012376].
- [6] A. G. Riess *et al.*, Astrophys. J. **826**, no. 1, 56 (2016) doi:10.3847/0004-637X/826/1/56 [arXiv:1604.01424 [astro-ph.CO]].
- [7] P. A. R. Ade *et al.* [Planck Collaboration], “Planck 2015 results. XIII. Cosmological parameters,” Astron. Astrophys. **594**, A13 (2016) doi:10.1051/0004-6361/201525830 [arXiv:1502.01589 [astro-ph.CO]].
- [8] S. H. Suyu *et al.*, Mon. Not. Roy. Astron. Soc. **468** (2017) no.3, 2590 doi:10.1093/mnras/stx483 [arXiv:1607.00017 [astro-ph.CO]].
- [9] D. Sluse *et al.*, Mon. Not. Roy. Astron. Soc. **470** (2017) 4838 doi:10.1093/mnras/stx1484 [arXiv:1607.00382 [astro-ph.CO]].
- [10] Cristian E. Rusu *et al.*, Mon. Not. Roy. Astron. Soc. **467** (2017) no.4, doi:10.1093/mnras/stx285 arXiv:1607.01047 [astro-ph.GA].
- [11] K. C. Wong *et al.*, Mon. Not. Roy. Astron. Soc. **465** (2017) no.4, 4895 doi:10.1093/mnras/stw3077 [arXiv:1607.01403 [astro-ph.CO]].
- [12] V. Bonvin *et al.*, Mon. Not. Roy. Astron. Soc. **465**, no. 4, 4914 (2017) doi:10.1093/mnras/stw3006 [arXiv:1607.01790 [astro-ph.CO]].
- [13] S. Refsdal, Mon. Not. Roy. Astron. Soc. **128**, 307 (1964).

- [14] V. V. Luković, R. D’Agostino and N. Vittorio, *Astron. Astrophys.* **595**, A109 (2016) doi:10.1051/0004-6361/201628217 [arXiv:1607.05677 [astro-ph.CO]].
- [15] E. Di Valentino, A. Melchiorri, E. V. Linder and J. Silk, *Phys. Rev. D* **96**, no. 2, 023523 (2017) doi:10.1103/PhysRevD.96.023523 [arXiv:1704.00762 [astro-ph.CO]].
- [16] J. Solà, A. Gómez-Valent and J. de Cruz Pérez, *Phys. Lett. B* **774**, 317 (2017) doi:10.1016/j.physletb.2017.09.073 [arXiv:1705.06723 [astro-ph.CO]].
- [17] E. Di Valentino, E. V. Linder and A. Melchiorri, *Phys. Rev. D* **97**, no. 4, 043528 (2018) doi:10.1103/PhysRevD.97.043528 [arXiv:1710.02153 [astro-ph.CO]].
- [18] R. J. Wilkinson, C. Boehm and J. Lesgourgues, *JCAP* **1405**, 011 (2014) doi:10.1088/1475-7516/2014/05/011 [arXiv:1401.7597 [astro-ph.CO]].
- [19] E. Di Valentino, C. Bøehm, E. Hivon and F. R. Bouchet, *Phys. Rev. D* **97**, no. 4, 043513 (2018) doi:10.1103/PhysRevD.97.043513 [arXiv:1710.02559 [astro-ph.CO]].
- [20] N. Alchera, M. Bonici and N. Maggiore, *Symmetry* **9**, no. 10, 202 (2017) doi:10.3390/sym9100202 [arXiv:1709.09055 [astro-ph.CO]].
- [21] Schneider, P., Ehlers, J., Falco, E.E., 1992, Springer, *Gravitational Lenses*
- [22] S. M. Carroll, “Spacetime and geometry: An introduction to general relativity,” San Francisco, USA: Addison-Wesley (2004) 513 p.
- [23] S. Weinberg, “Cosmology,” Oxford, UK: Oxford Univ. Pr. (2008) 593 p.
- [24] Robert Schmidt, Rachel L. Webster, Geraint F. Lewis; Weighing a galaxy bar in the lens Q2237 + 0305, *Monthly Notices of the Royal Astronomical Society*, Volume 295, Issue 2, 1 April 1998, Pages 488–496, <https://doi.org/10.1046/j.1365-8711.1998.01326.x>
- [25] M. Oguri, C. E. Rusu and E. E. Falco, *Mon. Not. Roy. Astron. Soc.* **439** (2014) 2494 doi:10.1093/mnras/stu106 [arXiv:1309.5408 [astro-ph.CO]].
- [26] S. H. Suyu, *Mon. Not. Roy. Astron. Soc.* **426** (2012) 868 doi:10.1111/j.1365-2966.2012.21661.x [arXiv:1202.0287 [astro-ph.CO]].

[27] CASTLES catalogue <https://www.cfa.harvard.edu/castles>