

Article

$(T; N)$ and Residual Fuzzy Co-Implication in Dual Heyting Algebra with Applications

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Abstract: Recently, many authors have been interested to introduce fuzzy implications over t -norms and t -conorms. In this paper, we introduce (S, N) and residuum fuzzy implication for Dubois t -norm and Hamacher's t -norm. Also, new concepts so-called (T, N) and residual fuzzy co-implication in dual Heyting Algebra are investigated. Some examples as well as application are discussed as well.

Keywords: Fuzzy implications; (S, N) implication; residuum t -norm; (T, N) co-implication; residual co-implication

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1 Introduction

In fuzzy logic, the basic theory of connective AND (\wedge), OR (\vee), NOT (\neg) are often modeled as (strong negations, t -norm, and t -conorms). An important notion in fuzzy set theory is that of t -norm (T), t -conorms (S) and strong negations (N_c) that are used to define a generalized intersection, union and negation of fuzzy sets (see [5] and [6]). Implication and co-implication functions play an important notion in fuzzy logic, approximate reasoning, fuzzy control, intuitionistic fuzzy logic and approximate reasoning of expert system (see ([1], [2], [3], [4], [7], [8], and [9])).

The notion of t -norm and t -conorm turned out to be basic tools for probabilistic metric spaces (see [10] and [11]), but also in several other parts and have found diverse applications in the theory of fuzzy sets, fuzzy decision making, in models of certain many-valued logics or in multivariate statistical analysis. (see [12], [13], and [10]).

2 Preliminaries

The logic connectives like negation \neg is interpreted by a strong negation, conjunction \wedge by a triangular norm and disjunction \vee by triangular conorm. [14]

2.1 Triangular Norm and Triangular conorm

The conjunction \wedge in fuzzy logic, it is often modeled as follows:

Definition 2.1. [10]: A mapping T from $[0,1]^2$ into $[0,1]$ is a triangular norm (in short, t-norm), iff T are commutative, nondecreasing in both arguments, associative and which satisfies $T(p,1) = p, \forall p \in [0,1]$.

Also, disjunction \vee in fuzzy logic is often modeled as follows:

Definition 2.2. [10] A mapping S from $[0,1]^2$ into $[0,1]$ is a triangular conorm (in short, t-conorm), iff S are commutative, nondecreasing in both arguments, associative and which satisfies $S(p,0) = p, \forall p \in [0,1]$.

Proposition 2.1. [10] A mapping S is a triangular conorm iff there exists a triangular norm T such that $S(p,q) = 1 - T(1-p, 1-q), \forall p, q \in [0,1]$. In this case S is called the dual t-conorm of T .

The standard examples of t-norm and dual t-conorms are stated in the following:

t--norm (T)	Dual t-conorm (S)
$M(p,q) = \min(p,q),$ (Minimum t-norm)	$S_M(p,q) = \max(p,q)$ (Maximum t-conorm)
$\Pi(p,q) = pq,$ (Probabilistic Product t-norm)	$S_{\Pi}(p,q) = p + q - pq,$ (Probabilistic sum t-conorm)
$W(p,q) = \begin{cases} p & \text{if } q = 1 \\ q & \text{if } p = 1 \\ 0 & \text{if } p, q \in [0,1). \end{cases}$ (Drastic or weak t-norm)	$S_W(p,q) = \begin{cases} p & \text{if } q = 1, \\ q & \text{if } p = 1, \\ 1 & \text{otherwise.} \end{cases}$ (Drastic or largest t-conorm)
$N(p,q) = \begin{cases} \min(p,q) & \text{if } p + q \geq 1 \\ 0 & \text{if } p + q < 1. \end{cases}$ (Nilpotent t-norm)	$S_N(p,q) = \begin{cases} \max(p,q) & \text{if } p + q < 1, \\ 0 & \text{if } p + q \geq 1. \end{cases}$ (Nilpotent t-conorm)
$L(p,q) = \max(p + q - 1, 0),$ (Lukasiewicz t-norm)	$S_L(p,q) = \min(p + q, 1),$ (Bounded Sum t-conorm)

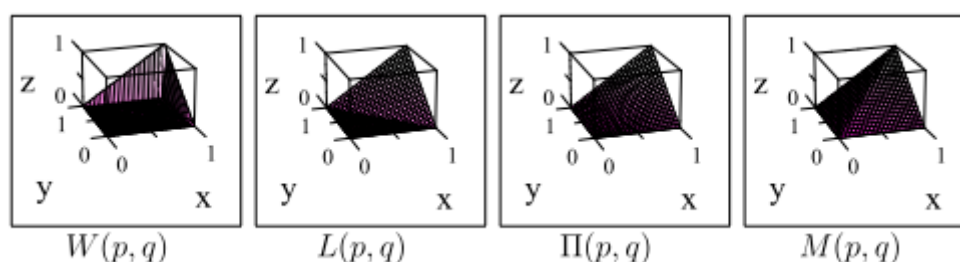
$H(p, q) = \begin{cases} 0 & \text{if } p = q = 0 \\ \frac{pq}{p + q - pq} & \text{otherwise.} \end{cases}$ <p>(Hamacher t-norm)</p>	$S_H(p, q) = \begin{cases} 0 & \text{if } p = q = 0, \\ \frac{p + q - 2pq}{1 - pq} & \text{otherwise.} \end{cases}$ <p>(Hamacher t-conorm)</p>
$D_\alpha(p, q) = \frac{pq}{\max(p, q, \alpha)}, \quad \alpha \in (0, 1),$ <p>(Dubois-Prade t-norm)</p>	$S_{D_\alpha}(p, q) = 1 - \frac{(1-p)(1-q)}{\max(1-p, 1-q, \alpha)},$ <p>$\alpha \in (0, 1)$. (Dubois-Prade t-conorm)</p>

For other family of t-norm (not needed here) we refer the reader to [12] for instance.

If $T_1 \leq T_2$ and there is at least one pair $(p, q) \in [0, 1]^2$ such that $T_1(p, q) < T_2(p, q)$

then we briefly write $T_1 < T_2$. With this, the above t-norms satisfy the next known chain of inequalities

$$W < L < \Pi < H < M.$$



Two t-norm T_1 and T_2 are called comparable if $T_1 \leq T_2$ or $T_2 \leq T_1$ holds. The above chain of inequalities shows that W, L, Π, H and M are comparable. It is not hard to see that, for example, Π and N are not comparable, while W, N and M comparable with

$$W < N < M.$$

2.3 Negation Function

The truth table of the classical negation N is given in following table.

p	Np
0	1
1	0

Definition 2.3. [15] A mapping N from $[0, 1]$ into $[0, 1]$ is a negation function, iff:

1. $N(0) = 1, N(1) = 0$;
2. $N(p) \leq N(q)$, if $p \geq q, \forall p, q \in [0, 1]$. (Monotonicity)

A negation function is strict, iff:

1. $N(p)$ is continuous;
2. $N(p) < N(q)$, if $p > q, \forall p, q \in [0, 1]$.

A strict negation function is strong or involutive, iff:

1. $N(N(p)) = p, \forall p \in [0, 1]$.

A negation function is weak, iff N is not strong.

Example 2.1. [15] The strong negation ($N_C(p) = 1 - p$), strict negation but not strong

($N_k(p) = 1 - p^2$), weaker negation ($N_{D_1}(p) = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}$) and strongest negation

($N_{D_2}(p) = \begin{cases} 1 & \text{if } p < 1, \\ 0 & \text{if } p = 1. \end{cases}$).

Definition 2.4. [13] Let T be a t-norm and S be a t-conorm. A mapping N_T, N_S from $[0, 1]$ into $[0, 1]$ defined by

$$N_T(p) = \sup\{r \in [0, 1] \mid T(p, r) = 0\}, \text{ for every } p \in [0, 1],$$

$$N_S(p) = \inf\{r \in [0, 1] \mid S(p, r) = 1\}, \text{ for every } p \in [0, 1],$$

are called the natural negation of T and S , respectively.

3 (T, N) Co-Implication

This section will be devoted to introduce the concept of (T, N) co-implication. The relation between classical logic and fuzzy logic as well as some examples are also discussed.

Definition 3.1. [15] A mapping I from $[0, 1]^2$ into $[0, 1]$ is fuzzy implication if, $\forall p, q, r \in [0, 1]$, the following conditions are satisfied:

$$I1: I(1,1) = I(0,1) = I(0,0) = 1 \text{ and } I(1,0) = 0.$$

$$I2: I(p,q) \geq I(r,q) \text{ if } p \leq r.$$

$$I3: I(p,q) \leq I(p,r) \text{ if } q \leq r.$$

The set of all fuzzy implications is denoted by FI .

In classical logic, the main two ways to defining an implication \Rightarrow in Boolean lattice (L, \wedge, \vee, \neg) are $p \Rightarrow q \equiv \neg p \vee q$ and $p \Rightarrow q \equiv \max\{r / p \wedge r \leq q\}$. [18]

The (S, N) implication and residual implication is generalization of these material implications to fuzzy logic.

Definition 3.2. [18] A mapping I from $[0,1]^2$ into $[0,1]$ is called an (S, N) implication if there exist a fuzzy negation N and a t-conorm S such that

$$I_{S,N}(p,q) = S(N(p), q), \forall p, q \in [0,1].$$

Definition 3.3. [18] Let T a left-continuous t-norm. Then, the residual implication or R -implication derived form T is given by

$$I_T(p,q) = \sup\{r \in [0,1] / T(r,p) \leq q\}, \forall p, q \in [0,1].$$

(R)

i.e. $T(r,p) \leq q \Leftrightarrow r \leq I_T(p,q), \forall p, q, r \in [0,1]$.

Remark 3.1. [18] It easy to check that for every left-continuous t-norm T , the only operation $I_T(p,q)$ satisfies (R) is called

$$I_T(p,q) = \max\{r \in [0,1] / T(r,p) \leq q\},$$

where the right side exists $\forall p, q \in [0,1]$.

Definition 3.4. [19] A mapping J from $[0,1]^2$ into $[0,1]$ is a fuzzy co-implication if, $\forall p, q, r \in [0,1]$, the following conditions are satisfied:

$$J1: J(1,1) = J(1,0) = J(0,0) = 0 \text{ and } J(0,1) = 1.$$

$$J2: J(p,q) \geq J(r,q) \text{ if } p \leq r.$$

$$J3: J(p,q) \leq J(p,r) \text{ if } q \leq r.$$

The set of all fuzzy co-implication is denoted by *Co-Fl*.

From the definition 3.4. $J(1,q) = J(p,0) = 0$ and $J(p,p) = 0, \forall p,q \in [0,1]$.

Lemma 3.1. If a mapping J from $[0,1]^2$ into $[0,1]$ satisfies (J1) and (J2), then the mapping $N_J : [0,1] \rightarrow [0,1]$ defined by

$$N_J(p) = J(p,1), \quad p \in [0,1],$$

is a fuzzy negation.

The following properties are generalization of fuzzy implication and fuzzy co-implication from classical logic.

Definition 3.5. [18] A fuzzy implications I and fuzzy co-implications J is said to satisfy the following most important properties, $\forall p,q,r \in [0,1]$.

$I(1,q) = q;$	(NP)	$J(0,q) = q;$	(Co-NP)
$I(p, I(q,r)) = I(q, I(p,r));$	(EP)	$J(p, J(q,r)) = J(q, J(p,r));$	(Co-EP)
$I(p,p) = 1;$	(IP)	$J(p,p) = 0;$	(Co-IP)
$I(p,q) = 1 \Leftrightarrow p \leq q;$	(OP)	$J(p,q) = 0 \Leftrightarrow p \geq q.$	(Co-OP)

Co-implication are extensions of the Boolean co-implication $\not\Leftarrow$ ($p \not\Leftarrow q$ meaning that p is not necessary for q). (see [20])

Proposition 3.1. The operator $\not\Leftarrow$ (material co-implication) is generated by Boolean negation \neg and conjunction \wedge :

$$q \not\Leftarrow p \equiv q \wedge \neg p.$$

The (T, N) co-implication is generalization of this material co-implication to fuzzy logic. In the following table we can see the truth table for the classical co-implication

p	q	$p \Rightarrow q$	$q \not\Leftarrow p$	$q \wedge \neg p$
0	0	1	0	0
0	1	1	1	1
1	0	0	0	0

1	1	1	0	0
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Definition 3.6. A mapping J from $[0,1]^2$ into $[0,1]$ is called an (T, N) co-implication if there exists a t-norm T and a fuzzy negation N such that

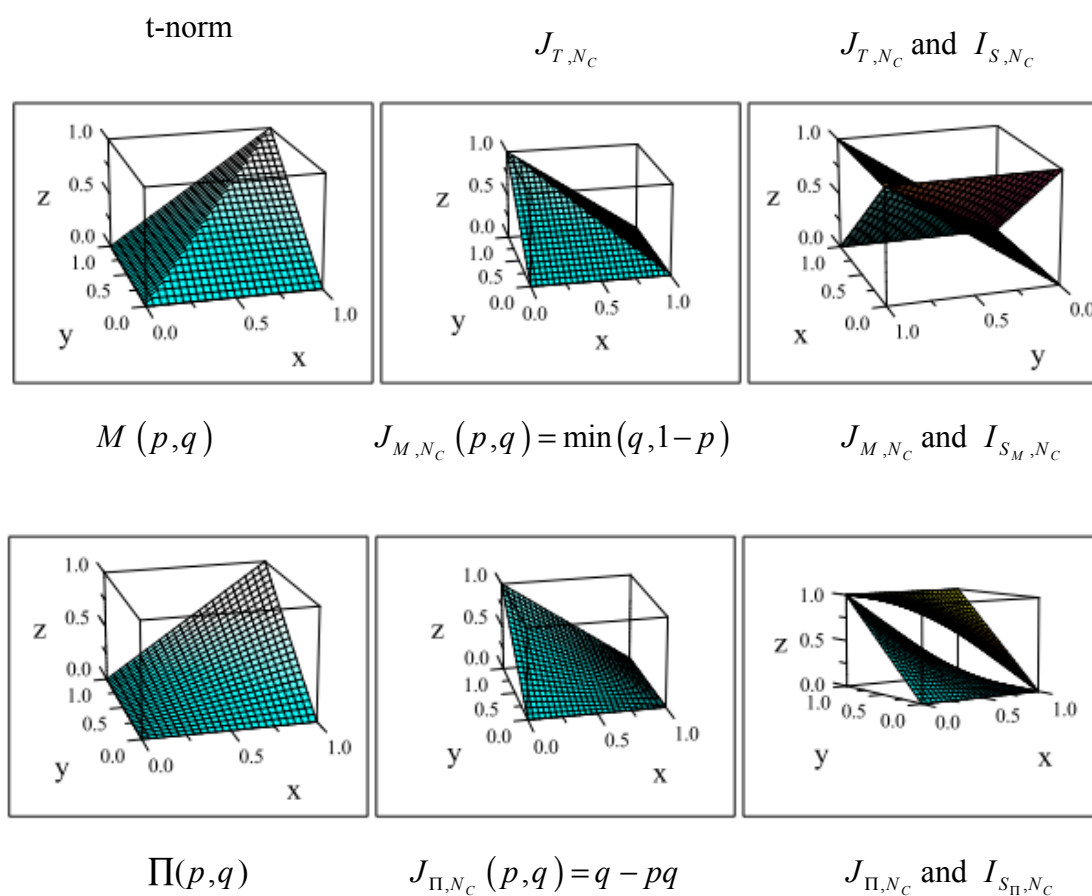
$$J_{T,N}(p,q) = T(q, N(p)), \quad \forall p, q \in [0,1].$$

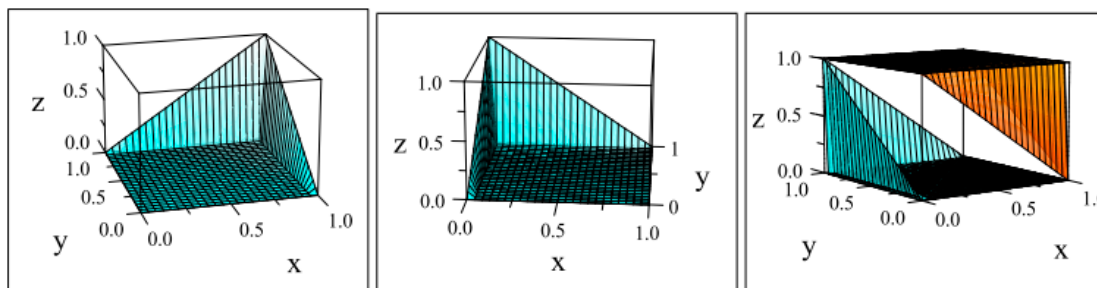
A relation between fuzzy negations and (S, N) implication is given in the next proposition.

Proposition 3.2 Let $J_{T,N}$ be an (T, N) implication, then $N_{J_{T,N}} = N$.

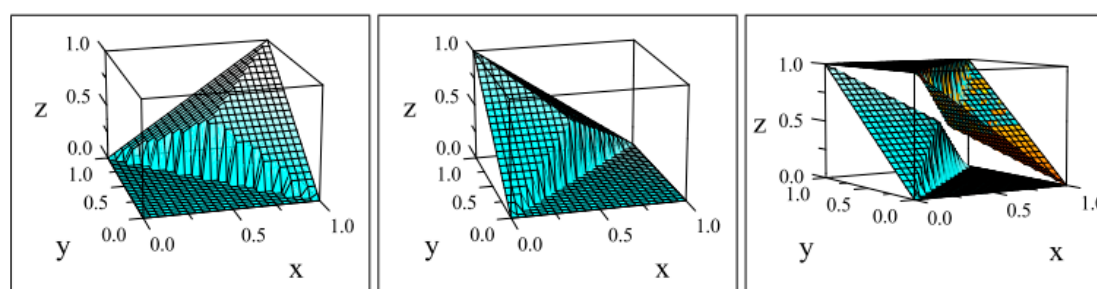
Proof. For any $p \in [0,1]$, $J_{T,N}(p) = J_{T,N}(p,1) = T(1, N(p)) = N(p)$. ■

Example 3.1. In the following examples we assume that N_C is a strong negation.

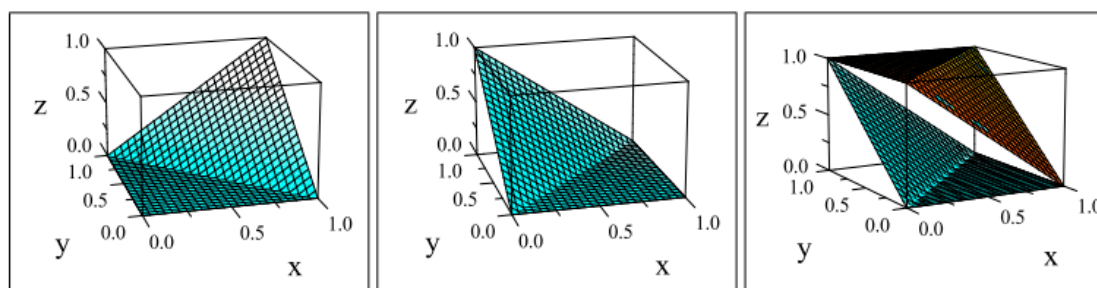




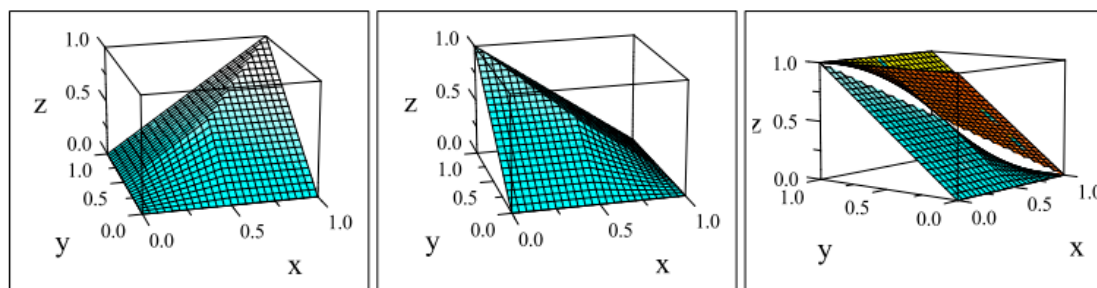
$$W(p,q) \quad J_{W,N_C}(p,q) = \begin{cases} q & \text{if } q = 1, \\ 1-p & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases} \quad J_{W,N_C} \text{ and } I_{S_W,N_C}$$



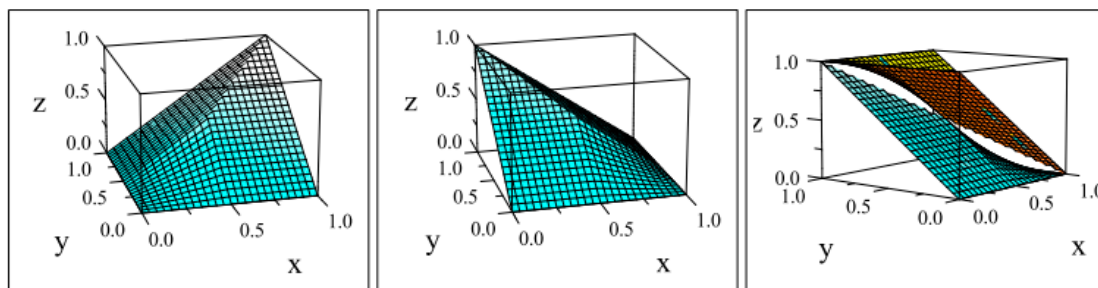
$$N(p,q) \quad J_{N,N_C}(p,q) = \begin{cases} \min(q, 1-p) & \text{if } p < q, \\ 0 & \text{if } p \geq q. \end{cases} \quad J_{N,N_C} \text{ and } I_{S_N,N_C}$$



$$L(p,q) \quad J_{L,N_C}(p,q) = \max(q - p, 0) \quad J_{L,N_C} \text{ and } I_{S_L,N_C}$$



$$H(p,q) \quad J_{H,N_C}(p,q) = \frac{q(1-p)}{1-p+qp} \quad J_{H,N_C} \text{ and } I_{S_H,N_C}$$

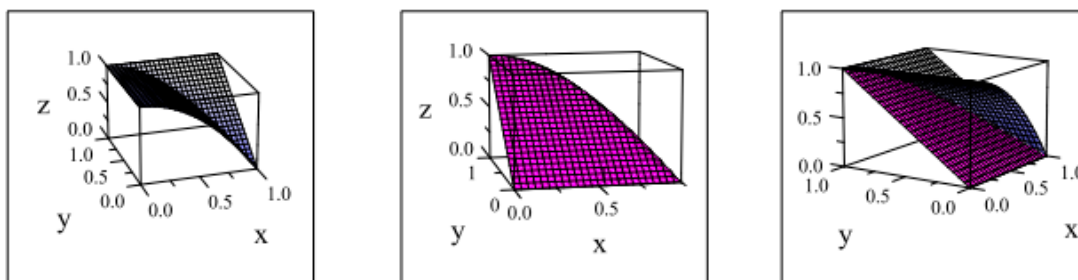


$$D_\alpha(p, q), \alpha \in (0, 1) \quad J_{D_\alpha, N_C}(p, q) = \frac{q(1-p)}{\max(q, 1-p, 0.5)} \quad J_{D_\alpha, N_C} \text{ and } I_{S_{D_\alpha, N_C}}$$

Example 3.2. For t-norm T and t-conorm S

1) A fuzzy negation $N_2(p) = 1 - p^2$, then, the basic (S, N_2) implications and

$I_{S_M, N_2}(T, N_2)$ co-implications $J_{M, N}$ are:

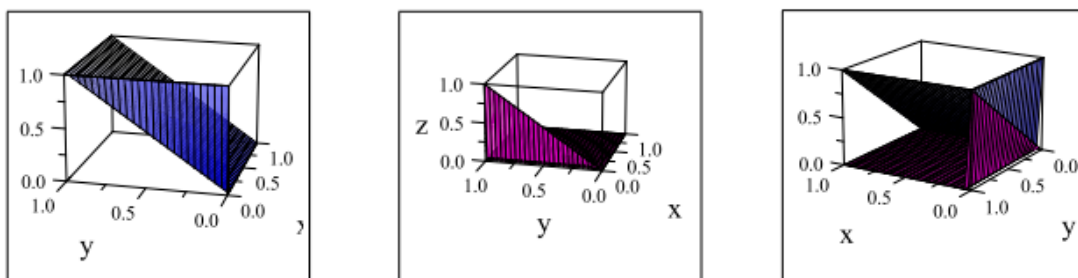


$$I_{S_M, N_2}(p, q) = \max(1 - p^2, q) \quad I_{M, N_2}(p, q) = \min(q, 1 - p^2) \quad I_{S_M, N_2} \text{ and } I_{M, N_2}$$

2) A fuzzy negation $N_3(p) = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$

Then, the basic (S, N_3) implications and $I_{S_M, N_3}(T, N_3)$ co-implications

J_{M, N_3} are:

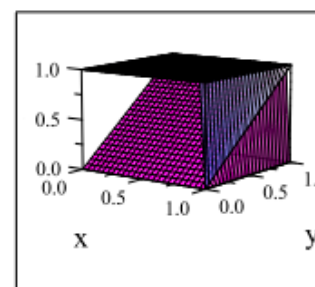
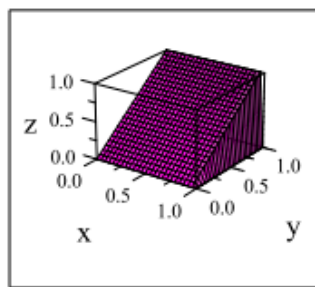
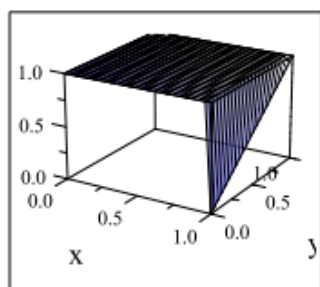


$$I_{S_M, N_3}(p, q) = \begin{cases} 1 & \text{if } p \leq 0, \\ q & \text{if } p > 0. \end{cases} \quad J_{M, N_3}(p, q) = \begin{cases} q & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases} \quad I_{S_M, N_3} \text{ and } J_{M, N_3}$$

As noted I_{S_M, N_3} is the least (S, N_3) implications and J_{M, N_3} is the least (T, N_3) co-implications.

3) A fuzzy negation $N_4(p) = \begin{cases} 1 & \text{if } p < 1, \\ 0 & \text{if } p = 1. \end{cases}$ then, the basic (S, N) implications

$I_{S_M, N}(T, N)$ co-implications $J_{M, N}$.



$$I_{S_M, N_4}(p, q) = \begin{cases} 1 & \text{if } p < 1, \\ q & \text{if } p = 0. \end{cases}$$

$$J_{M, N_4}(p, q) = \begin{cases} q & \text{if } p < 1, \\ 0 & \text{if } p = 1. \end{cases}$$

I_{S_M, N_4} and J_{M, N_4}

As noted I_{S_M, N_4} is the greatest (S, N_4) implications and J_{M, N_4} greatest (T, N_4) co-implications.

4. Residual Fuzzy Co-Implication in Dual Heyting

Heyting algebra logic is the system on Heyting algebras and Brouwerian algebras. Heyting algebra $\langle L, \wedge, \vee, \rightarrow, 0, 1 \rangle$ is lattice with the bottom 0, the top 1, and the binary operation called implication \rightarrow such that, $\forall p, q, r \in L, p \rightarrow q$ is the relative pseudocomplement of a with respect to c. That is to say

$$p \wedge r \leq q \leftrightarrow p \rightarrow q, \quad \forall p, q, r \in L.$$

In other words, the set of all $b \in L$ such that $p \wedge r \leq q$ contains the greatest element, denoted by $p \rightarrow q$. Precisely

$$p \rightarrow q = \sup \{r \in L \mid p \wedge r \leq q\}.$$

The dual of Heyting algebra is called Brouwerian algebra $(L, \wedge, \vee, \overset{*}{\longrightarrow}, 0, 1)$ is a lattice with 0 and 1, and the binary operation called co-implication $\overset{*}{\longrightarrow}$ in dual Heyting algebra. Satisfying $\forall p, q, r \in L$.

$$p \wedge r \geq q \Leftrightarrow p \overset{*}{\longrightarrow} q.$$

The set of all $r \in L$ such that $p \vee r \geq q$ contains the smallest element, denoted by $p \overset{*}{\longrightarrow} q$. Precisely

$$p \overset{*}{\longrightarrow} q = \inf \{r \in L \mid p \vee r \geq q\}.$$

Definition 4.1. Let S is the t-conorm of right continuous T . Then, the residual co-implication (R^* -implication) derived from S , is

$$J_S(p, q) = \inf \{r \in [0, 1] \mid S(r, p) \geq q\}, \forall p, q \in [0, 1]. \tag{R^*}$$

R^* -implication come from residuated lattices based on residuation property (R^*P) that can be written as

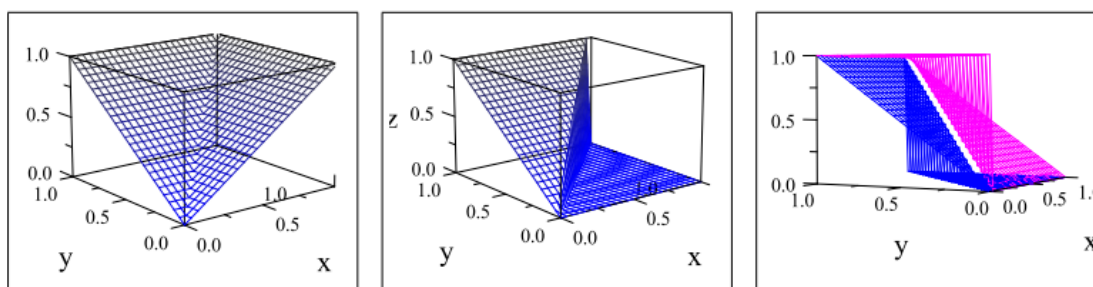
$$S(r, p) \geq q \text{ if and only if } r \geq J_S(p, q). \tag{R^*P}$$

The operation $J_S(x, y)$ is called residual co-implication of the t-conorm S .

We now list the residual co-implication associated to the standard left-continuous t-norms previously introduced.

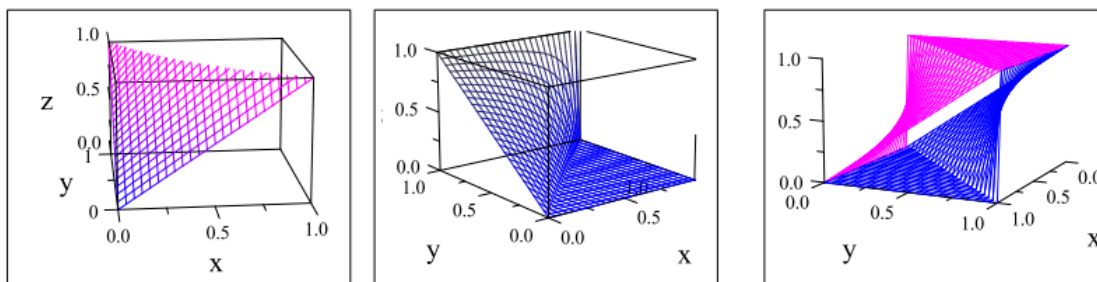
Applying the above concepts to the standard t-norms we obtain the following interesting results.

(1) Residuum of the Maximum t-conorm $S_M(p, q)$, is



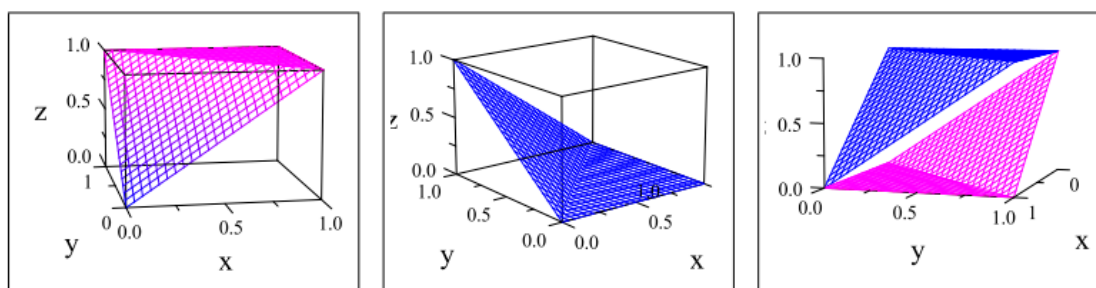
$$S_M(p, q) \qquad J_{S_M}(p, q) = \begin{cases} 0 & \text{if } p \geq q, \\ y & \text{otherwise.} \end{cases} \qquad I_M \text{ and } J_{S_M}$$

(2) Residuum of the Probabilistic sum t-conorm $S_{\Pi}(p, q)$ is



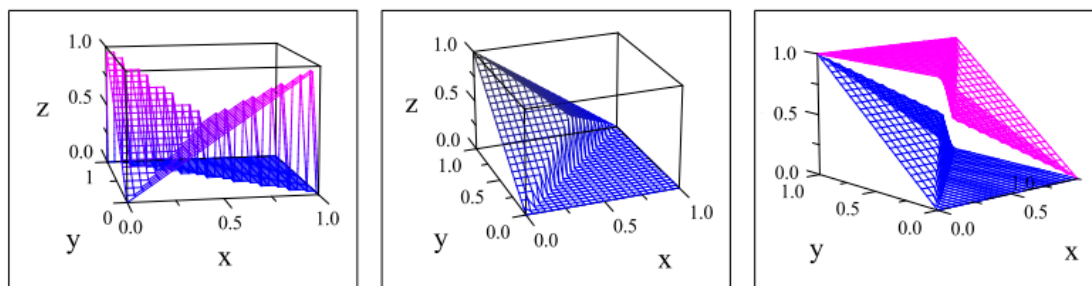
$$S_{\Pi}(p, q) \quad J_{S_{\Pi}}(p, q) = \begin{cases} 0 & \text{if } p \geq q, \\ q - p & \text{otherwise.} \end{cases} \quad I_{\Pi} \text{ and } J_{S_{\Pi}}$$

(3) Residuum of the Bounded Sum t-conorm $S_L(p, q)$ is



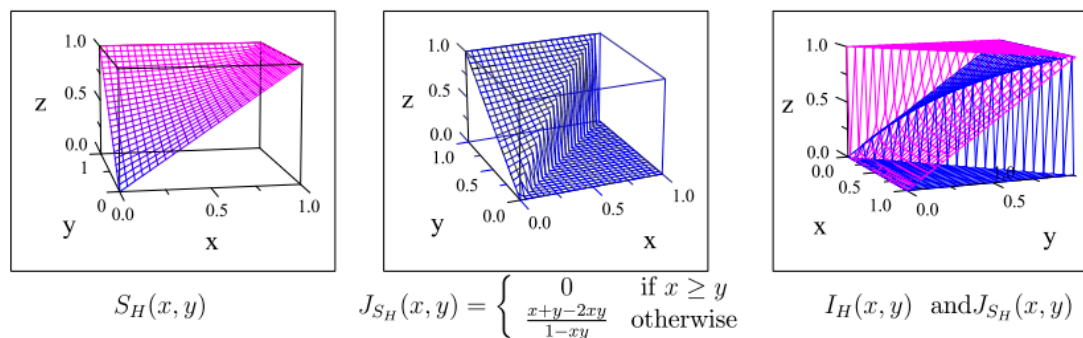
$$S_L(p, q) \quad J_{S_L}(p, q) = \max(0, q - p) \quad S_L \text{ and } J_{S_L}$$

(4) Residuum of the Nilpotent t-conorm $S_N(p, q)$ is

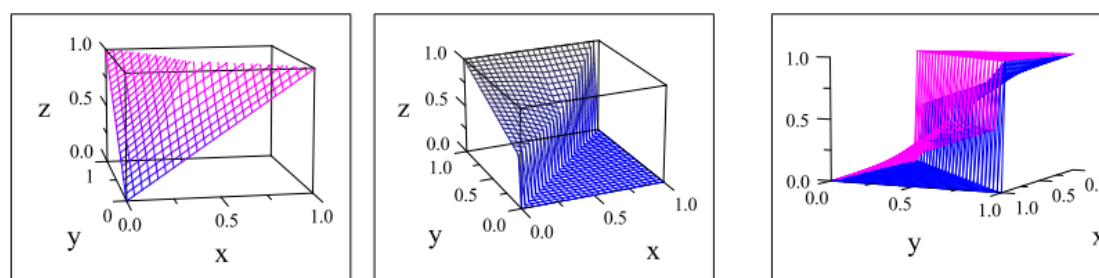


$$S_N(p, q) \quad J_{S_N}(p, q) = \begin{cases} 0 & \text{if } p \geq q \\ \min(1 - p, q) & \text{otherwise} \end{cases} \quad S_N \text{ and } J_{S_N}$$

(5) Residuum of the Hamacher t-conorm $S_H(p, q)$ is

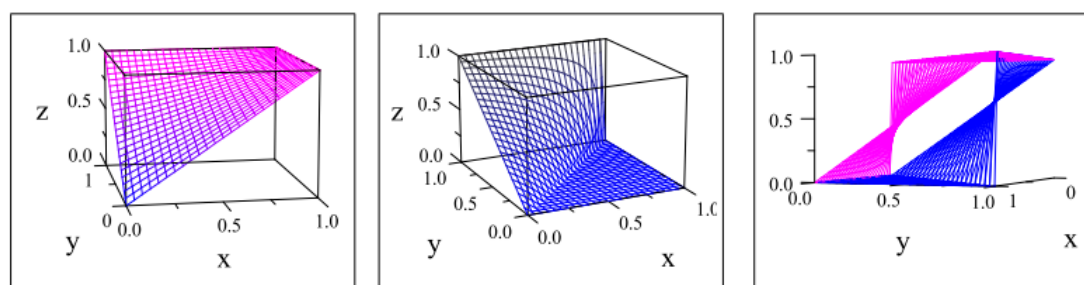


(6) Residuum of the Dubois-Prade t-conorm $S_D(p, q)$ is



$$S_{D_{0.5}}(p, q) \quad J_{S_{D_{0.5}}}(p, q) = \begin{cases} 0 & \text{if } p \geq q \\ \max\left(q, \frac{\alpha q - \alpha}{1-p} + 1\right) & \text{if } p < q \end{cases} \quad S_{D_{0.5}} \text{ and } J_{S_{D_{0.5}}}$$

(7) Residuum of the Hamacher's parametric t-conorm is $S_{T_\alpha}(p, q)$ is



$$S_{T_\alpha}(p, q) \quad J_{S_{T_\alpha}}(p, q) = \begin{cases} 0 & \text{if } p \geq q \\ \frac{\alpha q - p + (1-\alpha)q}{1-(2-\alpha)p + (1-\alpha)p} & \text{if } p < q \end{cases} \quad S_{T_\alpha} \text{ and } J_{S_{T_\alpha}}$$

5 (T, N) Co-Implication and Residual Co-implication Properties

In this section we introduce some properties for (T, N) co-implication and residual co-implication.

Proposition 5.1. For a left continuous t-norm T , then N_T is left-continuous.

Proof: Let T be left-continuous. Assume that there exist $p, q \in [0, 1]$ such that

$$N_T(p) < q \leq N_T(p - \varepsilon), \quad \forall \varepsilon \in (0, x].$$

This contradicts the left-continuity of T and hence, N_T must be left-continuous.

Proposition 5.2. For a left continuous t-norm T , then the supremum in the definition of N_T is the maximum, i.e.,

$$N_T(p) = \max \{t \in [0, 1] \mid T(p, q) = 0\}, \quad \forall p \in [0, 1],$$

where the left side exists for all $p \in [0, 1]$.

Proof: From the previous proposition and since $N_T(p) \geq N_T(p)$, for all $p \in [0, 1]$, one has $T(p, N_T(p)) = 0$, that means, by N_T definition that the supremum is the maximum.

Proposition 5.3. For a left continuous t-norm T , then $\forall p, q \in [0, 1]$ the following equivalence holds:

$$N_T(p) \geq q \Leftrightarrow T(p, q) = 0$$

Proof: Suppose that $N_T \geq q$ for some $p \in [0, 1]$ we consider two cases:

$$\begin{aligned} (i) \quad N_T > q &\Rightarrow \exists t' > q : T(p, t') = 0, \\ &\Rightarrow T(p, q) = 0. \quad (\text{By monotonicity of } T) \end{aligned}$$

$$\begin{aligned} (ii) \quad N_T = q &\Rightarrow q \in \{t \in [0, 1] \mid T(p, t) = 0\}, \\ &\Rightarrow T(p, q) = 0 \text{ or } q \notin \{t \in [0, 1] \mid T(p, t) = 0\}. \end{aligned}$$

Therefore, there exists an increasing sequence $(t_i)_{i \in \mathbb{N}}$ such that $t_i < q$ and $T(p, t_i) = 0$ for all $i \in \mathbb{N}$ and $\lim_{i \rightarrow \infty} t_i = q$. By the left continuity of T we get

$$T(p, q) = T(p, \lim_{i \rightarrow \infty} t_i) = T(p, t_i) \lim_{i \rightarrow \infty} t_i = 0,$$

which is a contradiction.

On the other side assume that T is a left continuous t-norm and for some $p, q \in [0, 1]$.

$$T(p, q) = 0 \Rightarrow q \in \{t \in [0, 1] \mid T(p, t) = 0\},$$

$$\Rightarrow q \leq \max \{t \in [0,1] \setminus T(p,t) = 0\},$$

$$\Rightarrow q \leq N_T(p).$$

The proof is complete.

Proposition below states how a (S,N) implications gives rise to a fuzzy (T,N) co-implication and vice-versa.

Proposition 5.4. A mapping $J_{T,N}$ from $[0,1]^2$ into $[0,1]$ is a (T,N) co-implication with strong negation iff

$$J_{T,N}(p,q) = N(I_{S_T,N}(q,p)),$$

for some $I_{S_T,N}$ and fuzzy (strong) negation N .

Conversely, $I_{S_T,N}$ from $[0,1]^2$ into $[0,1]$ is a (S,N) implication iff

$$I_{S_T,N}(p,q) = N(J_{T,N}(q,p)),$$

for some $J_{T,N}$ and fuzzy (strong) negation N .

Theorem 5.1. For t-norm T , then $J_{T,N} \in Co-FI$.

Proof: We have to show that J_1, J_2 , and J_3 in definition of fuzzy co-implication are satisfied for all $p, q, r \in [0,1]$.

$$J_1 : J_{T,N}(1,1) = J_{T,N}(1,0) = J_{T,N}(0,0) = 0 \text{ and } J_{T,N}(0,1) = 1.$$

$$J_2 : p \leq r \Rightarrow N(p) \geq N(r) \Rightarrow T(q, N(p)) \geq T(q, N(r)) \Rightarrow J_{T,N}(p,q) \geq J_{T,N}(r,q).$$

$$J_3 : q \leq r \Rightarrow T(q, N(p)) \leq T(r, N(p)) \Rightarrow J_{T,N}(p,q) \leq J_{T,N}(p,r).$$

Theorem 5.2. All (T,N) co-implications are fuzzy implications satisfy (Co-NP) and (Co-EP).

Proof: If an (T,N) co-implications then

$$J_{T,N}(0,q) = T(q,1) = q.$$

Also,

$$\begin{aligned} J_{T,N}(p, J(q, r)) &= T(J_{T,N}(q, r), N(p)) = T(T(r, N(q)), N(p)) \\ &= T(N(p), T(r, N(q))) = T(T(N(p), r), N(q)) \\ &= T(T(r, N(p)), N(q)) = T(J_{T,N}(p, r), N(q)) = J_{T,N}(q, J(p, r)). \end{aligned}$$

Theorem 5.3. If $I_{S_T, N}$ satisfies (IP) with strong negation then $J_{T, N}$ satisfies (Co-IP).

Proof: $J_{T, N}(p, p) = T(p, N(p)) = T(N(p), p) = N(S(p, N(p)))$

$$= N(S(N(p), p)) = N(I_{S_T, N}(p, p)) = N(1) = 0.$$

Theorem 5.4. If $I_{S_T, N}$ satisfies (OP) with strong negation then $J_{T, N}$ satisfies (Co-OP).

Proof: We would like to prove that

$$J_{T, N}(p, q) = 0 \Leftrightarrow p \geq q.$$

Let

$$\begin{aligned} p \geq q &\Leftrightarrow N(p) \leq N(q) \Leftrightarrow I_{S_T, N}(N(p), N(q)) = 1, \text{ by (OP)} \\ &\Leftrightarrow N(I_{S_T, N}(N(p), N(q))) = N(1) \Leftrightarrow N(S(p, N(q))) = 0 \\ &\Leftrightarrow N(N(T(N(p), q))) = 0 \Leftrightarrow T(q, N(p)) = 0 \\ &\Leftrightarrow J_{T, N}(p, q) = 0. \end{aligned}$$

Theorem 5.6. For t-norm T and a fuzzy negation N , then

$$J_{T, N}(p, p) = 0, \forall p \in [0, 1],$$

iff

$$T(p, N(p)) = 0, \text{ for all } p \in [0, 1].$$

Proof: If $J_{T, N}(p, p) = 0, \forall p \in [0, 1]$, then

$$T(p, N(p)) = J_{T, N}(p, p) = 0, \forall p \in [0, 1].$$

Conversely, if $T(p, N(p)) = 0, \forall p \in [0, 1]$, then

$$J_{T, N}(p, p) = T(p, N(p)) = 0, \forall p \in [0, 1].$$

Theorem 5.7. For a left continuous t-norm T and a continuous fuzzy negation N , then

$$J_{T,N}(p,p) = 0, \forall p \in [0,1],$$

iff

$$N(p) \leq N_T(p), \forall p \in [0,1].$$

Proof: Let T is a left continuous, t-norm and for a continuous fuzzy negation N , then

$$N_T(x) = \max\{t \in [0,1] : T(p,t) = 0\}, \forall p \in [0,1].$$

$$\text{Then } J_{T,N}(p,p) = T(p, N_T(p)) = 0.$$

By monotonicity of T and if

$$N(p) \leq N_T(p),$$

then $T(p, N(p)) = 0$.

Conversely, let $J_{T,N}(p,p) = 0, \forall p \in [0,1]$, then $T(p, N(p)) = 0, \forall p \in [0,1]$ and if

$$N(p) \in \{t \in [0,1] : T(p,t) = 0\},$$

then

$$N(p) \leq \max\{t \in [0,1] : T(p,t) = 0\} = N_T(p).$$

Theorem 5.8. For a right continuous t-conorm S , then $J_S \in Co-FI$.

Proof: We have to show that J_1, J_2 , and J_3 in definition of fuzzy co-implication are satisfied for

all $p, q, r \in [0,1]$.

$$J_1 : J_S(1,1) = \inf\{r \in [0,1] \setminus S(r,1) \geq 1\} = 0,$$

$$J_S(1,0) = \inf\{r \in [0,1] \setminus S(r,1) \geq 0\} = 0,$$

$$J_S(0,0) = \inf\{r \in [0,1] \setminus S(r,0) \geq 0\} = 0,$$

$$J_S(0,1) = \inf\{r \in [0,1] \setminus S(r,0) \geq 1\} = 1.$$

$$\begin{aligned} J_2 : p \leq r &\Rightarrow \{t \in [0,1] \setminus S(t,p) \geq q\} \subseteq \{t \in [0,1] \setminus S(t,r) \geq q\} \\ &\Rightarrow \inf\{t \in [0,1] \setminus S(t,p) \geq q\} \geq \inf\{t \in [0,1] \setminus S(t,r) \geq q\} \\ &\Rightarrow J_S(p,q) \geq J_S(r,q). \end{aligned}$$

$$\begin{aligned}
J_3 : q \leq r &\Rightarrow \{t \in [0,1] \setminus S(t,p) \geq q\} \supseteq \{t \in [0,1] \setminus S(t,p) \geq r\} \\
&\Rightarrow \inf \{t \in [0,1] \setminus S(t,p) \geq q\} \leq \inf \{t \in [0,1] \setminus S(t,p) \geq r\} \\
&\Rightarrow J_S(p,q) \leq J_S(p,r).
\end{aligned}$$

Theorem 5.9. A co-implications J_S satisfy (Co-NP) and (Co-IP).

Proof: For any S t-conorm and $p, q \in [0,1]$ we get

$$J_S(0,q) = \inf \{r \in [0,1] \setminus S(r,0) \geq q\} = \inf \{r \in [0,1] \setminus r \geq q\} = q.$$

Also,

$$J_S(p,p) = \inf \{r \in [0,1] \setminus S(r,p) \geq p\} = 0.$$

Theorem 5.10. If S is a right continuous, then J_S satisfy (Co-EP) and Co-OP).

Proof: For any right continuous t-conorm S and for all $p, q, r \in [0,1]$ and by using R^*

condition we have

$$\begin{aligned}
J_S(p, J_S(q,r)) &= \inf \{t \in [0,1] \setminus S(t,p) \geq J_S(q,r)\} = \inf \{t \in [0,1] \setminus S(S(t,p), q) \geq r\} \\
&= \inf \{t \in [0,1] \setminus S(t, S(p,q)) \geq r\} = \inf \{t \in [0,1] \setminus S(t, S(q,p)) \geq r\} \\
&= \inf \{t \in [0,1] \setminus S(S(t,q), p) \geq r\} = \inf \{t \in [0,1] \setminus S(t,q) \geq J_S(p,r)\} \\
&= J_S(q, J_S(p,r)).
\end{aligned}$$

Now, we would like to prove that

$$J_S(p,q) = 0 \Leftrightarrow p \geq q.$$

If $p \geq q$ then $S(p,0) = p \geq q$, so $J_S(p,q) = 0$. Conversely, if $J_S(p,q) = 0$, then because

of R^* condition we get $S(p,0) \geq q$, i.e., $p \geq q$.

Conclusion

There is four usual models of fuzzy implications that is (S,N) , residual, QL-operation $(I(p,q) = S(N(p), T(p,q)), p, q \in [0,1]$, and D-operations $(I(p,q) = S(T(N(p), N(q)), q)$, $p, q \in [0,1]$ implication. In this paper we introduced (T,N) and residual co-implication. Now, an interesting natural questions arises that to find Co-QL-operation and Co-D-operations.

Competing interests

The authors declare that they have no competing interests.

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